CC5 - Math 2 - vendredi 10 janvier 2014 - CORRIGÉ - Alifrabetti

(1)

Ex.1 $df_{(x,y)} = \frac{\partial f}{\partial x}(x,y)\,dx + \frac{\partial f}{\partial y}(x,y)\,dy = \frac{y^3}{2\sqrt{xy^3}}\,dx + \frac{3xy^2}{2\sqrt{xy^3}}\,dy$.

Ex. 2 $f(x,y) = x^3y - 4xy$

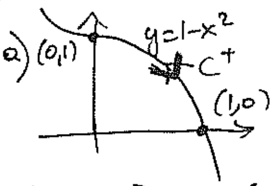
$$\vec{\nabla} f(x,y) = \begin{pmatrix} 3x^2y - 4y \\ x^3 - 4x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} y(3x^2-4) = 0 < \begin{matrix} y=0 \\ 3x^2-4=0 \ \text{IMP avec} \end{matrix} \\ x(x^2-4) = 0 < \begin{matrix} x=0 \\ x=\pm 2 \end{matrix} \end{cases} \Rightarrow y=0$$

$\Rightarrow$ Trois points critiques : $(0,0)$, $(-2,0)$, $(2,0)$.

$$\det \mathrm{Hess} f(x,y) = \det\begin{pmatrix} 6xy & 3x^2-4 \\ 3x^2-4 & 0 \end{pmatrix} = -(3x^2-4)^2 \leq 0$$

Dans les trois pts critiques on a $x \neq \pm\frac{2}{\sqrt{3}}$, donc $\det \mathrm{Hess} f(x,y) < 0$, il s'agit de trois points col.

Ex. 3 a)



$C^+ = \{(x,y) \mid y = 1-x^3,\ x: 0 \to 1\}$

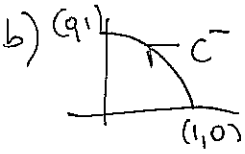
$\vec{V}(x,y) = \begin{pmatrix} 5xy \\ -(x^3+y) \end{pmatrix}$

$$\int_{C^+} \vec{V}\cdot\vec{d\ell} = \int_{C^+} 5xy\,dx - (x^3+y)\,dy = \int_{\substack{C^+ \\ x:0\to 1 \\ y=1-x^3}} 5xy\,dx - \int_{\substack{C^+ \\ x:0\to 1 \\ y=1-x^3 \Rightarrow dy=-3x^2dx}} (x^3+y)\,dy =$$

$$= \int_0^1 5x(1-x^3)\,dx - \int_0^1 (\cancel{x^3}+1-\cancel{x^3})\cdot(-3x^2)\,dx = \int_0^1 \left[5x - 5x^4 + 3x^2\right] dx =$$

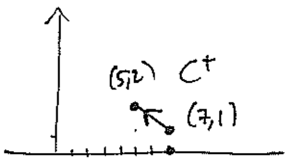
$$= \left[\frac{5}{2}x^2 - x^5 + x^3\right]_0^1 = \frac{5}{2} - 1 + 1 = \frac{5}{2}.$$

b)



$$\int_{C^-} \vec{V}\cdot\vec{d\ell} = -\int_{C^+} \vec{V}\cdot\vec{d\ell} = -\frac{5}{2}.$$

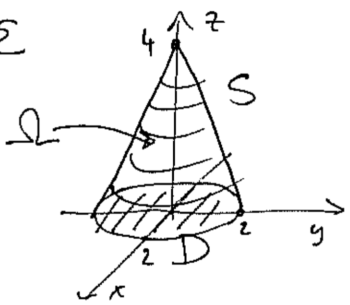
Ex 4



$$\int_{C^+} \vec{\nabla} f(x,y)\cdot\vec{d\ell} = f(5,2) - f(7,1) = 5^2\cdot 2^3 - 7^2\cdot 1^3 = 25\times 8 - 49$$
$$= 200 - 49 = 151.$$

2)

Ex. 5



$$D = \{ (x,y,z) \in \mathbb{R}^3 \mid z = 0 \ , \ x^2+y^2 \leq 4 \}$$

$$S = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 4 - z \ , \ 0 \leq z \leq 4 \}$$

$$\vec{V}(x,y,z) = z\vec{i} + 3y\vec{j} - 2x\vec{k}$$

a) $\mathrm{Vol}\,\Omega = \iiint_\Omega dx\,dy\,dz$ $\cancel{\int_0^4 dz \iint_D dx\,dy}$ où

$$\Omega = \{(x,y,z) \mid 0 \leq z \leq 4 \ , \ 0 \leq x^2+y^2 \leq 4-z \} = \{ (\rho,\theta,z) \mid 0 \leq z \leq 4 \ , 0 \leq \rho \leq \sqrt{4-z} \ , \ 0 \leq \theta \leq 2\pi \}$$

