Exercice 1 $\quad f(x,y) = \dfrac{x^2-y^2}{x+1}$

1. $D_f = \{(x,y)\in\mathbb{R}^2 \mid x \neq -1\}$ = plan moins une droite verticale.

2. $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{2x(x+1)-(x^2-y^2)}{(x+1)^2} = \dfrac{x^2+2x+y^2}{(x+1)^2}$

$\dfrac{\partial f}{\partial y}(x,y) = -\dfrac{2y}{x+1}$ $\quad\Longrightarrow\quad \vec{\nabla} f(x,y) = \begin{pmatrix} \dfrac{x^2+2x+y^2}{(x+1)^2} \\ -\dfrac{2y}{x+1} \end{pmatrix}.$

3. $df_{(x,y)} = \dfrac{x^2+2x+y^2}{(x+1)^2}\,dx - \dfrac{2y}{x+1}\,dy.$

4. $df_{(0,1)} = \dfrac{1}{1}\,dx - \dfrac{2}{1}\,dy = dx - 2dy.$

5. $df_{(0,1)}(7,2) = dx(7,2) - 2\,dy(7,2) = 1\cdot 7 - 2\cdot 2 = 7-4 = 3.$

6. $\dfrac{\partial^2 f}{\partial x^2}(x,y) = \dfrac{(2x+2)(x+1)^{\cancel{2}} - (x^2+2x+y^2)\cdot 2\cancel{(x+1)}}{(x+1)^{\cancel{4}\,3}} = \dfrac{\cancel{2x^2}+\cancel{2x}+\cancel{2x}+2-\cancel{2x^2}-\cancel{4x}-2y^2}{(x+1)^3} = \dfrac{2(1-y^2)}{(x+1)^3}$

$\dfrac{\partial^2 f}{\partial x\,\partial y}(x,y) = \dfrac{2y}{(x+1)^2}$

$\dfrac{\partial^2 f}{\partial y^2}(x,y) = -\dfrac{2}{x+1}$ $\quad\Longrightarrow\quad \mathrm{Hess}\, f(x,y) = \begin{pmatrix} \dfrac{2(1-y^2)}{(x+1)^3} & \dfrac{2y}{(x+1)^2} \\ \dfrac{2y}{(x+1)^2} & -\dfrac{2}{x+1} \end{pmatrix}$

7. (x,y) est pt. critique $\iff \vec{\nabla} f(x,y) = \begin{pmatrix}0\\0\end{pmatrix} \iff \begin{cases} x^2+2x+y^2=0 \\ y=0 \end{cases} \iff \begin{cases} x(x+2)=0 \\ y=0 \end{cases}$ $\quad x=0$ ou $x=-2$

Il y a donc deux points critiques : $(0,0)$ et $(-2,0)$.

8. $\det \mathrm{Hess}\, f(0,0) = \det\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = -4 < 0 \Longrightarrow (0,0)$ est un point col.

$\det \mathrm{Hess}\, f(-2,0) = \det\begin{pmatrix} \frac{2}{-1} & 0 \\ 0 & \frac{-2}{-1} \end{pmatrix} = \det\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = -4 < 0 \Longrightarrow (-2,0)$ aussi.

9. Taylor autour de $(-\frac{3}{2}, 0)$:

$f(x,y) = \dfrac{(-3/2)^2}{-1/2} + \dfrac{(-3/2)^2 - 2\cdot\frac{3}{2}}{(-\frac{1}{2})^2}\left(x+\frac{3}{2}\right) + 0\cdot y + \cancel{\frac{1}{2}}\cdot\cancel{2}\cdot\dfrac{1}{(-\frac{1}{2})^2}\left(x+\frac{3}{2}\right)^2 + 0\cdot\left(x+\frac{3}{2}\right)y - \cancel{\frac{1}{2}}\cdot\dfrac{\cancel{2}}{-\frac{1}{2}}\,y^2$ + reste

