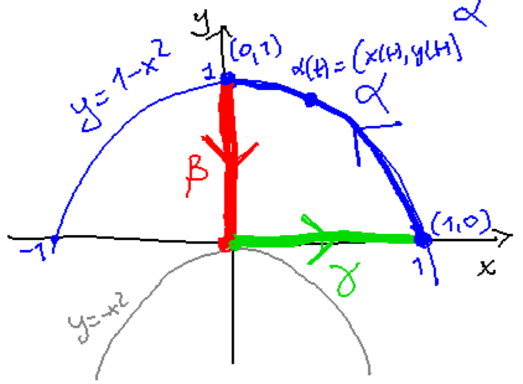
22 avril 2023 | Questions sur le CC3

1 Courbes – TD 6, Exo 18 b)

$$C^+ = \underbrace{\{(x,y) \mid \boxed{y = 1-x^2},\ x\in[0,1]\}}_{\text{orientée } x:\ 1\to 0\ \ \alpha} \cup \underbrace{\{(x,y) \mid \boxed{x=0},\ y\in[0,1]\}}_{\text{or. } y:1\to 0} \cup \underbrace{\{(x,y) \mid \boxed{y=0},\ x\in[0,1]\}}_{\text{or. } x:0\to 1}$$



$\alpha(t) = (\ \_\_\ , t)$ $x^2 = 1-y$, $x = \sqrt{1-y}$ $\boxed{\alpha(t) = (\sqrt{1-t},\ t)}$

$\alpha(t) = (t,\ 1-t^2)$ $t\in[0,1]$

$x = t$

$\alpha(0) = (0,1)$ $\alpha(1) = (1,0)$ } NON : mauvaise orientation

Astuce : $t \to 1-t$ $\alpha(t) = (1-t,\ 1-(1-t)^2) = (1-t,\ 2t-t^2)$ [$\cancel{1-1}+2t-t^2$]

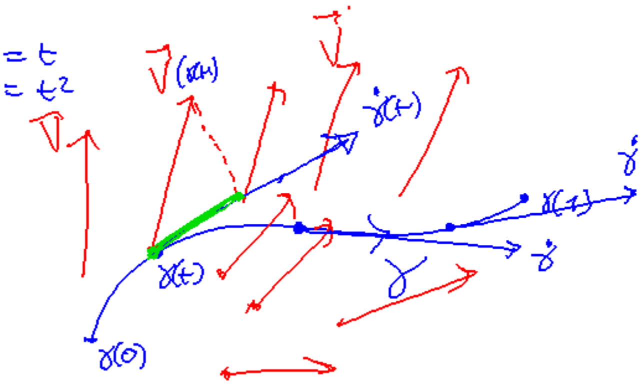
$\alpha(0) = (1,0)$ $\alpha(1) = (0,1)$ OK

$\beta(t) = (0,t)$, $t\in[0,1]$ $\beta(0) = (0,0)$ NON $\Rightarrow$ $\beta(t) = (0,1-t)$, $\beta(0) = (0,1)$ OK

$\gamma(t) = (t,0)$, $t\in[0,1]$ $\gamma(0) = (0,0)$ OK

Q3 $\vec V(x,y) = (x-y)\,\vec\imath + (x+y)\,\vec\jmath$

$\gamma(t) = (t, t^2)$ $t\in[0,1]$ $\leftarrow$ $\begin{cases} x = t \\ y = t^2 \end{cases}$



$$\int_\gamma \vec V \cdot \vec{dl} = \int_0^1 \underbrace{\vec V(\gamma(t)) \cdot \dot\gamma(t)}_{\text{projection}}\, dt$$

- $\vec V(\gamma(t)) = (t-t^2)\,\vec\imath + (t+t^2)\,\vec\jmath$
- $\dot\gamma(t) = \vec\imath + 2t\,\vec\jmath$
- $\vec V(\gamma(t)) \cdot \dot\gamma(t) = t-t^2 + 2t(t+t^2) = t-t^2+2t^2+2t^3 = \boxed{t+t^2+2t^3}$
- $\displaystyle\int_\gamma \vec V\cdot\vec{dl} = \int_0^1 (t+t^2+2t^3)\,dt = \left[\frac12 t^2 + \frac13 t^3 + \frac{2}{4} t^4\right]_0^1 = \frac12 + \frac13 + \frac12 = \frac{3+1}{3} = \frac43.$

Amine DM5 Q3

$\vec V(x,y,z) = z\vec i - y\vec j + x\vec k$ $\qquad \gamma(t) = (t, t, t^2) \quad t \in [0,1]$

- $\vec V(\gamma(t)) = t^2\vec i - t\vec j + t\vec k$
- $\dot\gamma(t) = \vec i + \vec j + 2t\vec k$
- $\vec V \cdot \dot\gamma = t^2 - t + 2t^2 = 3t^2 - t$
- $\displaystyle\int_\gamma \vec V \cdot \vec{dl} = \int_0^1 (3t^2 - t)\, dt = \left[t^3 - \tfrac{1}{2}t^2\right]_0^1 = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$

Q4 $\displaystyle\int_\gamma \underbrace{\boxed{\overrightarrow{grad}(f)}}_{\vec V} \cdot \vec{dl} = ?$ (*) connaissant $f(x,y,z) = xy^2 - yz^2$ avec γ courbe (quelconque) qui joint le point $A(0,1,2)$ au point $B(1,2,3)$.

