20/10/2026 Math2A TD4

Exo 20 Règle de la chaine : $f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$

t.q. $\frac{\partial f}{\partial x} = \frac{2x}{y-1}$, $\frac{\partial f}{\partial y} = -\frac{x^2}{(y-1)^2}$

a) Calculer les dérivées partielles de $F(u,v) = f(\underbrace{2u-v}_{x}, \underbrace{u-2v}_{y})$

$F: \mathbb{R}^2 \to \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$, $(u,v) \mapsto (x,y)$

$\begin{cases} x = 2u - v \\ y = u - 2v \end{cases}$

$$\frac{\partial F}{\partial u} = \partial_u F = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{2x}{y-1}\Big|_{\substack{x=2u-v\\y=u-2v}} \cdot 2 - \frac{x^2}{(y-1)^2}\Big|_{\substack{x=2u-v\\y=u-2v}} \cdot 1$$

$$= \frac{2\cdot 2(2u-v)}{u-2v-1} - \frac{(2u-v)^2}{(u-2v-1)^2} = \frac{4(2u-v)(u-2v-1) - (2u-v)^2}{(u-2v-1)^2}$$

$$= \frac{(2u-v)\left[4(u-2v-1) - (2u-v)\right]}{(u-2v-1)^2}$$

$4u - 8v - 4 - 2u + v$

$(2u - 8v - 4 + v)$

$$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{2(2u-v)}{u-2v-1}\cdot(-1) - \frac{(2u-v)^2}{(u-2v-1)^2}\cdot(-2) = \text{etc.}$$

b) Calculer la dérivée de $G(t) = f(\underbrace{t+1}_{x}, \underbrace{t^2}_{y})$

$$\frac{dG}{dt}(t) = G'(t) = \frac{\partial f}{\partial x}\cdot\dot{x}(t) + \frac{\partial f}{\partial y}\cdot\dot{y}(t)$$

$$= \frac{2x}{y-1}\Big|_{\substack{x=t+1\\y=t^2}} \cdot 1 - \frac{x^2}{(y-1)^2}\Big|_{\substack{x=t+1\\y=t^2}} \cdot 2t$$

$$= \frac{2(t+1)}{t^2-1} - \frac{2t(t+1)^2}{(t^2-1)^2} = \frac{2(t+1)}{t^2-1}\left[1 - \frac{t(t+1)}{t^2-1}\right]$$

$$= \frac{2(t+1)}{t^2-1}\cdot\frac{(t^2-1) - (t^2+t)}{t^2-1} = \frac{2(t+1)\left[t^2-1-t^2-t\right]}{(t^2-1)^2}$$

$$= -\frac{2(t+1)^2}{(t^2-1)^2} = -\frac{2(t+1)^2}{(t-1)^2(t+1)^2} = -\frac{2}{(t-1)^2}$$

$(t-1)(t+1))^2$

Exo 21 Calculer $\frac{\partial g}{\partial x}, \ldots$ et dg

a) $g(x,y) = f(\underbrace{x^2-y^2}_{u}, \underbrace{2xy}_{v})$ f inconnue ! ou (a,b), (α,β), (r,s), ...

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}\cdot 2x + \frac{\partial f}{\partial v}\cdot 2y$$

↑ inconnues — connues $\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$

$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}\cdot(-2y) + \frac{\partial f}{\partial v}\cdot 2x$$

différentielle : g ft de $(x,y) \Rightarrow dg_{(x,y)} = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy$

$$\Rightarrow dg_{(x,y)} = \left(2x\frac{\partial f}{\partial u} + 2y\frac{\partial f}{\partial v}\right)dx + \left(-2y\frac{\partial f}{\partial u} + 2x\frac{\partial f}{\partial v}\right)dy$$

b) $g(x,y,z) = f(\underbrace{2x-yz}_{u}, \underbrace{xy-3z}_{v})$ $\begin{cases} u = 2x - yz \\ v = xy - 3z \end{cases}$

