TD5

Exo 25 + Exo 26

a) $f(x,y) = x^3y + x^2y^2 + xy^3$

$\vec{\nabla} f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 3x^2y + 2xy^2 + y^3 \\ x^3 + 2x^2y + 3xy^2 \end{pmatrix}$

$Hf(x,y) = \begin{pmatrix} 6xy + 2y^2 & 3x^2 + 4xy + 3y^2 \\ 3x^2 + 6xy + 3y^2 & 2x^2 + 6xy \end{pmatrix}$

$\text{Hess}_f(x,y) = \det Hf(x,y) = (6xy+2y^2)(2x^2+6xy) - (3x^2+4xy+3y^2)^2$

$= \dots$

$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$

$(a+b)^2 = a^2+b^2+2ab$

Laplacien $\Delta f(x,y) = \text{tr}\, Hf(x,y) = 6xy + 2y^2 + 2x^2 + 6xy = 2x^2 + 2y^2 + 12xy$

$= 2(x^2+y^2+6xy)$

$\Delta f(1,-1) = -4 - 4 = -8$.

$x=1,\ y=-1$

$f(1,-1) = -1+1-1 = -1$

$\vec{\nabla} f(1,-1) = \begin{pmatrix} -3+2-1 \\ 1-2+3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

$Hf(1,-1) = \begin{pmatrix} -6+2 & 3-4+3 \\ \cdot & 2-6 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}$

$\text{Hess}_f(1,-1) = (-4)^2 - 2^2 = 16 - 4 = 12$

b) $g(\varphi,\theta) = \varphi \sin\theta - \theta \sin\varphi$

$\vec{\nabla} g(\varphi,\theta) = \begin{pmatrix} \sin\theta - \theta\cos\varphi \\ \varphi\cos\theta - \sin\varphi \end{pmatrix}$

$Hg(\varphi,\theta) = \begin{pmatrix} \theta\sin\varphi & \cos\theta - \cos\varphi \\ \cos\theta - \cos\varphi & -\varphi\sin\theta \end{pmatrix}$

$\Delta g(\varphi,\theta) = \theta\sin\varphi - \varphi\sin\theta$

$g(0,\frac{\pi}{2}) = 0 - \frac{\pi}{2}\sin 0 = 0$

$\vec{\nabla} g(0,\frac{\pi}{2}) = \begin{pmatrix} \sin\frac{\pi}{2} - \frac{\pi}{2}\sin 0 \\ 0 - \sin 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$Hg(0,\frac{\pi}{2}) = \begin{pmatrix} \frac{\pi}{2}\sin 0 & \cos\frac{\pi}{2} - \cos 0 \\ \text{idem} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$\Delta g(0,\frac{\pi}{2}) = 0$ $\quad \det Hg(0,\frac{\pi}{2}) = 0 \cdot 0 - (-1)^2 = -1$

c) $h(x,y,z) = xy^2 + yz^2$

$\vec{\nabla} h(x,y,z) = \begin{pmatrix} y^2 \\ 2xy + z^2 \\ 2yz \end{pmatrix}$

$Hh(x,y,z) = \begin{pmatrix} 0 & 2y & 0 \\ 2y & 2x & 2z \\ 0 & 2z & 2y \end{pmatrix}$ symétrique !

$\det Hh(x,y,z) = 0 - 2y\begin{vmatrix} 2y & 2z \\ 0 & 2y \end{vmatrix} + 0 = -2y(4y^2 - 0) = -8y^3$

$\Delta h(x,y,z) = 0 + 2x + 2y$

$h(0,1,2) = 0 + 1\cdot 2^2 = 4$

$\vec{\nabla} h(0,1,2) = \begin{pmatrix} 1 \\ 0+4 \\ 2\cdot 1\cdot 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$

$Hh(0,1,2) = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 4 \\ 0 & 4 & 2 \end{pmatrix}$

$\det Hh(0,1,2) = -8$

$\Delta h(0,1,2) = 0 + 0 + 2 = 2$

d) $F(u,v) = \frac{u^2 - v^2}{u^2 + v^2}$

$\vec{\nabla} F(u,v) = \begin{pmatrix} \frac{2u(u^2+v^2) - (u^2-v^2)2u}{(u^2+v^2)^2} \\ \frac{-2v(u^2+v^2) - (u^2-v^2)2v}{(u^2+v^2)^2} \end{pmatrix} = \begin{pmatrix} \frac{2u^3 + 2uv^2 - 2u^3 + 2uv^2}{(u^2+v^2)^2} \\ \frac{-2u^2v - 2v^3 - 2u^2v + 2v^3}{(u^2+v^2)^2} \end{pmatrix} = \begin{pmatrix} \frac{4uv^2}{(u^2+v^2)^2} \\ \frac{-4u^2v}{(u^2+v^2)^2} \end{pmatrix}$

