Bases of reproducing kernels in de-Branges spaces

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Abstract

This paper deals with geometric properties of sequences of reproducing kernels related to de-Branges spaces. If b is a nonconstant function in the unit ball of H^∞, and T_b is the Toeplitz operator, with symbol b, then the de-Branges space, $\mathcal{H}(b)$, associated to b, is defined by $\mathcal{H}(b) = (Id - T_b T_{\overline{b}})^{1/2} H^2$, where H^2 is the Hardy space of the unit disk. It is equiped with the inner product such that $(Id - T_b T_{\overline{b}})^{1/2}$ is a partial isometry from H^2 onto $\mathcal{H}(b)$. First, following a work of Ahern-Clark, we study the problem of orthogonal basis of reproducing kernels in $\mathcal{H}(b)$. Then we give a criterion for sequences of reproducing kernels which form an unconditionnal basis in their closed linear span. As far as concerns the problem of complete unconditionnal basis in $\mathcal{H}(b)$, we show that there is a dichotomy between the case where b is an extreme point of the unit ball of H^∞ and the opposite case.

Keywords: de-Branges spaces, Riesz bases, reproducing kernels.

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1 Introduction

This paper is devoted to geometric properties of sequences of reproducing kernels in de-Branges spaces. These spaces, first studied by L. de Branges and J. Rovnyak [6], are (not necessarily closed) subspaces of the Hardy space H^2 of the unit disk, $\mathbb{D}$. Recall first that

$$H^2 := \left\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic} : \sup_{0 \leqslant r < 1} \int_{\mathbb{T}} |f(r\zeta)|^2 \, dm(\zeta) < \infty \right\},$$

where $\mathbb{T}$ is the unit circle and dm is the normalized Lebesgue measure on $\mathbb{T}$. As usual, H^2 will be identified (via radial limits) with the space of $L^2 = L^2(\mathbb{T})$ functions whose negatively indexed Fourier coefficients vanish. Norm and inner product in L^2 or H^2 will be denoted by $\|\cdot\|_2$ and $\langle \cdot, \cdot \rangle_2$, respectively.

Let P_+ denote the orthogonal projection of L^2 onto H^2. For $\varphi \in L^\infty$, let T_φ denote the Toeplitz operator with symbol φ defined on H^2 by $T_\varphi f = P_+(\varphi f)$. The de-Branges space, $\mathcal{H}(\varphi)$, associated to φ consists of those H^2 functions which belong to the range of the operator $(Id - T_\varphi T_{\overline{\varphi}})^{1/2}$. It is a Hilbert space when equipped with the inner product

$$\langle f, g \rangle_\varphi := \langle P_{\mathrm{Ker}(Id - T_\varphi T_{\overline{\varphi}})^\perp} f_1, P_{\mathrm{Ker}(Id - T_\varphi T_{\overline{\varphi}})^\perp} g_1 \rangle_2,$$

where $f = (Id - T_\varphi T_{\overline{\varphi}})^{1/2} f_1$, $g = (Id - T_\varphi T_{\overline{\varphi}})^{1/2} g_1$ and $P_{\mathrm{Ker}(Id - T_\varphi T_{\overline{\varphi}})^\perp}$ denotes the orthogonal projection of H^2 onto $\mathrm{Ker}(Id - T_\varphi T_{\overline{\varphi}})^\perp$. Note that $\mathcal{H}(\varphi)$ is contained contractively in H^2 and the inner product is defined in order to make $(Id - T_\varphi T_{\overline{\varphi}})^{1/2}$ a partial isometry of H^2 onto $\mathcal{H}(\varphi)$. The norm of $\mathcal{H}(\varphi)$ will be denoted by $\|\cdot\|_\varphi$.

For $\lambda \in \mathbb{D}$, we let k_λ denote the kernel function for the functionnal on H^2 of evaluation at λ; it is given by $k_\lambda(z) = (1 - \overline{\lambda} z)^{-1}$ $(z \in \mathbb{D})$ and satisfies $f(\lambda) = \langle f, k_\lambda \rangle_2$ $(f \in H^2)$. Since $\mathcal{H}(\varphi)$ is contained contractively in H^2, the restriction to $\mathcal{H}(\varphi)$ of evaluation at λ is a bounded linear functionnal on $\mathcal{H}(\varphi)$. It is thus induced, relative to the inner product in $\mathcal{H}(\varphi)$, by a vector k_λ^φ in $\mathcal{H}(\varphi)$. It is easy to see ([19], (II-3)) that $k_\lambda^\varphi = (Id - T_\varphi T_{\overline{\varphi}}) k_\lambda$ and

$$f(\lambda) = \langle f, k_\lambda^\varphi \rangle_\varphi,$$

for all $f \in \mathcal{H}(\varphi)$. From now on, b will be a nonconstant function in the unit ball of H^∞, that is an holomorphic and bounded function in $\mathbb{D}$, with $\|b\|_\infty \leqslant 1$. Then since $T_{\overline{b}} k_\lambda = \overline{b(\lambda)} k_\lambda$, we have

$$k_\lambda^b = (Id - T_b T_{\overline{b}}) k_\lambda = \frac{1 - \overline{b(\lambda)} b}{1 - \overline{\lambda} z}.$$

