CONTRÔLE TERMINALE DE MATH 2 — 16/12/08 — CORRIGÉ — A. Frabetti

Ex. 1 $f(x,y) = x^2 - 2xy + y^3$, $D_f = \mathbb{R}^2$

1. Points critiques : $\vec{\nabla} f(x,y) = \begin{pmatrix} 2x-2y \\ -2x+3y^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x = y \\ y(2-3y) = 0 \end{cases} \Leftrightarrow$ $x=y=0$ ou $x=y=\frac{2}{3}$

Donc il y a deux pts critiques : $(0,0)$ et $(\frac{2}{3}, \frac{2}{3})$.

Nature des pts critiques : $\operatorname{Hess} f(x,y) = \begin{pmatrix} 2 & -2 \\ -2 & 6y \end{pmatrix}$, $\det \operatorname{Hess} f(x,y) = 12y - 4$.

En $(0,0)$ on a $\det \operatorname{Hess} f(0,0) = -4 < 0$, donc $(0,0)$ est un point col.

En $(\frac{2}{3}, \frac{2}{3})$ on a $\det \operatorname{Hess} f(\frac{2}{3}, \frac{2}{3}) = 12 \cdot \frac{2}{3} - 4 = 8 - 4 = 4 > 0$, donc $(\frac{2}{3}, \frac{2}{3})$ est un extrema local.

Puisque $\frac{\partial^2 f}{\partial x^2}(\frac{2}{3}, \frac{2}{3}) = 2 > 0$, le pt $(\frac{2}{3}, \frac{2}{3})$ est un minimum local.

2. Puisque $\mathbb{R}^2$ n'a pas de bord, la fct f a des extrema globaux seulement si les extrema locaux sont globaux. Le seul extrema loc. est le min. loc. $(\frac{2}{3}, \frac{2}{3})$: ce point n'est pas un min. absolu, car la fct f tend à $-\infty$, par ex. pour $x = 0$ et $y \to -\infty$.

Ex. 2 $\vec{V}(x,y) = (3x^2 + y^2)\,\vec{\imath} + 2xy\,\vec{\jmath}$, $D_{\vec{V}} = \mathbb{R}^2$.

1. Puisque $\mathbb{R}^2$ est simplement connexe, par le Thm. Poincaré on a $\vec{V} = \overrightarrow{\operatorname{grad}} f \Leftrightarrow \overrightarrow{\operatorname{rot}} \vec{V} = 0$.

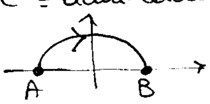
Calculons :
$$\overrightarrow{\operatorname{rot}} \vec{V}(x,y) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2+y^2 & 2xy & 0 \end{vmatrix} = \vec{k}\left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(3x^2+y^2)\right) = \vec{k}(2y - 2y) = \vec{0}.$$

Donc $\vec{V} = \overrightarrow{\operatorname{grad}} f$, calculons f :

$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 + y^2 \\ \frac{\partial f}{\partial y} = 2xy \end{cases} \Leftrightarrow \quad f(x,y) = \int (3x^2+y^2)\,dx + g(y) = x^3 + xy^2 + g(y)$$

$$\frac{\partial f}{\partial y} = 2xy + g'(y) = 2xy \Leftrightarrow g'(y) = 0 \Leftrightarrow g(y) = c. \quad \text{Donc } f(x,y) = x^3 + xy^2 + c.$$

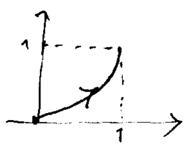
2. C^+ = demi-cercle.



La circulation de $\vec{V}$ le long de C^+ est
$$\int_{C^+} \vec{V} \cdot \vec{dl} = \int_{C^+} \overrightarrow{\operatorname{grad}} f \cdot \vec{dl} = f(B) - f(A) = f(1,0) - f(-1,0) = (1+c) - (-1+c) = 2.$$

Ex. 3 $\vec{U}(x,y) = xy\,\vec{\imath} + y^3\,\vec{\jmath}$, $D_{\vec{U}} = \mathbb{R}^2$.

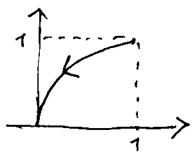
1. γ_1



$y = x^2$, $x : 0 \to 1$, $dy = 2x\,dx$

$$\int_{\gamma_1} \vec{U} \cdot \vec{dl} = \int_{\gamma_1} (xy\,dx + y^3\,dy) = \int_0^1 (x \cdot x^2 + (x^2)^3 \cdot 2x)\,dx = \int_0^1 (x^3 + 2x^7)\,dx = \left[\frac{x^4}{4} + \frac{2}{8}x^8\right]_0^1 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

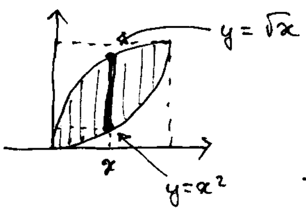
2. γ_2



$x = y^2$, $y : 1 \to 0$, $dx = 2y\,dy$

$$\int_{\gamma_2} \vec{U} \cdot \vec{dl} = \int_{\gamma_2} (xy\,dx + y^3\,dy) = \int_1^0 (y^2 \cdot y \cdot 2y + y^3)\,dy = \int_1^0 (2y^4 + y^3)\,dy = \left[\frac{2}{5}y^5 + \frac{1}{4}y^4\right]_1^0 = -\left(\frac{2}{5} + \frac{1}{4}\right) = -\frac{13}{20}.$$

3. D



$D = \{(x,y) \in \mathbb{R}^2,\ x \le y^2,\ y \ge x^2\}$

$$I = \iint_D x\,dx\,dy = \int_0^1 dx \int_{x^2}^{\sqrt{x}} x\,dy = \int_0^1 dx\,[xy]_{x^2}^{\sqrt{x}} = \int_0^1 \left(x^{\frac{3}{2}} - x^3\right)dx = \left[\frac{1}{\frac{3}{2}+1} x^{\frac{5}{2}} - \frac{1}{4}x^4\right]_0^1 = \frac{2}{5} - \frac{1}{4} = \frac{8-5}{20} = \frac{3}{20}.$$

4. $\vec{V} = \overrightarrow{\operatorname{rot}}\,\vec{U} = \vec{k}\left(\frac{\partial y^3}{\partial x} - \frac{\partial (xy)}{\partial y}\right) = -x\,\vec{k}$, donc $\iint_D \overrightarrow{\operatorname{rot}}\,\vec{U} \cdot \vec{dS} = -\iint_D x\,dx\,dy = -I$

Green-Riemann : $\iint_D \overrightarrow{\operatorname{rot}}\,\vec{U} \cdot \vec{dS} = \oint_{\partial D^+} \vec{U} \cdot \vec{dl}$ où $\partial D^+ = \gamma_1 \cup \gamma_2$. Alors on a :

$$I = -\iint_D \overrightarrow{\operatorname{rot}}\,\vec{U} \cdot \vec{dS} = -\oint_{\partial D^+} \vec{U} \cdot \vec{dl} = -\left(\int_{\gamma_1} \vec{U} \cdot \vec{dl} + \int_{\gamma_2} \vec{U} \cdot \vec{dl}\right) = -\left(\frac{1}{2} - \frac{13}{20}\right) = -\left(\frac{10-13}{20}\right) = \frac{3}{20} \quad \text{ok}.$$