EXCEPTIONAL SETS FOR THE DERIVATIVES OF BLASCHKE PRODUCTS

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ABSTRACT. We obtain growth estimates for the logarithmic derivative $B'(z)/B(z)$ of a Blaschke product as $|z| \to 1$ and z avoids some exceptional sets.

1. INTRODUCTION

Let f be a meromorphic function in the unit disc $\mathbb{D}$. Then its order is defined by

$$\sigma = \limsup_{r\to 1^-} \frac{\log^+ T(r)}{\log 1/(1-r)},$$

where

$$T(r) = \frac{1}{\pi} \int_{\{|z|<r\}} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} \log(\frac{r}{|z|}) \, dx\, dy$$

is the Nevanlinna characteristic of f [13]. Meromorphic functions of finite order have been extensively studied and they have numerous applications in pure and applied mathematics, e.g. in linear differential equations. In many applications a major role is played by the logarithmic derivative of meromorphic functions and we need to obtain sharp estimates for the logarithmic derivative as we approach to the boundary [7, 8]. In particular, the following result for the rate of growth of meromorphic functions of finite order in the unit disc has application in the study of linear differential equations [10, Theorem 5.1].

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Theorem 1.1. *Let f be a meromorphic function in the unit disc $\mathbb{D}$ of finite order σ and let $\varepsilon > 0$. Then the following two statements hold.*

(a) There exists a set $E_1 \subset (0,1)$ which satisfies

$$\int_{E_1} \frac{dr}{1-r} < \infty,$$

such that, for all $z \in \mathbb{D}$ with $|z| \notin E_1$, we have

$$\left| \frac{f'(z)}{f(z)} \right| \le \frac{1}{(1-|z|)^{3\sigma+4+\varepsilon}}. \tag{1.1}$$

(b) There exists a set $E_2 \subset [0,2\pi)$ whose Lebesgue measure is zero and a function $R(\theta) : [0,2\pi) \setminus E_2 \longrightarrow (0,1)$ such that for all $z = re^{i\theta}$ with $\theta \in [0,2\pi) \setminus E_2$ and $R(\theta) < r < 1$ the inequality (1.1) holds.

Clearly, the relation (1.1) can also be written as

$$\left| \frac{f'(z)}{f(z)} \right| = \frac{O(1)}{(1-|z|)^{3\sigma+4+\varepsilon}}$$

as $|z| \to 1$. But we should note that in case (b) it does not hold uniformly with respect to $|z|$.

Let $(z_n)_{n\ge1}$ be a sequence in the unit disc satisfying the Blaschke condition

$$\sum_{n=1}^{\infty} (1-|z_n|) < \infty. \tag{1.2}$$

Then the Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \, \frac{z_n - z}{1 - \bar{z}_n \, z}$$

is an analytic function in the unit disc with order $\sigma = 0$ and

$$\frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{(1-\bar{z}_n \, z)(z-z_n)}. \tag{1.3}$$

Thus Theorem 1.1 implies that, for any $\varepsilon > 0$,

$$\left| \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{(1-\bar{z}_n \, z)(z-z_n)} \right| = \frac{O(1)}{(1-|z|)^{4+\varepsilon}}$$

as $|z| \to 1^-$ in any of the two manners explained above. In this paper, instead of (1.2), we pose more restrictive conditions on the rate of convergence of zeros z_n and instead we improve the exponent $4+\varepsilon$. The most common condition is

$$\sum_{n=1}^{\infty} (1-|z_n|)^{\alpha} < \infty, \tag{1.4}$$

for some $\alpha \in (0,1]$. However, we consider a more general assumption

$$\sum_{n=1}^{\infty} h(1-|z_n|) < \infty, \tag{1.5}$$

where h is a positive continuous function satisfying certain smoothness conditions which will be described below. Our main prototype for h is

$$h(t) = t^{\alpha}\ (\log 1/t)^{\alpha_1}\ (\log_2 1/t)^{\alpha_2}\ \cdots\ (\log_n 1/t)^{\alpha_n}, \tag{1.6}$$

where $\log_n = \log\log\cdots\log$ (n times), $\alpha \in (0,1]$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$. If $\alpha = 1$ the first nonzero exponent among $\alpha_1, \alpha_2, \cdots, \alpha_n$ is positive [12].