It is easy to see that $\mathcal{H}(b)$ is a closed subspace of H^2 if and only if T_b is a partial isometry. That happens if and only if b is an inner function, that is a function in H^∞ whose radial limits are of modulus one almost everywhere. Then $\mathcal{H}(b)$ is the orthogonal complement of the Beurling invariant subspace bH^2, the typical nontrivial invariant subspace of the shift operator S. Hence, the space $\mathcal{H}(b)$, with b inner, are the nontrivial invariant subspaces of the backward shift S^*. In this case, starting with the work of S.V. Hruscev, N.K. Nikolski and B.S. Pavlov, a whole direction of research has investigated geometric properties of reproducting kernels in $\mathcal{H}(b)$ (see [4], [9], [10], [11]). One of the motivation to study geometric properties of reproducting kernels in $\mathcal{H}(b)$ is the link being with nontrigonometric exponentials systems. Recall that in the special case where $b(z) = \exp(a\frac{z+1}{z-1})$, $a > 0$, the reproducing kernels k_λ^b, with $\lambda \in \mathbb{D}$, arise as the range of the exponential functions $\exp(-i\overline{\mu}w)\chi_{(0,a)}$, with $\mu = i\frac{1+\lambda}{1-\lambda}$, under a natural unitary map going from $L^2(0,a)$ to $\mathcal{H}(b)$. Geometric properties of family of exponentials arise in many problems such as scaterring theory, controllability and analysis of convolution equations (see [3] and [11] for details). We intend to provide a comprehensive treatment of geometric properties of reproducing kernels of $\mathcal{H}(b)$, emphasing the parallel with the particular case where b is an inner function.

We now recall some basic definitions concerning geometric properties of sequences in an Hilbert space. For most of the definitions and facts below, one can use [14] as a main reference.

Let $\mathcal{H}$ be a complex Hilbert space. If $(x_n)_{n\geqslant 1} \subset \mathcal{H}$, we denote by $\mathrm{Span}(x_n : n \geqslant 1)$ the closure of the linear hull generated by $(x_n)_{n\geqslant 1}$. The sequence $(x_n)_{n\geqslant 1}$ is called:

(Co) *complete* if $\mathrm{Span}(x_n : n \geqslant 1) = \mathcal{H}$;

(M) *minimal* if for all $n \geqslant 1$, $x_n \notin \mathrm{Span}(x_m : m \neq n)$;

(UM) *uniformly minimal* if $\inf_{n\geqslant 1} \mathrm{dist}\left(\frac{x_n}{\|x_n\|}, \mathrm{Span}(x_m : m \neq n)\right) > 0$;

(UBS) *an unconditionnal basis in its closed linear span* if every element $x \in \mathrm{Span}(x_n : n \geqslant 1)$ can be uniquely decomposed in an uncondionnal convergent series $x = \sum_{n\geqslant 1} a_n x_n$;

(RS) *a Riesz basis in its closed linear span* if there are positive constants c, C such that

$$c\sum_{n\geqslant 1} |a_n|^2 \leqslant \Big\|\sum_{n\geqslant 1} a_n x_n\Big\|^2 \leqslant C\sum_{n\geqslant 1} |a_n|^2, \qquad (1)$$

finite complex sequences $(a_n)_{n\geq 1}$;

(UB) *an unconditionnal basis of* $\mathcal{H}$ if it is complete and an unconditionnal basis in its closed linear span.

Obviously we have

$$(\mathrm{UB}) \Longrightarrow (\mathrm{RS}) \Longrightarrow (\mathrm{USB}) \Longrightarrow (\mathrm{UM}) \Longrightarrow (M).$$

In general, all the converse implications are false but Köthe-Topelitz theorem asserts that if $\|x_n\| \asymp 1$, then (USB) $\Longleftrightarrow$ (RS).

The *Gram matrix* of the sequence $(x_n)_{n\geq 1}$ is $\Gamma = (\langle x_n, x_m\rangle)_{n,m\geq 1}$. Unconditionnal basis are characterized by the fact that Γ defines an invertible operator on ℓ^2.

We recall some well-known facts concerning reproducing kernels in H^2. Let $\Lambda = (\lambda_n)_{n\geqslant 1}$ be a sequence of distinct points in $\mathbb{D}$ and denote by $x_n = \dfrac{k_{\lambda_n}}{\|k_{\lambda_n}\|_2}$ the normalized reproducing kernel. Then we have

- $(k_{\lambda_n})_{n\geqslant 1}$ is minimal if and only if $(\lambda_n)_{n\geqslant 1}$ is Blaschke sequence (which means that $\sum_{n\geqslant 1}(1-|\lambda_n|) < \infty$). As usual, we denote by $B = B_\Lambda = \prod_{n\geqslant 1} b_{\lambda_n}$, where $b_{\lambda_n}(z) = \frac{|\lambda_n|}{\lambda_n}\frac{\lambda_n - z}{1-\overline{\lambda}_n z}$.
- If $(\lambda_n)_{n\geqslant 1}$ is a Blaschke sequence, then $(k_{\lambda_n})_{n\geqslant 1}$ is complete in $\mathcal{H}(B)$.
- $(x_n)_{n\geqslant 1}$ is a Riesz basis of $\mathcal{H}(B)$ if and only if it is uniformly minimal which is equivalent to $(\lambda_n)_{n\geqslant 1}$ satisfies the Carleson condition

  $$\inf_{n\geqslant 1} |B_n(\lambda_n)| > 0,$$

  where $B_n = B/b_{\lambda_n}$; we will write in this case $(\lambda_n)_{n\geqslant 1} \in (C)$.

In this paper, we intend to study the property of unconditionnal basis for sequences of reproducing kernels in $\mathcal{H}(b)$. The study of the spaces $\mathcal{H}(b)$ frequently bifurcates into two cases depending b is an extreme point of the unit ball of H^∞ or not. We will show that for the property of unconditionnal basis in $\mathcal{H}(b)$, there exists a dichotomy between the two cases. Recall that K. de Leeuw and W. Rudin [7] proved that b is an extreme point of the unit ball of H^∞ (abbreviated by b is extreme) if and only if

$$\int_{\mathbb{T}} \log(1-|b|^2)\, dm = -\infty.$$

We now precise some notations that will be used in this paper. For a positive finite Borel measure ν on $\mathbb{T}$ and q a function in $L^2(\nu)$, we let

$$(K_\nu q)(z) := \int_{\mathbb{T}} \frac{q(e^{i\theta})}{1-e^{-i\theta}z}\, d\nu(e^{i\theta}), \qquad z \in \mathbb{C}\setminus\mathbb{T},$$

and we think of K_ν as a linear transformation of $L^2(\nu)$ into the space of holomorphic functions in $\mathbb{C}\setminus\mathbb{T}$. Moreover, we let $H^2(\nu)$ be the closed linear span of z^n,