Thm $\displaystyle\int_A^B \overrightarrow{grad}(f) \cdot \vec{dl} = f(B) - f(A) = \left[f(x,y,z)\right]_A^B$

f est "primitive" de $\overrightarrow{grad} f$

analogue au thm. Fonda de l'analyse : $\boxed{\displaystyle\int_a^b f'(x)\,dx = \left[f(x)\right]_a^b = f(b) - f(a)}$

f est une primitive de f'

car f' est la dérivée de f

(*) $= f(B) - f(A) = f(1,2,3) - f(0,1,2)$

$= (4 - 18) - (-4) = 4 - 18 + 4 = -10.$

• Même qst. mais γ est fermée (ex. un cercle $\circlearrowright$) : $A = B$

$$\boxed{\oint_\gamma \vec{\operatorname{grad}} f \cdot \vec{dl}} = \underset{A=B}{f(B) - f(A)} = f(A) - f(A) = \boxed{0} .$$

• Même qst mais on donne $\gamma(t) = (3\cos t, 3\sin t, t)$, $t \in [0, \frac{\pi}{2}]$

$$\int_\gamma \vec{\operatorname{grad}} f \cdot \vec{dl} \overset{\text{Thm}}{=} f(B) - f(A) = (*)$$

$A = \gamma(0) = (3, 0, 0)$ $\qquad$ $B = \gamma(\pi/2) = (0, 3, \frac{\pi}{2})$ $\qquad$ $f(x,y,z) = xy^2 - yz^2$

$$(*) = f(0,3,\tfrac{\pi}{2}) - f(3,0,0) = -3\frac{\pi^2}{4} - 0 = -\frac{3}{4}\pi^2 .$$

2 Surfaces

Q5 Paramétrisation $f(\rho, \varphi)$ de $S = \{(x,y,z) \mid \boxed{z^2 = 2(x^2+y^2)}\}$:

choix entre

~~(a) $\left(\overset{x}{\cos\varphi}, \overset{y}{\sin\varphi}, \overset{z}{\frac{\rho}{\sqrt{2}}}\right)$~~ $\qquad$ ~~(b) $(2\rho^2\cos\varphi, 2\rho^2\sin\varphi, \rho)$~~

OK (c) $\boxed{\left(\overset{x}{\rho\cos\varphi}, \overset{y}{\rho\sin\varphi}, \overset{z}{\sqrt{2}\rho}\right)}$ $\qquad$ (d) $(\cos\varphi, \sin\varphi, \sqrt{2}\rho)$

(a): $2(x^2+y^2) = 2(\cos^2\varphi + \sin^2\varphi) = 2 \overset{?}{=} z^2 = \frac{\rho^2}{2}$ NON

(b): $2(x^2+y^2) = 2(4\rho^4\cos^2\varphi + 4\rho^4\sin^2\varphi) = 2 \cdot 4\rho^4 \underbrace{(\cos^2\varphi + \sin^2\varphi)}_{1} = 8\rho^4 \neq z^2 = \rho^2$ NON

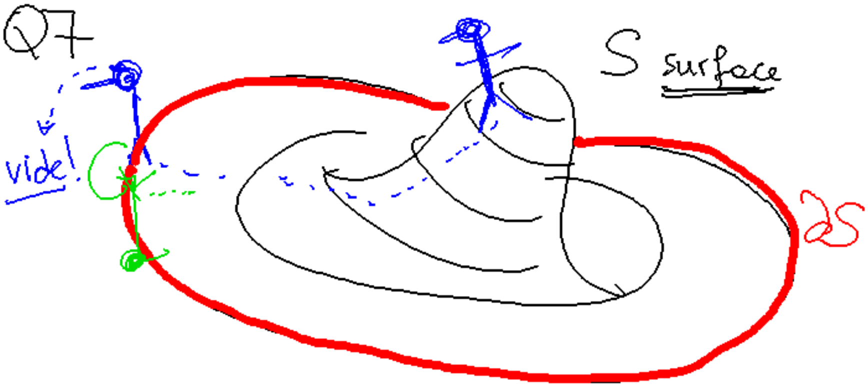
(c): $2(x^2+y^2) = 2(\rho^2\cos^2\varphi + \rho^2\sin^2\varphi) = 2\rho^2 \overset{!}{=} z^2 = 2\rho^2$ OK