The function h is usually defined in an open interval $(0,\epsilon)$. Of course, by extending its domain of definition, we may assume that h is defined on the interval $(0,1)$, or if required, on the entire positive real axis. Moreover, since a Blaschke sequence satisfies (1.2), the condition (1.5) will provide further information about the rate of increase of the zeros provided that $h(t) \geq C\,t$ as $t \to 0$.

The condition (1.4) has been extensively studied by many authors [1, 2, 3, 9, 11, 14] to obtain estimates for the integral means of the derivative of Blaschke products. We [6] have recently shown that many of these estimates can be generalized for Blaschke products satisfying (1.5).

2. Circular Exceptional Sets

The function h given in (1.6) satisfies the following conditions:

a) h is continuous, positive and increasing with $h(0+) = 0$;

b) $h(t)/t$ is decreasing;

In the following, we just need these conditions. Hence, we state our results for a general function h satisfying *a*) and *b*).

Theorem 2.1. *Let $(z_n)_{n\geq 1}$ be a sequence in the unit disc satisfying*

$$\sum_{n=1}^{\infty} h(1-|z_n|) < \infty$$

and let B be the Blaschke product formed with zeros z_n, $n \geq 1$. Let $\beta \geq 1$. Then there is an exceptional set $E \subset (0,1)$ such that

$$\int_E \frac{dt}{(1-t)^\beta} < \infty$$

and that

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1-|z|)^\beta \; h^2(1-|z|)}$$

as $|z| \to 1^-$ with $|z| \notin E$.

Proof. Without loss of generality, assume that $h(t) < 1$ for $t \in (0,1)$. Let

$$E = \bigcup_{n=1}^{\infty} \Big(|z_n| - (1-|z_n|)^\beta h(1-|z_n|), \; |z_n| + (1-|z_n|)^\beta h(1-|z_n|) \Big).$$

In the definition of E we implicitly assume that $|z_n| - (1-|z_n|)^\beta h(1-|z_n|) > 0$ in order to have $E \subset (0,1)$. Certainly this condition holds for large values of n. If it does not hold for some small values of n, we simply remove those intervals from the definition of E.

Let $z \in \mathbb{D}$ with $|z| \notin E$ and fix $0 < \delta \leq (1-|z|)/2$. By (1.3), we have

$$\frac{B'(z)}{B(z)} = \Bigg(\sum_{\big| |z|-|z_n| \big| \geq \delta} + \sum_{\big| |z|-|z_n| \big| < \delta} \Bigg) \frac{1-|z_n|^2}{(1-\bar{z}_n\, z)(z-z_n)}.$$

We use different techniques to estimate each sum. For the first sum we have

$$\sum_{\big| |z|-|z_n| \big| \geq \delta} \frac{1-|z_n|^2}{|1-\bar{z}_n\, z|\,|z-z_n|} \;\leq\; \frac{2}{\delta} \sum_{\big| |z|-|z_n| \big| \geq \delta} \frac{1-|z_n|}{1-|z_n|\,|z|}.$$

But

$$\frac{1-|z_n|}{1-|z|\,|z_n|} = \left(\frac{1-|z_n|}{h(1-|z_n|)} \, \frac{h(1-|z|\,|z_n|)}{1-|z|\,|z_n|} \right) \left(\frac{h(1-|z_n|)}{h(1-|z|\,|z_n|)} \right).$$