$n \geqslant 0$, (for the norm of $L^2(\nu)$) and we denote by Z_ν the operator of multiplication by the independant variable on $H^2(\nu)$. If ν is absolutely continuous and ρ is its Radon-Nikodym derivative with respect to nomalized Lebesgue measure, we write K_ρ in place of K_ν, $H^2(\rho)$ in place of $H^2(\nu)$ and Z_ρ in place of Z_ν. Notice that if $q \in L^2(\rho)$ then $q\rho \in L^2$ and

$$K_\rho q = P_+(q\rho).$$

The plan of the paper is the following: the next section deals with the problem of orthogonal basis of reproducing kernels in $\mathcal{H}(b)$. As for the classical case where b is inner, this problem depends on the spectral study of a rank one perturbation of X^*, where $X = S^*{}_{|\mathcal{H}(b)}$. In particular, we prove (Corollary 2.2) that if b is not an inner function, then $\mathcal{H}(b)$ does not possess orthogonal basis of reproducing kernels. In section 3, we give a criterion for the property of unconditionnal basis in its closed linear span (Theorem 3.1 and Theorem 3.2). Then we give some applications of this criterion, which are generalizations of results concerning the classical case. In section 4, we study the case where b is extreme and prove that $Id - T_b T_{\overline{b}}$ is an invertible operator from $\mathcal{H}(u)$ onto $\mathcal{H}(b)$, with u an inner function, if and only if $\operatorname{dist}(\overline{u}b, H^\infty) < 1$ and $\operatorname{dist}(\overline{z}\overline{u}b, H^\infty) = 1$ (Theorem 4.1). Then we use this result to caracterize sequences $(k^b_{\lambda_n})_{n\geqslant 1}$ which form an unconditionnal basis of $\mathcal{H}(b)$ (Theorem 4.2). In section 5, we study the case where b is not an extreme point. Contrary to the extreme case, we show that $\mathcal{H}(b)$ cannot possess unconditionnal basis of reproducing kernels (Corollary 5.1). Then, we get a caracterization of completeness (Proposition 5.2) and finally making further assumption on the multiplier of $\mathcal{H}(b)$, we give a result concerning summation basis of reproducing kernels (Theorem 5.1).

2 Orthogonal bases of reproducing kernel

It is clear that if $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$, then the family $(k^b_{\lambda_n})_{n\geqslant 1}$ cannot be orthogonal. In some cases, it is possible, however, to consider reproducing kernels with poles on the unit circle. Let

$$b(z) = z^N \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z} \exp\left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta) \right),$$

be the canonical factorisation of b, where $\sum_n (1 - |a_n|) < \infty$ and where μ is a positive Borel measure on $\mathbb{T}$ and set

$$E_b := \left\{ \zeta \in \mathbb{T} : \sum_n \frac{1 - |a_n|^2}{|\zeta - a_n|^2} + \int_{\mathbb{T}} \frac{d\mu(t)}{|t - \zeta|^2} < +\infty \right\}.$$

Recall that we say that b has an angular derivative in the sense of Carathéodory at the point λ of $\mathbb{T}$ if b and b' have a nontangential limit at λ and $|b(\lambda)| = 1$.

Then we have the following criterion for the inclusion $k_\lambda^b := \dfrac{1-\overline{b(\lambda)}b}{1-\overline{\lambda}z} \in \mathcal{H}(b)$, $\lambda \in \mathbb{T}$.

Theorem A (Ahern-Clark-Sarason, see [2] and [19]) *Let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ and $\lambda \in \mathbb{T}$. Then the following assertions are equivalent:*

(i) *there is a complex number c of unit modulus such that the function $\dfrac{1-\overline{c}b(z)}{1-\overline{\lambda}z}$ is in $\mathcal{H}(b)$;*

(ii) *$\lambda \in E_b$;*

(iii) $\liminf_{z\to\lambda} \dfrac{1-|b(z)|}{1-|z|} < +\infty$*;*

(iv) *b has an angular derivative in the sense of Carathéodory at λ;*

(v) *every function in $\mathcal{H}(b)$ has a nontangential limit at the point λ.*

Moreover, in this case, the number c is unique and is given by $c = b(\lambda) := \lim_{r\to 1} b(r\lambda)$. If $k_\lambda^b := \dfrac{1-\overline{b(\lambda)}b}{1-\overline{\lambda}z}$, then for all $f \in \mathcal{H}(b)$, we have

$$f(\lambda) = \langle f, k_\lambda^b\rangle_b.$$

Let now $\lambda, \lambda' \in E_b$, $\lambda \neq \lambda'$ and assume that $b(\lambda) = b(\lambda') = \alpha$, $|\alpha| = 1$. Then

$$\langle k_\lambda^b, k_{\lambda'}^b\rangle_b = k_\lambda^b(\lambda') = \frac{1-\overline{b(\lambda)}b(\lambda')}{1-\overline{\lambda}\lambda'} = 0.$$

So if we want to get an orthogonal sequence of reproducing kernel $(k_{\lambda_n}^b)_{n\geqslant 1}$, we have to choose sequence $(\lambda_n)_{n\geqslant 1}$ such that $(\lambda_n)_{n\geqslant 1} \subset E_b$ and $b(\lambda_n) = \alpha$, $n \geqslant 1$, $|\alpha| = 1$. Following the work of Ahern-Clark [1] concerning the classical case where b is an inner function, we proceed first to a study of rank one perturbations of X^* which are isometry, where $X = S^*{}_{|\mathcal{H}(b)}$. Recall that if $\varphi \in H^\infty$, then $\mathcal{H}(b)$ is invariant under $T_{\overline{\varphi}}$ and the norm of $T_{\overline{\varphi}}$ as an operator in $\mathcal{H}(b)$ does not exceed $\|\varphi\|_\infty$. Hence $S^* = T_{\overline{z}}$ acts as a contraction in $\mathcal{H}(b)$ (see [19], (II-7)). Recall also that we have (see [19], (II-9))

$$X^*h = Sh - \langle h, S^*b\rangle_b b \qquad (h \in \mathcal{H}(b)). \tag{2}$$