(d) $2(x^2+y^2) = 2(\cos^2\varphi + \sin^2\varphi) = 2 \neq z^2 = 2\rho^2$ NON

Autre choix : $x = u$, $y = v$, $z = \sqrt{2(u^2+v^2)} \Rightarrow f(u,v) = (u, v, \sqrt{2(u^2+v^2)})$

Q7

S surface



le bord ∂S est une courbe

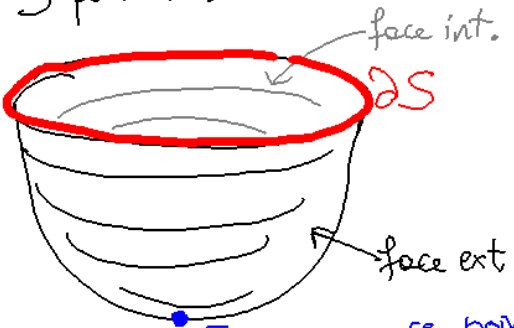
S sphère

∂S vide !

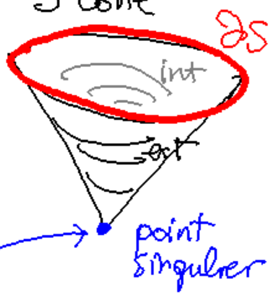
S est fermée

S entoure un solide Ω ! (la boule)

S paraboloïde



S cône



S cylindre

ce point n'est pas dans ∂S

Q9 Flux de $\vec{V}$ à travers S^+

Déf : $\displaystyle\iint_{S^+} \vec{V}\cdot\vec{dS} = \iint_{U\times V} \vec{V}(f(u,v))\cdot\vec{n}(u,v)\,du\,dv$

S^+ ← param. $(x,y,z) = f(u,v)$, $u\in U$, $v\in V$ $\quad \vec{n} = \dfrac{\partial f}{\partial u}\wedge\dfrac{\partial f}{\partial v}$

Cas particuliers :

Thm Stokes : (Q9) $\boxed{\text{Si } \vec{V} = \vec{\text{rot}}\,\vec{U}}$, alors $\displaystyle\iint_{S^+} \vec{\text{rot}}\,\vec{U}\cdot\vec{dS} = \oint_{\gamma} \vec{U}\cdot\vec{d\ell}$

$S^+ \longrightarrow \gamma = \partial S^+$ courbe fermée !

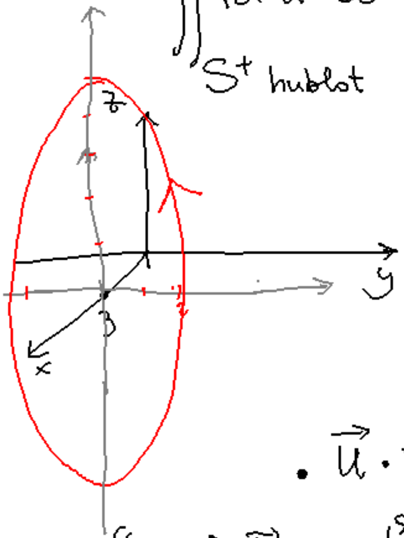
Thm Gauss : (Q10)

$\boxed{\begin{array}{l}\text{Si } S \text{ est fermée} \\ \text{donc } \partial S = \text{vide} \\ S = \partial\Omega \text{ où } \Omega \text{ solide}\end{array}}$

alors $\displaystyle\oiint_{S^+} \vec{V}\,\vec{dS} = \iiint_{\Omega\subset\mathbb{R}^3} \text{div}\,\vec{V}\,dx\,dy\,dz.$

Q9 $\iint_{S^+} \overrightarrow{rot}\,\vec{U}\cdot\vec{dS} = ?$ où $\vec{U}(x,y,z) = y^2\,\vec{i} + z\,\vec{j} + x\,\vec{k}$

S^+ hublot avec bord $\partial S^+ = \{\gamma(t) = (3,\ 2\cos t,\ 5\sin t),\ t\in[0,2\pi]\}$



Thm Stokes : flux $= \oint_\gamma \vec{U}\cdot\vec{dl}$

par parties: $u=\sin t$ $v'=\sin t$

$$\boxed{\int \sin^2 t\,dt = \frac{1}{2}(t-\sin t\cos t)}$$

- $\vec{U}(\gamma(t)) = 4\cos^2 t\,\vec{i} + 5\sin t\,\vec{j} + 3\,\vec{k}$
- $\dot{\gamma}(t) = -2\sin t\,\vec{j} + 5\cos t\,\vec{k}$
- $\vec{U}\cdot\dot{\gamma} = -10\sin^2 t + 15\cos t$
- $\iint_{S^+} \overrightarrow{rot}\,\vec{U}\cdot\vec{dS} = \int_0^{2\pi}(-10\sin^2 t + 15\cos t)\,dt = \left[-\overset{5}{\cancel{\frac{10}{2}}}(t-\cancel{\sin t\cos t}) + \cancel{15\sin t}\right]_0^{2\pi} = -5\cdot 2\pi = -10\pi$