Since $h(t)$ is increasing and $h(t)/t$ is decreasing, we get

$$\frac{1-|z_n|}{1-|z|\,|z_n|} \leq \frac{h(1-|z_n|)}{h(1-|z|)}$$

and thus

$$\sum_{\big|\, |z|-|z_n| \,\big| \geq \delta} \frac{1-|z_n|^2}{|1-\bar{z}_n\, z|\, |z-z_n|} \leq \frac{2 \sum_{\big|\, |z|-|z_n| \,\big| \geq \delta} h(1-|z_n|)}{\delta\; h(1-|z|)} \leq \frac{C}{\delta\; h(1-|z|)}.$$

A generalized version of this estimation technique has been used in [6, Lemma 2.1]. To estimate the second sum, we see that

$$\begin{aligned} \left| \frac{1-|z_n|^2}{(1-\bar{z}_n\, z)(z-z_n)} \right| &\leq \frac{2}{|z-z_n|} \leq \frac{2}{(1-|z_n|)^{\beta}\, h(1-|z_n|)} \\ &\leq \frac{C}{(1-|z|)^{\beta}\, h(1-|z|)}, \end{aligned}$$

and thus

$$\left| \sum_{\big|\, |z|-|z_n| \,\big| < \delta} \frac{1-|z_n|^2}{(1-\bar{z}_n\, z)(z-z_n)} \right| \leq C \; \frac{n(|z|+\delta)-n(|z|-\delta)}{(1-|z|)^{\beta}\, h(1-|z|)},$$

where $n(t)$ is the number of points z_n lying in the disc $\{\, z \,:\, |z| \leq t \,\}$. Therefore

$$\left| \frac{B'(z)}{B(z)} \right| \leq \frac{C}{h(1-|z|)} \left(\frac{1}{\delta} + \frac{n(|z|+\delta)-n(|z|-\delta)}{(1-|z|)^{\beta}} \right) \tag{2.1}$$

provided that $z \in \mathbb{D}$ with $|z| \notin E$. The best choice of δ depends on the counting function $n(t)$. We make a choice for the most general case.

Assume that $\delta = (1-|z|)/2$. Our assumption (1.5) on the rate of increase of zeros z_n is equivalent to

$$\int_0^1 h(1-t)\; dn(t) < \infty,$$

and it is well known that this condition implies

$$n(t) = \frac{o(1)}{h(1-t)} \tag{2.2}$$

as $t \to 1^-$. Therefore,

$$n(|z|+\delta) - n(|z|-\delta) \le \frac{o(1)}{h(1-|z|)}. \tag{2.3}$$

Hence, by (2.1) and (2.3), we get the promised growth for B'/B. To verify the size of E, note that

$$\begin{aligned} \int_E \frac{dt}{(1-t)^\beta} &= \sum_{n=1}^{\infty} \int_{|z_n|-(1-|z_n|)^\beta h(1-|z_n|)}^{|z_n|+(1-|z_n|)^\beta h(1-|z_n|)} \frac{dt}{(1-t)^\beta} \\ &= \sum_{n=1}^{\infty} \int_{(1-|z_n|)-(1-|z_n|)^\beta h(1-|z_n|)}^{(1-|z_n|)+(1-|z_n|)^\beta h(1-|z_n|)} \frac{d\tau}{\tau^\beta} \\ &\le \sum_{n=1}^{\infty} \frac{2(1-|z_n|)^\beta h(1-|z_n|)}{(\,(1-|z_n|)-(1-|z_n|)^\beta h(1-|z_n|)\,)^\beta} \\ &\le C \sum_{n=1}^{\infty} h(1-|z_n|) < \infty. \end{aligned}$$

□

Remark 1: As the counting function $n(t) = 1/(1-t)^\alpha$ suggests, the assumption

$$n(|z|+\delta) - n(|z|-\delta) \le C\,\frac{\delta\, n(|z|)}{1-|z|} \tag{2.4}$$

is fulfilled by a wide class of distribution of zeros. If (2.4) holds, by (2.3) and (2.1) with

$$\delta = (1-|z|)^{\frac{1+\beta}{2}}\, h^{\frac{1}{2}}(1-|z|),$$

we obtain

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{O(1)}{(1-|z|)^{\frac{1+\beta}{2}}\, h^{\frac{3}{2}}(1-|z|)}$$

as $|z| \to 1^-$ with $|z| \notin E$.