2.1 Spectral properties of rank one perturbation of X^*

In this subsection, we proceed to an investigation of spectral properties of rank one perturbations of X^* which are isometry. Actually, our study goes beyond what is necessary for our treatment of the existence of orthogonal basis. First we give results concerning spectral properties for X^*. We will see that these properties depend whether b is an extreme point or not (for the analogue results for X, see [19], (IV-5), (V-7) and (V-8)).

Lemma 2.1 *Let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ and $h \in \mathcal{H}(b)$. Then*

$$\|X^*h\|_b = \|h\|_b \Longleftrightarrow \langle h, S^*b\rangle_b = 0$$

Proof: using the relation (2), we get

$$XX^*h = S^*(Sh - \langle h, S^*b\rangle_b b) = h - \langle h, S^*b\rangle_b S^*b.$$

Hence

$$\|X^*h\|_b^2 = \langle XX^*h, h\rangle_b = \|h\|_b^2 - |\langle h, S^*b\rangle_b|^2, \tag{3}$$

which gives the lemma. □

Lemma 2.2 *Let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Then $\sigma_p(X^*) \subset \mathbb{D}$.*

Proof: the inclusion $\sigma_p(X^*) \subset \overline{\mathbb{D}}$ follows from the fact that X^* is a contraction. Assume that there exist $\lambda \in \mathbb{T} \cap \sigma_p(X^*)$ and let $h \in \mathcal{H}(b)$, $h \not\equiv 0$ such that $X^*h = \lambda h$. Then $\|X^*h\|_b = \|h\|_b$ and Lemma 2.1 implies that $\langle h, S^*b\rangle_b = 0$. Hence $X^*h = Sh = \lambda h$, which gives that $\lambda \in \sigma_p(S)$, which is absurd and proves the lemma. □

Proposition 2.1 (a) *If b is extreme then*

$$\sigma_p(X^*) = \{\lambda \in \mathbb{D} : b(\lambda) = 0\} \quad \text{and} \quad \sigma(X^*) = \sigma_p(X^*) \cup \sigma(b),$$

where $\sigma(b) := \mathbb{T} \setminus \rho(b)$ and $\rho(b)$ denotes the set of points $\zeta \in \mathbb{T}$ such that there exist an open arc I, $\zeta \in I$ and b can be continued analytically across I with $|b| = 1$ on I. Moreover if $b(\lambda) = 0$, then

$$\operatorname{Ker}(X^* - \lambda Id) = \mathbb{C}\left(\frac{b}{z - \lambda}\right). \tag{4}$$

(b) *If b is nonextreme then $\sigma(X^*) = \overline{\mathbb{D}}$.*

Proof: recall that X^* is completely nonunitary and if Θ_{X^*} denotes the characteristic operator function of X^*, in the theory of Sz-Nagy and Foias, we have (see [17]) $\Theta_{X^*} = b$ (in the extreme case) and $\Theta_{X^*} = \begin{pmatrix} b \\ a \end{pmatrix}$ (in the nonextreme case). Spectral properties of X^* follow now from a theorem of Sz-Nagy and Foias (see [20], Theorem 4.1. p. 247). It just remains to check equality (4). Let $\lambda \in \mathbb{D}$, $b(\lambda) = 0$ and $f \in \mathrm{Ker}(X^* - \lambda Id)$, $f \not\equiv 0$. Then using (2), we have

$$(z - \lambda)f = \langle f, S^*b\rangle_b b,$$

which implies that $f \in \mathbb{C}(b/(z - \lambda))$. Thus $\mathrm{Ker}(X^* - \lambda Id) \subset \mathbb{C}\left(\frac{b}{z-\lambda}\right)$ and an argument of dimension shows that there is equality. □

Rank one perturbations of X^* that we will interest in are defined as follows.

Definition 2.1 *If λ is a complex number of modulus 1, define the operator U_λ of $\mathcal{H}(b)$ by*

$$U_\lambda := X^* + \lambda(1 - \lambda\overline{b(0)})^{-1} k_0^b \otimes S^*b.$$

Proposition 2.2 *The operator U_λ is an isometry of $\mathcal{H}(b)$. Moreover, it is a unitary operator of $\mathcal{H}(b)$ if and only if b is extreme and in this case, the U_λ are the only one-dimensional perturbations of X^* which are unitary.*

Proof: denote by μ_λ the measure on $\mathbb{T}$ whose Poisson integral is the real part of $\dfrac{1+\overline{\lambda}b}{1-\overline{\lambda}b}$, denote by $V_{\overline{\lambda}b}$ the transformation defined on $L^2(\mu_\lambda)$ by $V_{\overline{\lambda}b}q(z) = (1 - \overline{\lambda}b(z))K_{\mu_\lambda}q(z)$, and finally denote by Z_{μ_λ} the operator of multiplication by the independant variable on $H^2(\mu_\lambda)$. We know (see [19], (III-8)) that we have

$$U_\lambda = V_{\overline{\lambda}b} Z_{\mu_\lambda} V_{\overline{\lambda}b}^{-1} \tag{5}$$

and moreover $V_{\overline{\lambda}b}$ is an isometry of $H^2(\mu_\lambda)$ onto $\mathcal{H}(b)$. Hence U_λ is cleary an isometry of $\mathcal{H}(b)$. We see also that this isometry is onto if and only if Z_{μ_λ} is onto, which is equivalent to $H^2(\mu_\lambda) = L^2(\mu_\lambda)$. But a theorem of Szegö says that $H^2(\mu_\lambda) = L^2(\mu_\lambda)$ if and only if the Radon-Nikodym derivative of the absolutely continuous part of μ_λ with respect to normalized Lebesgue measure is not log-integrable. Now a theorem of Fatou shows that this Radon-Nikodym derivative equals to $\dfrac{1-|b|^2}{|1-\overline{\lambda}b|^2}$. Since $\log|1-\overline{\lambda}b|^2$ is always integrable (being the logarithm of the modulus of the H^∞ function $1-\overline{\lambda}b$), we see that $H^2(\mu_\lambda) = L^2(\mu_\lambda)$ if and only if $\log(1-|b|^2)$ is not integrable, which is exactely the condition that b is extreme.

