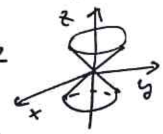
Géométrie et Calcul Diff. – Corrigé du DM2 – 17 avril 2014 – A. Frabetti –

Exercice 1 $\quad \gamma(t) = (t\cos t, t\sin t, t) = t(\cos t, \sin t, 1)\ , \ t\in\mathbb{R}$

1. $\gamma'(t) = (\cos t - t\sin t,\ \sin t + t\cos t,\ 1) \neq (0,0,0)\ \forall t\in\mathbb{R} \Rightarrow \gamma$ est régulière.

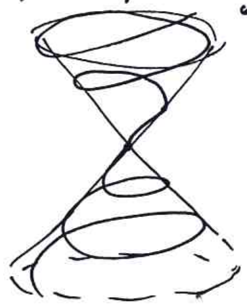
2. $\begin{cases} x = t\cos t \\ y = t\sin t \\ z = t\end{cases} \Rightarrow x^2+y^2 = t^2 = z^2$ donc $\Gamma\subset$ cône d'éq. $x^2+y^2=z^2$



$\gamma(0) = 0\cdot(1,0,1) = (0,0,0)$

$\gamma(\tfrac{\pi}{2}) = \tfrac{\pi}{2}\cdot(0,1,1)$ $\qquad \gamma(\tfrac{3\pi}{2}) = \tfrac{3\pi}{2}(0,-1,1)$

$\gamma(\pi) = \pi\cdot(-1,0,1)$ $\qquad \gamma(2\pi) = 2\pi(1,0,1)$



hélice enroulée sur le cône.

$t\in[0,\pi/2] \Rightarrow \frac{y}{x} = \tan t = \tan z \Rightarrow \Gamma_{[0,\pi/2]} = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=z^2$ et $y = x\tan z$, $z\in[0,\pi/2]\}$

3. $\|\gamma'(t)\|^2 = \cos^2 t - \cancel{2t\cos t\sin t} + t^2\sin^2 t + \sin^2 t + \cancel{2t\sin t\cos t} + t^2\cos^2 t + 1 = 2+t^2$

$$L_0^{\sqrt2}(\gamma) = \int_0^{\sqrt2}\|\gamma'(t)\|\,dt = \int_0^{\sqrt2}\sqrt{2+t^2}\,dt = \sqrt2\int_0^{\sqrt2}\sqrt{1+\left(\tfrac{t}{\sqrt2}\right)^2}\,dt = 2\int_0^{\sqrt2}\sqrt{1+\left(\tfrac{t}{\sqrt2}\right)^2}\,d\left(\tfrac{t}{\sqrt2}\right)$$

$\frac{t}{\sqrt2} = \operatorname{sh} u \Rightarrow 1+\frac{t^2}{2} = 1+\operatorname{sh}^2 u = \operatorname{ch}^2 u$ et $d(\frac{t}{\sqrt2}) = \operatorname{ch} u\, du$

$$L_0^{\sqrt2}(\gamma) = 2\int_{\operatorname{argsh} 0}^{\operatorname{argsh} 1}\operatorname{ch}^2 u\,du = \Big[\operatorname{ch} u\operatorname{sh} u + u\Big]_{\operatorname{argsh} 0}^{\operatorname{argsh} 1}$$

$\operatorname{argsh} 1 \to \operatorname{sh} u = 1,\ \operatorname{ch} u = \sqrt2,\ u = \ln(1+\sqrt2)$

$\operatorname{argsh} 0 \to \operatorname{sh} u = 0,\ \operatorname{ch} u = 1,\ u = 0$

$\Rightarrow L_0^{\sqrt2}(\gamma) = \sqrt2 + \ln(1+\sqrt2)$.

4. $\gamma''(t) = (-(2\sin t + t\cos t),\ 2\cos t - t\sin t,\ 0)$

$\gamma'(t)\wedge\gamma''(t) = (-2\cos t + t\sin t,\ -(2\sin t + t\cos t,\ (\cos t - t\sin t)(2\cos t - t\sin t) + (2\sin t + t\cos t)(\sin t + t\cos t))$

$= (-2\cos t + t\sin t,\ -(2\sin t + t\cos t),\ 2+t^2)$.