Remark 2: Let us call φ almost increasing if $\varphi(x) \leq \mathrm{Const}\, \varphi(y)$ provided that $x \leq y$. Almost decreasing functions are defined similarly. As it can be easily verified, Theorem 2.1 (and also Theorem 3.1) is still true if we assume that $h(t)$ is almost increasing and $h(t)/t$ is almost decreasing.

Corollary 2.2. *Let $\alpha \in (0,1]$, and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$. Let $(z_n)_{n\geq 1}$ be a sequence in the unit disc with*

$$\sum_{n=1}^{\infty}(1-|z_n|)^{\alpha}\ (\log 1/(1-|z_n|))^{\alpha_1}\ \cdots\ (\log_n 1/(1-|z_n|))^{\alpha_n} < \infty$$

and let B be the Blaschke product formed with zeros z_n, $n \geq 1$. Let $\beta \geq 1$. Then there is an exceptional set $E \subset (0,1)$ such that

$$\int_E \frac{dt}{(1-t)^{\beta}} < \infty$$

and that

$$\left|\frac{B'(z)}{B(z)}\right| = \frac{o(1)}{(1-|z|)^{\beta+2\alpha}\,(\log 1/(1-|z|))^{2\alpha_1}\ \cdots\ (\log_n 1/(1-|z|))^{2\alpha_n}} \tag{2.5}$$

as $|z| \to 1^-$ with $|z| \notin E$.

In particular, if

$$\sum_{n=1}^{\infty}(1-|z_n|)^{\alpha} < \infty, \tag{2.6}$$

then, for any $\beta \geq 1$, there is an exceptional set $E \subset (0,1)$ such that

$$\int_E \frac{dt}{(1-t)^{\beta}} < \infty \tag{2.7}$$

and that

$$\left|\frac{B'(z)}{B(z)}\right| = \frac{o(1)}{(1-|z|)^{\beta+2\alpha}}$$

as $|z| \to 1^-$ with $|z| \notin E$. If $(|z_n|)_{n\geq 1}$ is an interpolating sequence then

$$1-|z_{n+1}| \leq c\,(1-|z_n|)$$

for a constant $c < 1$ [4, Theorem 9.2]. Hence, (2.6) is satisfied for any $\alpha > 0$ and thus, for any $\beta \geq 1$ and for any $\varepsilon > 0$, there is an exceptional set E satisfying (2.7) such that

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1-|z|)^{\beta+\varepsilon}} \tag{2.8}$$

as $|z| \to 1^-$ with $|z| \notin E$. It is interesting to know if in (2.8) we are able to replace ε by zero.

3. Radial Exceptional Sets

Contrary to the preceding section, we now study the behavior of

$$\left| \frac{B'(re^{i\theta})}{B(re^{i\theta})} \right|$$

as $r \to 1$ for a *fixed* θ. We obtain an upper bound for the quotient B'/B as long as $e^{i\theta} \in \mathbb{T} \setminus E$ where E is an exceptional set of Lebesgue measure zero.

Theorem 3.1. *Let B be the Blaschke product formed with zeros $z_n = r_n e^{i\theta_n}$, $n \geq 1$, satisfying*

$$\sum_{n=1}^{\infty} h(1-r_n) < \infty.$$

Then there is an exceptional set $E \subset \mathbb{T}$ whose Lebesgue measure $|E|$ is zero such that for all $z = re^{i\theta}$ with $e^{i\theta} \in \mathbb{T} \setminus E$

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1-|z|)\, h(1-|z|)}$$

as $|z| \to 1^-$.