Now, assume that b is extreme and that $U := X^* + h \otimes k$, $h, k \in \mathcal{H}(b)$, is a unitary operator. If $f \perp k$, then we have $Uf = X^*f$, which gives $\|X^*f\|_b = \|f\|_b$.

Lemma 2.1 implies that $f \perp S^*b$. It follows that there exist $c \in \mathbb{C}$ such that $k = cS^*b$, which gives $U = X^* + h_1 \otimes S^*b$, with $h_1 = \overline{c}h$. Taking the adjoint of this relation, we see that if $f \perp h_1$, then $\|Xf\|_b = \|f\|_b$. Now recall (see [19], (VIII-4)) that $\|Xf\|_b^2 = \|f\|_b^2 - |f(0)|^2$, which gives $f(0) = 0$, that is $f \perp k_0^b$. It follows that there exist $c_1 \in \mathbb{C}$ such that $h_1 = c_1 k_0^b$ and thus $U = X^* + c_1 k_0^b \otimes S^*b$. It remains to show that there exist $\lambda \in \mathbb{T}$ such that $c_1 = \lambda(1 - \lambda\overline{b(0)})^{-1}$. Notice that for all $f \in \mathcal{H}(b)$, we have

$$\|f\|_b^2 = \|Uf\|_b^2 = \|X^*f\|_b^2 + |c_1|^2 |\langle f, S^*b\rangle_b|^2 \|k_0^b\|_b^2 + 2\,\mathrm{Re}\left(c_1 \langle f, S^*b\rangle_b \langle k_0^b, X^*f\rangle_b\right).$$

In particular for $f = S^*b$, using relation (3), we get

$$0 = -\|S^*b\|_b^2 + |c_1|^2 \|S^*b\|_b^2 (1 - |b(0)|^2) + 2\,\mathrm{Re}(c_1 \overline{(X^*S^*b)(0)}).$$

Since $X^*S^*b = SS^*b - \|S^*b\|_b^2 b$, it follows that $(X^*S^*b)(0) = -\|S^*b\|_b^2 b(0)$, which implies that

$$0 = -1 + |c_1|^2 (1 - |b(0)|^2) - 2\,\mathrm{Re}(c_1 \overline{b(0)}).$$

Now define $\lambda := \overline{c_1}^{-1} + b(0)$. Using the previous equality, easy computations show that $\lambda \in \mathbb{T}$ and $c_1 = \lambda(1 - \lambda\overline{b(0)})^{-1}$, which ends the proof of the proposition. $\square$

The following lemma is a generalization of a result of Ahern-Clark [1] for the case where b is an inner function.

Lemma 2.3 *Let $\zeta \in \mathbb{T}$. The following assertions are equivalent:*

(i) *b has an angular derivative in the sense of Carathéodory at ζ;*

(ii) *$k_0^b \in \mathrm{Im}(Id - \overline{\zeta}X^*)$.*

Moreover, in that case, we have $(Id - \overline{\zeta}X^*)k_\zeta^b = k_0^b$.

Proof: (i) $\Longrightarrow$ (ii): since b has angular derivative in the sense of Carathéodory at ζ, we know (see [19], (VI-4)) that k_z^b tends to k_ζ^b in norm as z tends nontangentially to ζ. Notice we have

$$\begin{aligned}\|(Id - \overline{z}X^*)k_z^b - (Id - \overline{\zeta}X^*)k_\zeta^b\| =& \|(Id - \overline{z}X^*)(k_z^b - k_\zeta^b) + ((Id - \overline{z}X^*) - (Id - \overline{\zeta}X^*))k_\zeta^b\| \\ \leqslant& 2\|k_z^b - k_\zeta^b\| + |z - \zeta| \|X^*\| \|k_\zeta^b\|.\end{aligned}$$

Hence $(Id - \overline{z}X^*)k_z^b$ tends to $(Id - \overline{\zeta}X^*)k_\zeta^b$ as z tends nontangentially to ζ. Moreover we have

$$(Id - \overline{z}X^*)k_z^b = k_0^b,$$

(see [19], (V-8)) which implies that $k_0^b = (Id - \overline{\zeta}X^*)k_\zeta^b$.

(ii) $\Longrightarrow$ (i): assume that there exists $g \in \mathcal{H}(b)$ such that $k_0^b = (Id - \overline{\zeta}X^*)g$. We have

$$\begin{aligned}k_z^b =& (Id - \overline{z}X^*)^{-1} k_0^b \\ =& (Id - \overline{z}X^*)^{-1}(Id - \overline{\zeta}X^*)g \\ =& g + (\overline{z} - \overline{\zeta})(Id - \overline{z}X^*)X^*g,\end{aligned}$$

which gives that

$$\|k_z^b\| \leqslant \|g\| \left(1+|z-\zeta|\|(Id-\overline{z}X^*)^{-1}\|\|X^*\|\right).$$

Using the fact that $\|(Id-\overline{z}X^*)^{-1}\| \leqslant (1-|z|)^{-1}$, we deduce that

$$\|k_z^b\| \leqslant \|g\| \left(1+\frac{|z-\zeta|}{1-|z|}\|X^*\|\right).$$

As $|z-\zeta|/(1-|z|)$ stays bounded as z tends nontangentially to ζ, we get that $\|k_z^b\|$ stays bounded as z tends nontangentially to ζ, which by Theorem A, implies that b has an angular derivative in the sense of Carathéodory at ζ. □

Since U_λ is an isometry, its point spectrum is located on the unit circle. The notion of angular derivative will lead us to caracterize it. This result was obtained by Ahern-Clark [1] for the case where b is inner.