$$\mathrm{Vol}\,\Omega = \int_0^4 dz \int_0^{\sqrt{4-z}} \rho\, d\rho \int_0^{2\pi} d\theta = 2\pi \int_0^4 \left[\tfrac{1}{2}\rho^2\right]_0^{\sqrt{4-z}} dz = \frac{2\pi}{2} \int_0^4 (4-z)\, dz =$$

$$= \pi \cdot \left[4z - \tfrac{1}{2}z^2\right]_0^4 = \pi\left(16 - \tfrac{1}{2}\cdot 16\right) = \pi(16-8) = 8\pi .$$

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b) $\sigma(\theta,\rho) = (\rho\cos\theta, \rho\sin\theta, 0)$ $\qquad \theta \in [0,2\pi]$, $\rho \in [0,2]$

$$\begin{cases} \dfrac{\partial \sigma}{\partial \theta} = (-\rho\sin\theta, \rho\cos\theta, 0) \\ \dfrac{\partial \sigma}{\partial \rho} = (\cos\theta, \sin\theta, 0) \end{cases} \Rightarrow \frac{\partial \sigma}{\partial \theta} \wedge \frac{\partial \sigma}{\partial \rho} = (0, 0, -\rho\sin^2\theta - \rho\cos^2\theta) = (0,0,-\rho)$$

$$\iint_{D^+} \vec{V}\cdot\vec{dS} = \int_0^{2\pi} d\theta \int_0^2 d\rho \left(\vec{V} \cdot \overrightarrow{\frac{\partial\sigma}{\partial\theta}\wedge\frac{\partial\sigma}{\partial\rho}}\right) = \int_0^{2\pi} d\theta \int_0^2 d\rho \cdot \begin{pmatrix} 0 \\ 3\rho\sin\theta \\ -2\rho\cos\theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -\rho \end{pmatrix}$$

$$= \int_0^{2\pi} d\theta \int_0^2 d\rho \cdot 2\rho^2\cos\theta = 2\int_0^2 \rho^2 d\rho \int_0^{2\pi} \cos\theta\, d\theta = 2\cdot\left[\tfrac{1}{3}\rho^3\right]_0^2 \cdot \underbrace{\left[\sin\theta\right]_0^{2\pi}}_{0} = 0 .$$

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c) $\Sigma^+ = S^+ \cup D^+$

$$\iint_\Sigma \vec{V}\cdot\vec{dS} = \iiint_\Omega \mathrm{div}\vec{V} \cdot dx\,dy\,dz$$

$$\mathrm{div}\vec{V} = \frac{\partial z}{\partial x} + \frac{\partial(3y)}{\partial y} + \frac{\partial(-2x)}{\partial z} = 0 + 3 + 0 = 3 \Rightarrow \iint_\Sigma \vec{V}\cdot\vec{dS} = 3\iiint_\Omega dx\,dy\,dz$$

$$= 3\,\mathrm{Vol}(\Omega) = 24\pi .$$

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d) $$\iint_S \vec{V}\cdot\vec{dS} = \iint_\Sigma \vec{V}\cdot\vec{dS} - \iint_D \vec{V}\cdot\vec{dS} = 24\pi - 0 = 24\pi .$$

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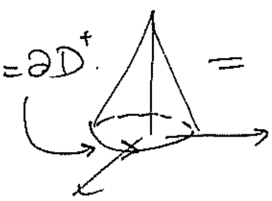
e) $$\vec{W} = \vec{\mathrm{rot}}\,\vec{V} \Rightarrow \iint_{S^+} \vec{W}\cdot\vec{dS} = \int_{\partial S^+ = \partial D = \{x^2+y^2=4\}} \vec{V}\cdot\vec{dl} = \int_0^{2\pi} \vec{V}(2\cos\theta, 2\sin\theta, 0)\cdot\gamma'(\theta)\, d\theta$$

$\partial S^+ = \partial D^+$



$$= \{ \underbrace{(2\cos\theta, 2\sin\theta, 0)}_{\gamma(\theta)} \mid \theta \in [0,2\pi] \}$$

$$= \int_0^{2\pi} \begin{pmatrix} 0 \\ 6\sin\theta \\ -2\cos\theta \end{pmatrix} \cdot \begin{pmatrix} -2\sin\theta \\ 2\cos\theta \\ 0 \end{pmatrix} d\theta$$

$$= \int_0^{2\pi} 12\sin\theta\cos\theta\, d\theta = 6\left[\sin^2\theta\right]_{\theta=0}^{\theta=2\pi}$$

$$= 0 .$$