From gauge fields to direct connections on gauge groupoids

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Women at the intersection of Mathematics and Theoretical Physics
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Based on a joint work with

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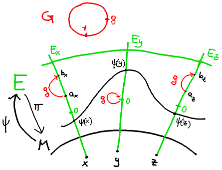
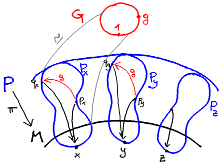
Geometric model underlying field and gauge theories



- space-time manifold M
- vector/spinor bundle $E \to M$ with fibres V
- (matter) field $\psi : M \to E$ section of E
  $\Rightarrow$ configuration space $\mathcal{E} = \Gamma(M, E)$
- dynamics via action $S[\psi]$ (isolated particles)
  $\Rightarrow$ use derivatives $D_X \psi$ i.e. linear connection on E

- symmetries by Lie group G acting on fibres of E
- principal G-bundle $P \to M$ s.t. $E = P \times_G V$
  (associated bundle)
- gauge bosons (force carriers) $A \in \Omega^1_{\mathrm{loc}}(M, \mathfrak{g})$
  local form of principal connection on P
- gauge group $\widehat{G} = \mathrm{Aut}_M(P) = \Gamma(M, P^{ad}) = \mathrm{Aut}_M(E)$
  $\Rightarrow$ acts on ψ and A
- dynamics via $\widehat{G}$-inv. action $S[\psi, A]$ (interacting particles)
  $\Rightarrow$ new covariant derivative $D^A_X \psi$



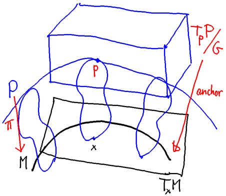
Idea: replace $\left\{ \begin{array}{l} \text{connection on } P \\ + \text{ gauge group} \\ (+ \text{ Lie algebra } \mathfrak{g}) \end{array} \right.$ by $\left\{ \begin{array}{l} \text{direct connection} \\ \text{on gauge groupoid of } P \\ (+ \text{ Lie algebroid of } P) \end{array} \right.$.

Infinitesimal structure: Lie algebroids

- **Lie algebroid**: vector bundle $\boxed{A \to M}$ with a Lie bracket $[\ ,\ ]_A$ on sections $\Gamma(A)$ and an **anchor map** $\boxed{a : A \to TM}$ inducing a derivation on sections w.r.t. vector fields on M:

$$[X, fY]_A = f[X, Y]_A + a(X)(f)Y$$

- Atiyah Lie algebroid of a principal G-bundle $\pi : P \to M$:
  $\boxed{A(P) = TP/G \to M}$ with fibres $A_x(P) \cong T_{p_x}P$,
  anchor $A(P) \to TM$ induced by $d\pi : TP \to TM$
  via quotient map $j : TP \to TP/G$
  and Lie bracket of G-invariant vector felds on P.



- Trivial G-bundle: $P = M \times G \to M \quad \Rightarrow \quad A(M \times G) = TM \oplus (M \times \mathfrak{g}) \overset{id+0}{\to} TM$
  where $\mathfrak{g} = \mathrm{Lie}(G)$.

- Frame bundle of a vector bundle $E \to M$ of rank r: $\quad F(E) = \bigcup_{x \in M} \mathrm{Iso}(\mathbb{R}^r, E_x) \to M$

  $\Rightarrow \quad A(F(E)) = \mathrm{Der}(E) \to TM$ bundle of derivative endomorphisms
  s.t. $\Gamma(\mathrm{Der}(E)) =$ derivations of $\Gamma(E)$.

Principal connections, gauge fields and covariant derivative

- **Principal connection on P:** five equivalent presentations

1) G-equivariant **horizontal subbundle** $\boxed{HP \subset TP \to P}$ s.t. $TP = HP \oplus VP$, where the **vertical bundle** VP (spaces tangent to the fibres) is canonical.