Theorem 2.1 *Let $\lambda \in \mathbb{T}$. Then a complex number ζ is an eigenvalue of U_λ if and only if b has an angular derivative in the sense of Caratheodory at ζ and $b(\zeta)=\lambda$. Moreover we have* $\mathrm{Ker}(U_\lambda-\zeta Id)=\mathbb{C}k_\zeta^b$.

Proof: assume that b has an angular derivative in the sense of Caratheodory at ζ and $b(\zeta)=\lambda$. Using Lemma 2.3, we have $k_0^b=(Id-\overline{\zeta}X^*)k_\zeta^b$. Hence

$$\begin{aligned}(U_\lambda-\zeta Id)k_\zeta^b =&(X^*-\zeta Id)k_\zeta^b+\lambda(1-\lambda\overline{b(0)})^{-1}\langle k_\zeta^b,S^*b\rangle_b k_0^b\\ =&-\zeta k_0^b+\lambda(1-\lambda\overline{b(0)})^{-1}\langle k_\zeta^b,S^*b\rangle_b k_0^b.\end{aligned}$$

Take now a sequence $(z_n)_n$ which tends nontangentially to ζ and notice that

$$\langle S^*b,k_\zeta^b\rangle_b=\lim_{n\to+\infty}\langle S^*b,k_{z_n}^b\rangle_b=\lim_{n\to+\infty}\frac{b(z_n)-b(0)}{z_n}=\frac{\lambda-b(0)}{\zeta}.$$

That implies

$$(U_\lambda-\zeta Id)k_\zeta^b=-\zeta k_0^b+\lambda(1-\lambda\overline{b(0)})^{-1}\zeta(\overline{\lambda}-\overline{b(0)})k_0^b=0,$$

which proves that $\zeta\in\sigma_p(U_\lambda)$ and $\mathbb{C}k_\zeta^b\subset\mathrm{Ker}(U_\lambda-\zeta Id)$.

Reciprocally, let $\zeta\in\sigma_p(U_\lambda)$ and $f\in\mathcal{H}(b)$, $f\not\equiv 0$ such that $(U_\lambda-\zeta Id)f=0$. Then, we have $(X^*-\zeta Id)f=-\lambda(1-\lambda\overline{b(0)})^{-1}\langle f,S^*b\rangle_b k_0^b$. Notice that if $\langle f,S^*b\rangle_b=0$, then $\zeta\in\sigma_p(X^*)$, which is absurd thanks to Lemma 2.2. Hence $\langle f,S^*b\rangle_b\neq 0$, and there exists c$\in\mathbb{C}$, $c\neq 0$, such that $k_0^b=(Id-\overline{\zeta}X^*)(cf)$. Lemma 2.3 implies that b has an angular derivative in the sense of Carathédory at ζ and $k_0^b=(Id-\overline{\zeta}X^*)k_\zeta^b$. We deduce that $k_\zeta^b-cf\in\mathrm{Ker}(X^*-\overline{\zeta}Id)$ and Lemma 2.2 implies that $k_\zeta^b=cf$. Hence $k_\zeta^b\in\mathrm{Ker}(U_\lambda-\zeta Id)$. But previous computations show that

$$(U_\lambda-\zeta Id)k_\zeta^b=\left(-\zeta+\lambda(1-\lambda\overline{b(0)})^{-1}\frac{\overline{b(\zeta)}-\overline{b(0)}}{\overline{\zeta}}\right)k_0^b,$$

which implies that $\lambda(\overline{b(\zeta)} - \overline{b(0)})(1 - \lambda\overline{b(0)})^{-1} = 1$, hence $b(\zeta) = \lambda$. Moreover as $k_\zeta^b = cf$, we have that $\mathrm{Ker}(U_\lambda - \zeta Id) \subset \mathbb{C}k_\zeta^b$. □

As in the classical case where b is inner, we can deduce from this result the description of the spectrum of U_λ.

Corollary 2.1 *Let $\lambda \in \mathbb{T}$.*

a) *If b is extreme, then $\sigma(U_\lambda) \subset \mathbb{T}$ and*

$$\zeta \in \sigma(U_\lambda) \Longleftrightarrow \begin{array}{ll} (i) & \zeta \in \sigma(b) \\ or & \\ (ii) & \zeta \in {}^c\sigma(b) \text{ and } b(\zeta) = \lambda. \end{array}$$

b) *If b is nonextreme, then $\sigma(U_\lambda) = \overline{\mathbb{D}}$.*

Proof: a): assume that b is extreme. Proposition 2.2 shows that U_λ is unitary, so $\sigma(U_\lambda) \subset \mathbb{T}$. Let $\zeta \in \sigma(U_\lambda)$, $\zeta \in {}^c\sigma(b)$. Using the fact that $\sigma(b) = \sigma(X^*) \cap \mathbb{T}$ (see Proposition 2.1) and the fact that U_λ is a rank-one perturbation of X^*, we deduce that $U_\lambda - \zeta Id$ is a Fredholm operator of index 0. As $\zeta \in \sigma(U_\lambda)$, we get that $\zeta \in \sigma_p(U_\lambda)$ and Theorem 2.1 implies that $b(\zeta) = \lambda$.