$\|\gamma'\wedge\gamma''(t)\|^2 = 8 + 5t^2 + t^4$

$k_\gamma(t) = \frac{\|\gamma'\wedge\gamma''\|}{\|\gamma'\|^3} = \frac{\sqrt{8+5t^2+t^4}}{(2+t^2)\sqrt{2+t^2}} \neq 0\ \forall t \Rightarrow \gamma$ est birégulière.

5. $\gamma'''(t) = (-3\cos t + t\sin t,\ -(3\sin t + t\cos t),\ 0)$

$\tau_\gamma(t) = \frac{\det(\gamma',\gamma'',\gamma''')}{\|\gamma'\wedge\gamma''\|^2} = \frac{6+t^2}{8+5t^2+t^4} \neq 0$ et même $>0\ \forall t$.

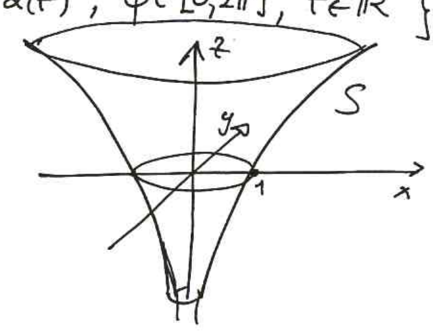
$\det(\gamma',\gamma'',\gamma''') = (2\sin t + t\cos t)(3\sin t + t\cos t) - (-3\cos t + t\sin t)(2\cos t - t\sin t) = 6+t^2$.

2/ Exercice 2

1. $\alpha(t) = (e^t, 0, t)$, $t \in \mathbb{R}$, $\quad S = \{ f(t,\varphi) = R_\varphi^{\vec{Oz}}\alpha(t)$, $\varphi \in [0,2\pi]$, $t \in \mathbb{R} \}$

$f(t,\varphi) = (e^t\cos\varphi, e^t\sin\varphi, t)$.

dessin de α : $x = e^z$ dans le plan xOz



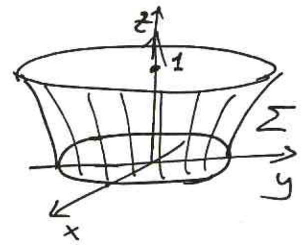
2. $\begin{cases} \dfrac{\partial f}{\partial t} = (e^t\cos\varphi, e^t\sin\varphi, 1) \\ \dfrac{\partial f}{\partial \varphi} = (-e^t\sin\varphi, e^t\cos\varphi, 0) \end{cases}$

$\dfrac{\partial f}{\partial t} \wedge \dfrac{\partial f}{\partial \varphi} = (-e^t\cos\varphi, -e^t\sin\varphi, e^{2t}\cos^2\varphi + e^{2t}\sin^2\varphi) = (-e^t\cos\varphi, -e^t\sin\varphi, e^{2t}) \neq \vec{0}$ $\forall t$ $\forall\varphi$

$\Longrightarrow$ S est régulière partout.

$\left\| \dfrac{\partial f}{\partial t} \wedge \dfrac{\partial f}{\partial \varphi} \right\|^2 = e^{2t}(\cos^2\varphi + \sin^2\varphi) + e^{4t} = e^{2t} + e^{4t} = e^{2t}(1 + e^{2t})$

$\vec{n}_S(t,\varphi) = \dfrac{\frac{\partial f}{\partial t} \wedge \frac{\partial f}{\partial \varphi}}{\left\| \frac{\partial f}{\partial t} \wedge \frac{\partial f}{\partial \varphi} \right\|} = \dfrac{\cancel{e^t}}{\cancel{e^t}\sqrt{1+e^{2t}}}(-\cos\varphi, -\sin\varphi, e^t) = \dfrac{1}{\sqrt{1+e^{2t}}}(-\cos\varphi, -\sin\varphi, e^t)$.