$= -2\cdot\frac{9}{4} + 4\left(\frac{9}{4}-3\right)\left(x+\frac{3}{2}\right) + 4\left(x+\frac{3}{2}\right)^2 + 2y^2$ + reste

$\Longrightarrow \dfrac{x^2-y^2}{x+1} \underset{(x,y)\to(-\frac{3}{2},0)}{\sim} -\dfrac{9}{2} - 3\left(x+\frac{3}{2}\right) + 4\left(x+\frac{3}{2}\right)^2 + 2y^2.$

10. Taylor autour de $(1,1)$:

$f(x,y) = \cancel{\dfrac{1-1}{2}} + \dfrac{1+2+1}{2^2}(x-1) - \dfrac{2}{2}(y-1) + \cancel{\dfrac{1}{2}\cdot\dfrac{2(1-1)}{2^3}(x-1)^2} + \dfrac{2}{2^2}(x-1)(y-1) - \dfrac{1}{2}\cdot\dfrac{\cancel{2}}{\cancel{2}}(y-1)^2$ + reste

$\Longrightarrow \dfrac{x^2-y^2}{x+1} \underset{(x,y)\to(1,1)}{\sim} (x-1) - (y-1) + \dfrac{1}{2}(x-1)(y-1) - \dfrac{1}{2}(y-1)^2.$

2) <u>Exercice 2</u>

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = -\frac{2x}{(x^2-y^2)^2} \\ \frac{\partial f}{\partial y}(x,y) = \frac{2y}{(x^2-y^2)^2} \end{cases} \qquad x \neq y \quad \text{et} \quad x \neq -y$$

1. $$\begin{cases} x(u,v) = \frac{u+v}{2} \\ y(u,v) = \frac{u-v}{2} \end{cases} \Rightarrow F(u,v) := f\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

$$x^2-y^2 = \left(\frac{u+v}{2}\right)^2 - \left(\frac{u-v}{2}\right)^2 = \frac{u^2+2uv+v^2}{4} - \frac{u^2-2uv+v^2}{4} = \frac{\cancel{u^2}+2uv+\cancel{v^2}-\cancel{u^2}+2uv-\cancel{v^2}}{4}$$

$$= \frac{4uv}{4} = uv.$$

$$\frac{\partial F}{\partial u}(u,v) = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{-\cancel{2}\frac{(u+v)}{\cancel{2}}}{(uv)^2}\cdot\frac{1}{2} + \frac{\cancel{2}\frac{(u-v)}{\cancel{2}}}{(uv)^2}\cdot\frac{1}{2}$$

$$= \frac{\cancel{-u}-v+\cancel{u}-v}{2u^2v^2} = \frac{-2v}{2u^2v^2} = -\frac{1}{u^2v}.$$

$$\frac{\partial F}{\partial v}(u,v) = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{-(u+v)}{u^2v^2}\cdot\frac{1}{2} + \frac{u-v}{u^2v^2}\cdot\left(-\frac{1}{2}\right)$$

$$= \frac{-u\cancel{-v}-u\cancel{+v}}{2u^2v^2} = \frac{-2u}{2u^2v^2} = -\frac{1}{uv^2}.$$

2. $$\begin{cases} x(t) = \operatorname{ch} t \\ y(t) = \operatorname{sh} t \end{cases} \Rightarrow G(t) := f(\operatorname{ch} t, \operatorname{sh} t)$$

$$x^2-y^2 = \operatorname{ch}^2 t - \operatorname{sh}^2 t = 1.$$

$$G'(t) = \frac{\partial f}{\partial x}\cdot x'(t) + \frac{\partial f}{\partial y}\cdot y'(t)$$

$$= \frac{-2\operatorname{ch} t}{1}\cdot \operatorname{sh} t + \frac{2\operatorname{sh} t}{1}\cdot \operatorname{ch} t = -2\operatorname{ch} t \operatorname{sh} t + 2\operatorname{sh} t \operatorname{ch} t = 0.$$