Reciprocally let $\zeta \in \sigma(b)$ and assume that $\zeta \in {}^c\sigma(U_\lambda)$. Using once more the fact that U_λ is a rank-one perturbation of X^*, we get that $X^* - \zeta Id$ is a Fredholm operator of index 0. Thanks to Lemma 2.2, we have that $\mathrm{Ker}(X^* - \zeta Id) = \{0\}$. Hence $X^* - \zeta Id$ is invertible, which gives $\zeta \in {}^c\sigma(X^*) = {}^c\sigma(b)$, which is absurd.

On the other hand, let $\zeta \in {}^c\sigma(b)$ and $b(\zeta) = \lambda$. By definition, there exist an open arc I, $\zeta \in I$ such that b can be continued analytically across I and $|b| = 1$ on I. In particular, b has an angular derivative in the sense of Caratheodory at ζ and since $b(\zeta) = \lambda$, thanks to Theorem 2.1, we get that $\zeta \in \sigma_p(U_\lambda) \subset \sigma(U_\lambda)$.

b): assume that b is nonextreme. Since U_λ is an isometry, we clearly have $\sigma(U_\lambda) \subset \overline{\mathbb{D}}$. Now let $\zeta \in \mathbb{D}$ and assume that $\zeta \in {}^c\sigma(U_\lambda)$. Recall that when b is nonextreme, then $b \in \mathcal{H}(b)$ and the space $\mathcal{H}(b)$ is invariant under the unilateral shift S (see [19], (IV-5)). Hence if we denote by $Y := S_{|\mathcal{H}(b)}$, we have, using the formula (2),

$$U_\lambda = X^* + \lambda(1 - \lambda\overline{b(0)})^{-1} k_0^b \otimes S^*b = Y - b \otimes S^*b + \lambda(1 - \lambda\overline{b(0)})^{-1} k_0^b \otimes S^*b.$$

Thus we get that $Y - \zeta Id$ is a Fredholm operator of index 0. Since $\mathrm{Ker}(Y - \zeta Id) = \{0\}$, the following lemma gives a contradiction; hence $\mathbb{D} \subset \sigma(U_\lambda)$, which ends the proof of the corollary. □

Lemma 2.4 *Assume that b is nonextreme and let $Y = S_{|\mathcal{H}(b)}$. Then for $\mu \in \mathbb{D}$, we have* $\mathrm{Ker}(Y^* - \overline{\mu} Id) = \mathbb{C}k_\mu^b$.

Proof of Lemma 2.4: for all $f \in \mathcal{H}(b)$, we have

$$\langle Y^* k_\mu^b, f\rangle_b = \langle k_\mu^b, Yf\rangle_b = \langle k_\mu^b, zf\rangle_b = \overline{\mu}\overline{f(\mu)} = \langle \overline{\mu} k_\mu^b, f\rangle_b,$$

which proves that $Y^* k_\mu^b = \overline{\mu} k_\mu^b$. Hence $k_\mu^b \in \mathrm{Ker}(Y^* - \overline{\mu} Id)$.

Let now $f \in \mathrm{Ker}(Y^* - \overline{\mu} Id)$ and $g \in (\mathbb{C} k_\mu^b)^\perp$. Define $h := \frac{g}{z-\mu}$. Since $g(\mu) = 0$, we get that $h \in \mathcal{H}(b)$ (see [19], (II-8)). Hence

$$g = (z-\mu)h = (Y - \mu Id)h \in (Y - \mu Id)\mathcal{H}(b) \subset (\mathrm{Ker}(Y^* - \overline{\mu} Id))^\perp .$$

That implies $\langle f, g\rangle_b = 0$, and thus $f \in ((\mathbb{C} k_\mu^b)^\perp)^\perp = \mathbb{C} k_\mu^b$. □

2.2 orthogonal bases of reproducing kernels in $\mathcal{H}(b)$

Let $\lambda \in \mathbb{T}$. The function $\dfrac{1-|b(z)|^2}{|\lambda - b(z)|^2}$ is a nonnegative harmonic function in $\mathbb{D}$, so it can be represented as a Poisson integral

$$\frac{1-|b(z)|^2}{|\lambda - b(z)|^2} = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\overline{\zeta} z|^2}\, d\mu_\lambda(\zeta); \qquad z \in \mathbb{D},$$

where μ_λ is a nonnegative Borel measure in $\mathbb{T}$.

The following result gives a criterion in terms of the measure μ_λ for the existence of an orthogonal basis of reproducing kernels in $\mathcal{H}(b)$. In the particular case where b is inner, this result was obtained by Ahern-Clark [1].

Theorem 2.2 *Let $\lambda \in \mathbb{T}$. The following assertions are equivalent:*

(i) *the family $\{k_\zeta^b : \zeta \in E_b, b(\zeta) = \lambda\}$ forms an orthogonal basis of $\mathcal{H}(b)$;*

(ii) *the measure μ_λ is purely atomic.*

Proof: if $\zeta \in E_b$, $b(\zeta) = \lambda$, then Theorem 2.1 implies that $U_\lambda k_\zeta^b = \zeta k_\zeta^b$. Hence the family $\{k_\zeta^b : \zeta \in E_b, b(\zeta) = \lambda\}$ forms an orthogonal system of eigenvectors of U_λ in $\mathcal{H}(b)$.
(i) $\Longrightarrow$ (ii): since $\mathcal{H}(b)$ is separable, $\{\zeta \in E_b : b(\zeta) = \lambda\}$ is countable. Denote by $(\zeta_n)_{n\geqslant 1} := \{\zeta \in E_b : b(\zeta) = \lambda\}$. Since $V_{\overline{\lambda} b}$ is an isometry from $H^2(\mu_\lambda)$ onto $\mathcal{H}(b)$, the family $(V_{\overline{\lambda} b}^{-1} k_{\zeta_n}^b)_{n\geqslant 1}$ is an orthogonal basis of $H^2(\mu_\lambda)$. Moreover, using (5), we have

$$Z_{\mu_\lambda} V_{\overline{\lambda} b}^{-1} k_{\zeta_n}^b = V_{\overline{\lambda} b}^{-1} U_\lambda k_{\zeta_n} = \zeta_n V_{\overline{\lambda} b}^{-1} k_{\zeta_n}^b .$$

That means that $H^2(\mu_\lambda)$ has an orthogonal basis of eigenvectors of Z_{μ_λ}, the operator of multiplication by the independant variable on $L^2(\mu_\lambda)$. It is now a well-known fact that implies that $\mu_\lambda = \sum_{n\geqslant 1} a_n \delta_{\{\zeta_n\}}$, $a_n := \mu_\lambda(\zeta_n)$.