3. Parallèle à $t=0$: cercle $\beta(\varphi) = (\cos\varphi, \sin\varphi, 0)$ $\quad \varphi \in [0, 2\pi]$

parallèle à $t=1$: cercle $\gamma(\varphi) = (e\cos\varphi, e\sin\varphi, 1)$

$\text{Aire}(\Sigma) = \displaystyle\iint_\Sigma \left\| \frac{\partial f}{\partial t} \wedge \frac{\partial f}{\partial \varphi} \right\| dt\, d\varphi = \int_0^{2\pi} d\varphi \int_0^1 dt\, e^t\sqrt{1+e^{2t}} = 2\pi \int_0^1 \sqrt{1+e^{2t}} \cdot \underbrace{e^t dt}_{d(e^t)}$

$e^t = \text{sh}\, u \Rightarrow \sqrt{1+e^{2t}} = \text{ch}\, u$, $\quad d(e^t) = \text{ch}\, u\, du$

$\text{Aire}(\Sigma) = 2\pi \displaystyle\int_{\text{argsh}(1)}^{\text{argsh}(e)} \text{ch}^2 u\, du = \cancel{2}\pi \cdot \frac{1}{\cancel{2}} \left[\text{ch}u\,\text{sh}u + u \right]_{\text{argsh}(1)}^{\text{argsh}(e)}$ $\leftarrow$ $\text{sh}\,u = e$, $\text{ch}\,u = \sqrt{1+e^2}$; $\text{sh}\,u = 1$, $\text{ch}\,u = \sqrt{1+1} = \sqrt{2}$

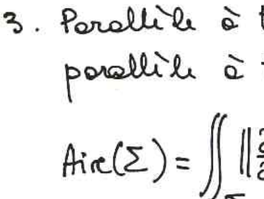
$= \pi \left[e\sqrt{1+e^2} + \text{argsh}(e) - \sqrt{2} - \text{argsh}(1) \right]$.

4. $\begin{cases} x = e^t\cos\varphi \\ y = e^t\sin\varphi \\ z = t \end{cases} \Rightarrow x^2 + y^2 = e^{2t}(\cos^2\varphi + \sin^2\varphi) = e^{2t} = e^{2z} \Rightarrow$ éq. $\boxed{x^2 + y^2 = e^{2z}}$

ou bien $\boxed{z = \frac{1}{2}\ln(x^2+y^2) = \ln\sqrt{x^2+y^2}}$

L'équation $z = \ln\sqrt{x^2+y^2}$ dit que S est le graphe de la fct $g(x,y) = \ln\sqrt{x^2+y^2}$.

5.



$\Omega = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 1$ et $z \geq \ln\sqrt{x^2+y^2} \}$

$\text{Vol}(\Omega) = \displaystyle\iint_D |g(x,y)|\, dx\,dy = \iint_{0 < x^2+y^2 \leq 1} \left| \ln\sqrt{x^2+y^2} \right| dx\,dy = \int_0^{2\pi} d\theta \int_0^1 |\ln\rho| \cdot \rho\, d\rho$

$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$ $\quad dx\,dy = \rho\, d\rho\, d\theta$

puisque $\ln\rho \leq 0$ pour $\rho \in]0,1]$, on a

$\text{Vol}(\Omega) = -2\pi \displaystyle\int_0^1 \ln\rho \cdot \rho\, d\rho = -2\pi \left(\left[\frac{1}{2}\rho^2 \ln\rho \right]_0^1 - \frac{1}{2}\int_0^1 \frac{\rho^2}{\rho} d\rho \right) = -2\pi \left[\frac{1}{2}\rho^2\ln\rho - \frac{1}{4}\rho^2 \right]_0^1$

$u = \ln\rho \quad v' = \rho$

$u' = \frac{1}{\rho} \quad v = \frac{1}{2}\rho^2$

$= -\pi \left[\rho^2 \ln\rho - \frac{1}{2}\rho^2 \right]_0^1 = -\pi\left(1 \cdot 0 - \frac{1}{2} - 0 + 0\right) = \dfrac{\pi}{2}$.

$\lim\limits_{\rho \to 0} \rho^2 \ln\rho = 0 \cdot -\infty$ indet mais $\lim\limits_{\rho\to 0} \rho^2\ln\rho = \lim\limits_{\rho \to 0} \dfrac{\ln\rho}{\frac{1}{\rho^2}} \underset{\text{l'Hôpital}}{=} \lim\limits_{\rho \to 0} \dfrac{\frac{1}{\rho}}{-\frac{2}{\rho^3}} = \lim\limits_{\rho \to 0} -\dfrac{\rho^{\cancel{3}}}{2\cancel{\rho}} = 0$