2) G-equivariant **connection 1-form** $\boxed{\omega \in \Omega^1(P, \mathfrak{g})}$ s.t. $\omega_p(\hat{A}_p) = A$ if $\hat{A}$ is the (vertical) fundamental vector field on P det. by $A \in \mathfrak{g}$, and $\omega_p(B_p) = 0$ if $B_p \in H_p P$ is horizontal.

3) **infinitesimal connection** $\boxed{\delta : TM \to A(P) \text{ s.t. } j^{-1}(\delta(X)) \in HP}$ is the horizontal lift of $X \in TM$.

4) **parallel transport** $\boxed{\tau_\gamma(y,x) : P_x \xrightarrow{\simeq} P_y}$ along a curve γ of M from x to y given by the horizontal lift of γ.

5) (local) **gauge fields** $\boxed{\{A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})\}}$ = pull back of ω along local sections of P.

- **Covariant derivative on sections of E:** bundle map $\boxed{D^A : TM \to \mathrm{Der}(E)}$ equivalent to $C^\infty(M)$-derivation $D^A{}_X : \Gamma(E) \to \Gamma(E)$ for $X \in \Gamma(TM)$ given only locally by

$$\boxed{D^A{}_X(\psi)_{|U_\alpha} = \sum_{\mu,i,j} \left(X^\mu \partial_\mu \psi^i + X^\mu A^i_{\mu j} \psi^j\right) e_i}$$

if $X = X^\mu \partial_\mu$ in coordinates x^μ on $U_\alpha \subset M$
$\psi = \psi^i e_i$ on a local basis (e^i) of E_{U_α}
and $A^i_j = A^i_{\mu j} dx^\mu$ are the components of the gauge field A in terms of generators of $\mathfrak{g}$.

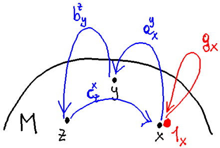
Lie groupoids

- **Lie groupoid** $\boxed{\mathcal{G} \rightrightarrows M}$: bi-fibred manifold $\mathcal{G} = \bigcup_{(y,x)\in M\times M} \mathcal{G}_x^y$

  with elements $a_{yx} \in \mathcal{G}_x^y$ called **arrows**

  projections $\quad s, t : \mathcal{G} \to M \begin{cases} \textbf{source} & s(a_{yx}) = x \\ \textbf{target} & t(a_{yx}) = y \end{cases}$

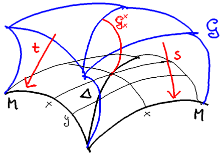
  such that



- arrows can be **composed**: $b_{zy} a_{yx} \in \mathcal{G}_x^z$ if $s(b_{zy}) = t(a_{yx})$ (composition is associative),
- there are **units** $u(x) = 1_x \in \mathcal{G}_x^x$ and $M \equiv u(M) \subset \mathcal{G}$,
- each arrow $a_{yx} \in \mathcal{G}_x^y$ has an **inverse** $a_{yx}^{-1} \in \mathcal{G}_y^x$.

The induced map $(t, s) : \mathcal{G} \to M \times M$ is called the **anchor**.

Each $\mathcal{G}_x^x$ is a Lie group, called the **vertex group** or **isotropy**.



- Infinitesimal structure of Lie groupoid = Lie algebroid: $\boxed{\mathcal{L}\mathcal{G} = \bigcup_{x\in M} T_{1_x}\mathcal{G}_x \to TM}$

Gauge groupoids

- Gauge groupoid of principal G-bdl $P \to M$: $\boxed{\mathcal{G}(P) = P \times_G P \rightrightarrows M}$
  contains equivalence classes $[p,q]$ under $(p,q) \sim (p\,g, q\,g)$ for $g \in G$.
  $\Rightarrow$ 1) $\boxed{\mathcal{L}(\mathcal{G}(P)) = A(P) \to TM}$
  2) $\mathcal{G}(P)$ **acts on** $E = P \times_G V$: $\boxed{\mathcal{G}(P) \times_M E = \{([p,q],[r,v]),\ \pi(q) = \pi(r)\} \to E}$
  with action $\rho_{[p,q]}([r,v]) = [p, gv]$ where $g \in G$ s.t. $r = qg$,
  3) $\boxed{\widehat{G} = \mathrm{Aut}_M(P) \subset \mathcal{G}(P)}$ given by $\boxed{\Phi \mapsto [\Phi(p), p]}$ for any $p \in P$.