Proof. Let us consider the open set

$$U_n = \{\, z \in \mathbb{D} \,:\, (1-|z|) > C|z-z_n| \,\}$$

with $C > 1$, and we define

$$I_n = \{\, \zeta \in \mathbb{T} \,:\, \exists z \in U_n \;\&\; \zeta = z/|z| \,\}.$$

In other words, I_n is the radial projection of U_n on the unit circle $\mathbb{T}$. Then we know that

$$|I_n| \le C'(1 - r_n), \tag{3.1}$$

where C' is a constant just depending on C. Let

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} I_k.$$

By (3.1), we see that $|E| = 0$.

Fix $z \in \mathbb{D}$ with $z/|z| \notin E$. Hence, there is N such that $z/|z| \notin I_k$ for all $k \ge N$. Let $R = (1 + |z|)/2$. Now, we write

$$\frac{B'(z)}{B(z)} = \Bigg(\sum_{|z_n| \ge R} + \sum_{|z_n| < R,\, n \ge N} + \sum_{n=1}^{N-1} \Bigg) \frac{1 - |z_n|^2}{(1 - \bar{z}_n\, z)(z - z_n)},$$

and as in the preceding case

$$\sum_{|z_n| \ge R} \frac{1 - |z_n|^2}{|1 - \bar{z}_n\, z|\, |z - z_n|} \le \frac{o(1)}{(1 - |z|)\ h(1 - |z|)}. \tag{3.2}$$

To estimate the second sum, we see that

$$\left| \frac{1 - |z_n|^2}{(1 - \bar{z}_n\, z)(z - z_n)} \right| \le \frac{2}{|z - z_n|} \le \frac{2C}{1 - |z|}, \qquad (|z| \notin E),$$

and thus, by (2.2),

$$\left| \sum_{|z_n| < R,\, n \ge N} \frac{1 - |z_n|^2}{(1 - \bar{z}_n\, z)(z - z_n)} \right| \le \frac{2C\, n(R)}{1 - |z|} \le \frac{o(1)}{(1 - |z|)\, h(1 - |z|)}. \tag{3.3}$$

Since the last sum is uniformly bounded (θ is fixed), (3.2) and (3.3) give the required result. □

Corollary 3.2. *Let $\alpha \in (0, 1]$, and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$. If $\alpha = 1$ the first nonzero number among $\alpha_1, \alpha_2, \cdots, \alpha_n$ is positive. Let B be the Blaschke product formed with* *zeros* $z_n = r_n e^{i\theta_n}$, $n \geq 1$, *satisfying*

$$\sum_{n=1}^{\infty} (1-r_n)^{\alpha}\ (\log 1/(1-r_n))^{\alpha_1}\ \cdots\ (\log_n 1/(1-r_n))^{\alpha_n} < \infty.$$

Then there is an exceptional set $E \subset \mathbb{T}$ *whose Lebesgue measure* $|E|$ *is zero such that for all* $z = re^{i\theta}$ *with* $e^{i\theta} \in \mathbb{T} \setminus E$

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1-|z|)^{1+\alpha} (\log 1/(1-|z|))^{\alpha_1}\ \cdots\ (\log_n 1/(1-|z|))^{\alpha_n}} \tag{3.4}$$

as $|z| \to 1^-$.

In particular, if

$$\sum_{n=1}^{\infty} (1-r_n)^{\alpha} < \infty,$$

then there is an exceptional set $E \subset \mathbb{T}$ whose Lebesgue measure $|E|$ is zero such that for all $z = re^{i\theta}$ with $e^{i\theta} \in \mathbb{T} \setminus E$

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1-|z|)^{1+\alpha}} \tag{3.5}$$

as $|z| \to 1^-$.

Remark: Theorems 2.1 and 3.1 can be easily generalized to obtain estimates for

$$\frac{B^{(k)}(z)}{B^{(j)}(z)}$$

as $|z| \to 1^-$. This is a standard technique which can been find for example in [9, 11].

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