$\frac{\partial^2 F}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{4uv^2}{(u^2+v^2)^2}\right) = \frac{4v^2(u^2+v^2)^2 - 4uv^2 \cdot 2(u^2+v^2)\cdot 2u}{(u^2+v^2)^4}$

$= \frac{(u^2+v^2)\left[4v^2(u^2+v^2) - 4uv^2 \cdot 4u\right]}{(u^2+v^2)^{4\,3}}$

$= \frac{4u^2v^2 + 4v^4 - 16u^2v^2}{(u^2+v^2)^3} = \frac{4v^4 - 12u^2v^2}{(u^2+v^2)^3} = \frac{4v^2(v^2-3u^2)}{(u^2+v^2)^3}$

$\frac{\partial^2 F}{\partial u \partial v} = \frac{\partial}{\partial v}\left(\frac{4uv^2}{(u^2+v^2)^2}\right) = \frac{8uv(u^2+v^2)^2 - 4uv^2 \cdot 2(u^2+v^2)\cdot 2v}{(u^2+v^2)^4}$

$= \frac{(u^2+v^2)\left[8uv(u^2+v^2) - 4uv^2 \cdot 4v\right]}{(u^2+v^2)^{4\,3}}$

$= \frac{8u^3v + 8uv^3 - 16uv^3}{(u^2+v^2)^3} = \frac{8u^3v - 8uv^3}{(u^2+v^2)^3} = \frac{8uv(u^2-v^2)}{(u^2+v^2)^3}$

$\frac{\partial^2 F}{\partial v^2} = \frac{\partial}{\partial v}\left(-\frac{4u^2v}{(u^2+v^2)^2}\right) = -\frac{4u^2(u^2+v^2)^2 - 4u^2v \cdot 2(u^2+v^2)\cdot 2v}{(u^2+v^2)^4}$

$= -\frac{(u^2+v^2)\left[4u^2(u^2+v^2) - 4u^2v \cdot 4v\right]}{(u^2+v^2)^{4\,3}}$

$= -\frac{4u^4 + 4u^2v^2 - 16u^2v^2}{(u^2+v^2)^3} = -\frac{4u^4 - 12u^2v^2}{(u^2+v^2)^3} = -\frac{4u^2(u^2-3v^2)}{(u^2+v^2)^3}$

$HF(u,v) = \frac{\times 4}{(u^2+v^2)^3}\begin{pmatrix} 4v^2(v^2-3u^2) & 8uv(u^2-v^2) \\ 8uv(u^2-v^2) & -4u^2(u^2-3v^2) \end{pmatrix}$

$HF(1,1) = \frac{1}{2\,8}\begin{pmatrix} 1\cdot 3 & 2(-1) \\ 0 & -(1\cdot 3) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

$\det HF(u,v) = \dots$ $\quad \det HF(1,1) = -1$

$\Delta F(u,v) = \frac{4}{(u^2+v^2)^3}\left[v^4 - 3u^2v^2 - u^4 + 3u^2v^2\right] = \frac{4(v^4-u^4)}{(u^2+v^2)^3} = \frac{4(v^2-u^2)(v^2+u^2)}{(u^2+v^2)^{3\,2}}$

$= \frac{4(v^2-u^2)}{(u^2+v^2)^2}$. $\quad \Delta F(1,1) = \frac{4(1-1)}{\text{non}} = 0$.

Exo 27 : Trouver $c \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$ t.q. $u(x,y,t) = x^2 + y^2 - c^2t^2$ est harmonique : $\Delta u(x,y,t) = 0 \quad \forall (x,y,t) \in \mathbb{R}^3$

Il faut calculer le laplacien de u.

$\vec{\nabla} u(x,y,t) = \begin{pmatrix} 2x \\ 2y \\ -2c^2t \end{pmatrix}$ $\quad Hu(x,y,t) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2c^2 \end{pmatrix}$

$\Delta u(x,y,t) = 2 + 2 - 2c^2 = 4 - 2c^2 = 2(2-c^2)$

u harmonique $\Leftrightarrow \Delta u = 2(2-c^2) = 0 \quad \forall (x,y,t)$

$\Leftrightarrow c^2 = 2 \Leftrightarrow c = \pm\sqrt{2}$.