- Pair groupoid: $\boxed{\mathrm{Pair}(M) = M \times M \rightrightarrows M}$ for $P = M \times \{1\} \to M$
  $\Rightarrow$ Lie algebroid $= \mathcal{L}(\mathrm{Pair}(M)) = TM \xrightarrow{id} TM$.

- Trivial Lie groupoid with fibre G: $\boxed{M \times G \times M \rightrightarrows M}$ for $P = M \times G \to M$
  $\Rightarrow$ Lie algebroid $= TM \oplus (M \times \mathfrak{g}) \xrightarrow{id+0} TM$.

- Frame groupoid of $E \to M$: $\boxed{\mathrm{Iso}(E) = \bigcup_{x,y} \mathrm{Iso}(E_y, E_x)}$ for $P = F(E)$
  $\Rightarrow$ Lie algebroid $= \mathrm{Der}(E) \to TM$

  If the structure gp of E reduces to $G \subset GL_r(\mathbb{R})$ and $P \subset F(E)$ then $\boxed{\mathcal{G}(P) \hookrightarrow \mathrm{Iso}(E)}$

Direct connections on Lie groupoids

- **Local map** $\boxed{\psi : \mathcal{G} \ast\!\!\to \mathcal{G}'}$ between two groupoids: map $\psi : \mathcal{U} \subset \mathcal{G} \to \mathcal{G}'$ defined on an open neighborhood $\mathcal{U}$ of the units $u(M) \subset \mathcal{G}$, which commutes with s, t and u.

  Local morphism: local map which also preserves composition (hence inversion).

- [Teleman 2004 in the linear case, Kock 2007 similar, ABFP general]

  Direct connection on $\mathcal{G} \rightrightarrows M$: local right inverse of the anchor which preserves units, i.e. $\boxed{\Gamma : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathcal{G}}$ defined on an open n. $\mathcal{U}_\Delta$ of the diagonal $\Delta \subset \mathrm{Pair}(M)$ s.t.

  $\boxed{\Gamma(y,x) \in \mathcal{G}_x^y}$ for all $(y,x) \in \mathcal{U}_\Delta$ and $\boxed{\Gamma(x,x) = 1_x \in \mathcal{G}_x^x}$ for all $x \in M$.

- $\boxed{\text{A Lie groupoid with a direct connection is a gauge groupoid.}}$

- If $\mathcal{G} \times_M E \to E$ is a linear action, then a direct connection Γ on $\mathcal{G}$ induces a transport on fibres $E_x \to E_y$ which is not necessarily a parallel displacement!

- Γ **natural** if $\boxed{\Gamma(x,y)\,\Gamma(y,x) = 1_x}$ for all $x \in M$ and suitable y.

- **Curvature of Γ at x**: $\boxed{R^\Gamma(z,y,x) = \Gamma(z,x)^{-1}\Gamma(z,y)\,\Gamma(y,x) \in \mathcal{G}_x^x}$ for suitable y, z.

  Γ is **flat** if $R^\Gamma(\_,\_,x) = 1_x$ for any x, i.e. **Γ is a groupoid morphism**.

Relationship to usual connections

Assume M is a manifold with affine connection ∇^M and local geodesics.

- Parallel displacement τ on $P \to M$ along small geodesics (equivalent to a principal connection ω on P hence to gauge fields A) defines a direct connection Γ^τ on $\mathcal{G}(P) \rightrightarrows M$ by

$$\Gamma^\tau(y,x) = \left[\tau(y,x)(p), p\right] \quad \text{for any choice of } p \in P_x$$

Same for $E \to M$ and $\mathrm{Iso}(E)$ [Teleman 2004].