$\mathbb{R}^3 \to \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$, $(x,y,z) \mapsto (x,y)$

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}\cdot 2 + \frac{\partial f}{\partial v}\cdot y$$

$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}\cdot(-z) + \frac{\partial f}{\partial v}\cdot x$$

$$\frac{\partial g}{\partial z} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} = \frac{\partial f}{\partial u}\cdot(-y) + \frac{\partial f}{\partial v}\cdot(-3)$$

$$dg_{(x,y,z)} = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz$$

$$= \left(2\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v}\right)dx + \left(-z\frac{\partial f}{\partial u} + x\frac{\partial f}{\partial v}\right)dy - \left(y\frac{\partial f}{\partial u} + 3\frac{\partial f}{\partial v}\right)dz.$$

Exo 22 Jacobienne

$h: \mathbb{R}^2 \to \mathbb{R}^2$ inconnue, $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$

a) $g(x,y) = h(\underbrace{x^2-y^2}_{u}, \underbrace{2xy}_{v})$,

$\mathbb{R}^2 \to \mathbb{R}^2 \xrightarrow{h} \mathbb{R}^2$, $(x,y) \mapsto (u,v)$, g : $\begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix}$

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial h}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial h}{\partial u}\cdot 2x + \frac{\partial h}{\partial v}\cdot 2y$$

$\left(\frac{\partial h_1}{\partial u}\right)$

$$\frac{\partial g}{\partial y} = \frac{\partial h}{\partial u}\cdot(-2y) + \frac{\partial h}{\partial v}\cdot 2x$$

$g_1(x,y) = h_1(x^2-y^2, 2xy)$

$g_2(x,y) = h_2(x^2-y^2, 2xy)$

$$Jg(x,y) = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial u}2x + \frac{\partial h_1}{\partial v}\cdot 2y & \frac{\partial h_1}{\partial u}\cdot(-2y) + \frac{\partial h_1}{\partial v}\cdot 2x \\ \frac{\partial h_2}{\partial u}2x + \frac{\partial h_2}{\partial v}2y & \frac{\partial h_2}{\partial u}(-2y) + \frac{\partial h_2}{\partial v}\cdot 2x \end{bmatrix}$$

b) Idem, voir le corrigé écrit du TD

Exo 23

$F: \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto \begin{pmatrix} xe^y \\ ye^x \end{pmatrix}$ $\quad$ $G: \mathbb{R}^2 \to \mathbb{R}^2$, $(u,v) \mapsto \begin{pmatrix} u+v \\ u-v \end{pmatrix}$

- $\boxed{f = G \circ F}$ $\quad f(x,y) = G\left(F(x,y)\right) = G\left(\underbrace{xe^y}_{u}, \underbrace{ye^x}_{v}\right) = \begin{pmatrix} xe^y + ye^x \\ xe^y - ye^x \end{pmatrix}$

$\boxed{g = F \circ G}$ $\quad g(u,v) = F\left(G(u,v)\right) = F\left(\underbrace{u+v}_{x}, \underbrace{u-v}_{y}\right) = \begin{pmatrix} (u+v)e^{u-v} \\ (u-v)e^{u+v} \end{pmatrix}$

- $JF(x,y) = \begin{pmatrix} e^y & xe^y \\ ye^x & e^x \end{pmatrix}$ $\quad JG(u,v) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

- $Jf(x,y) = \begin{pmatrix} e^y + ye^x & xe^y + e^x \\ e^y - ye^x & xe^y - e^x \end{pmatrix} = J(G \circ F)$

$$JG \cdot JF = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\cdot\begin{pmatrix} e^y & xe^y \\ ye^x & e^x \end{pmatrix} = \begin{pmatrix} e^y + ye^x & xe^y + e^x \\ e^y - ye^x & xe^y - e^x \end{pmatrix}$$

$$J(G \circ F) = JG \cdot JF$$

composée de fonctions — produit de matrices

- Idem : vérifier que $J(\underbrace{F \circ G}_{g}) \overset{!}{=} JF \cdot JG$!