- Viceversa, a direct connection Γ on $\mathcal{G}(P) \rightrightarrows M$ induces an infinitesimal connection on the Lie algebroid $A(P) \to TM$ by

$$\nabla^\Gamma(\dot{\gamma}(0)) = D\Gamma_{|M}(\dot{\gamma}(0)) = \frac{d}{dt}_{|t=0}\Gamma(\gamma(t), x),$$

hence a principal connection ω^Γ on P.

- Apply maps $\omega \mapsto \tau \mapsto \Gamma^\tau \mapsto \omega^{\Gamma^\tau}$, then $\boxed{\omega^{\Gamma^\tau} = \omega}$ on P.
- Viceversa, if apply maps $\Gamma \mapsto \omega^\Gamma \mapsto \tau^\Gamma \mapsto \Gamma^{\tau^\Gamma}$, then $\boxed{\Gamma^{\tau^\Gamma} \neq \Gamma}$ on $\mathcal{G}(P)$ in general. There are direct connections on $\mathcal{G}(P)$ which are not parallel displacements!

Examples

- $M = \mathbb{R}$ with flat connection $\nabla^M_{\partial_x}(h(x)\,\partial_x) = h'(x)\,\partial_x$.
- $E = M \times \mathbb{R} \to M$ with global section $e_1(x) = (x, 1) \in E_x$ and linear connection $\nabla^E_{\partial_x} : \Gamma(E) \to \Gamma(E)$ given by $f \in C^\infty(M)$ s.t. $\nabla^E_{\partial_x} e_1 = f\, e_1$.
- The induced **parallel transport** along a geodesic from x to y is the isomorphism $\tau(y, x) : E_x \to E_y$ defined by $\tau(y, x)\,\xi_0\, e_1(x) = \xi(y)\, e_1(y)$ solution of the ODE

$$\nabla^E_{\partial_x}\big(\xi(x)\, e_1(x)\big) = \big(\xi'(x) + \xi(x) f(x)\big) e_1(x) = 0$$

with initial value $\xi(x)\, e_1(x) = \xi_0\, e_1(x)$. Set $F(x) = \fint f(x) dx$.
Then the direct connection on $\mathrm{Iso}(E)$ is

$$\boxed{\tau(y, x) : E_x \to E_y, \quad e_1(x) \mapsto \tau(y, x)\, e_1(x) = e^{F(y) - F(x)}\, e_1(y)}$$

The associated direct connection is flat. For instance:
$\nabla^E_{\partial_x} e_1(x) = -2x\, e_1(x) \quad \Rightarrow \quad \tau(y, x) e_1(x) = e^{y - x + y^2 - x^2}\, e_1(y)$,
$\nabla^E_{\partial_x} e_1(x) = -3x^2\, e_1(x) \quad \Rightarrow \quad \tau(y, x) e_1(x) = e^{y - x + y^3 - x^3}\, e_1(y)$.

- Instead, the following direct connections **are not parallel transports**:
$\Gamma(y, x) = e^{y - x + (y - x)^2}$ non natural $\big(\Gamma(x, y)\Gamma(y, x) = e^{2(y-x)^2} \neq 1_x\big)$,
$\Gamma(y, x) = e^{y - x + (y - x)^3}$ natural but non-flat.

Conclusion: there is a surjective functor

Gauge groupoids with direct connections	$\longrightarrow$	Principal bundles with connections

which admits an inverse, but it is not an equivalence of categories.

Further results:

- Jet prolongation of direct connections to jet groupoids $J^n\mathcal{G} \rightrightarrows M$ (existence, examples).
- Applications to geometric regularity structures for solving stochastic PDEs (cf. M. Hairer and coll.).

Next:

- Look for more examples of direct connections which are not parallel displacements.
- Adapt to α-Hölder sections of bundles i.e. define distributional direct connections and compare to usual propagators.
- Study the whole geometry of **groupoids with direct connections** and compare with **usual gauge theory**.

Thank you for the attention!