(ii) $\Longrightarrow$ (i): assume that μ_λ is purely atomic, that is $\mu_\lambda = \sum_{n\geqslant 1} a_n \delta_{\{\zeta_n\}}$, with $a_n = \mu_\lambda(\{\zeta_n\}) > 0$. In particular, for all f in $H^2(\mu_\lambda) = L^2(\mu_\lambda)$, we have

$$\|f\|^2 = \sum_{n\geqslant 1} a_n |f(\zeta_n)|^2.$$

Using this equality, it is easy to see that $(\chi_{\{\zeta_n\}})_{n\geqslant 1}$ is an orthogonal basis of $L^2(\mu_\lambda)$ and we get that $(V_{\overline{\lambda}b}\chi_{\{\zeta_n\}})_{n\geqslant 1}$ is an orthogonal basis of $\mathcal{H}(b)$. Using once more (5), we have $U_\lambda(V_{\overline{\lambda}b}\chi_{\{\zeta_n\}}) = V_{\overline{\lambda}b} Z_{\mu_\lambda}\chi_{\{\zeta_n\}} = \zeta_n V_{\overline{\lambda}b}\chi_{\{\zeta_n\}}$. Theorem 2.1 implies that $\zeta_n \in E_b$, $b(\zeta_n) = \lambda$ and there exits $c_n \in \mathbb{C}^*$ such that $V_{\overline{\lambda}b}\chi_{\{\zeta_n\}} = c_n k^b_{\zeta_n}$. Hence $(k^b_{\zeta_n})_{n\geqslant 1}$ is an orthogonal basis of $\mathcal{H}(b)$. It remains to notice that $\{\zeta \in E_b : b(\zeta) = \lambda\} = (\zeta_n)_{n\geqslant 1}$. The inclusion $(\zeta_n)_{n\geqslant 1} \subset \{\zeta \in E_b : b(\zeta) = \lambda\}$ has already been proved. Assume that there exists $\zeta \in E_b$, $b(\zeta) = \lambda$, $\zeta \neq \zeta_n$, $n \geqslant 1$. Theorem A implies that $k^b_\zeta \in \mathcal{H}(b)$ and

$$\langle k^b_{\zeta_n}, k^b_\zeta\rangle = \frac{1 - \overline{b(\zeta_n)}b(\zeta)}{1 - \overline{\zeta_n}\zeta} = 0.$$

Hence $k^b_\zeta \in \mathcal{H}(b) \ominus \mathrm{Span}(k^b_{\zeta_n} : n \geqslant 1)$, which is absurd. $\square$

Corollary 2.2 *Let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Assume that b is not an inner function. Then $\mathcal{H}(b)$ does not have an orthogonal basis of reproducing kernels.*

Proof: let $\lambda \in \mathbb{T}$. Since b is not an inner function, there exists $A \in \mathrm{Bor}(\mathbb{T})$, $m(A) > 0$ such that for all $\zeta \in A$, $|b(\zeta)| \neq 1$. Now if

$$\frac{1 - |b(z)|^2}{|\lambda - b(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \overline{\zeta}z|^2}\, d\mu_\lambda(\zeta); \qquad z \in \mathbb{D},$$

and if $\mu_\lambda^{(a)}$ denotes the absolutely component part of the measure μ_λ, we know that $\dfrac{1 - |b(\zeta)|^2}{|\lambda - b(\zeta)|^2} = \dfrac{d\mu_\lambda^{(a)}}{dm}(\zeta)$, for almost $\zeta \in A$ with respect to the Lebesgue measure. Hence $\mu_\lambda^{(a)} \neq 0$ and the measure μ_λ cannot be purely atomic. Theorem 2.2 implies that $\mathcal{H}(b)$ does not have an orthogonal basis of reproducing kernels. $\square$

3 Unconditionnal bases of reproducing kernels in $\mathcal{H}(b)$

Let us first remark that if $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and $(k^b_{\lambda_n})_{n\geqslant 1}$ is minimal, then $(\lambda_n)_{n\geqslant 1}$ is a Blaschke sequence of distincts points. Thus from now on, we assume that $(\lambda_n)_{n\geqslant 1}$ is a Blaschke sequence of distincts points of the unit disk and we denote by B the Blaschke product associated to $(\lambda_n)_{n\geqslant 1}$.

The problem of unconditionnal basis of reproducing kernels of $\mathcal{H}(b)$, in the case where b is inner, was solved by S.V. Hruscev, N.K. Nikolski and B.S. Pavlov [11]. They prove that if b is inner and $\sup_{n\geqslant 1}|b(\lambda_n)|<1$, then $(k^b_{\lambda_n})_{n\geqslant 1}$ is an unconditionnal basis in its closed linear span (resp. of $\mathcal{H}(b)$) if and only if $(\lambda_n)_{n\geqslant 1}\in(C)$, $\mathrm{dist}(\overline{B}b,H^\infty)<1$ (resp. plus $\mathrm{dist}(\overline{B}b,H^\infty)<1$). The key point to get this criterion is the following formulae

$$bJT_{b\overline{B}}J\overline{B}=Id_{H^2_-}\oplus P_{b|\mathcal{H}(B)},\quad \text{in the space } BH^2_-=H^2_-\oplus\mathcal{H}(B), \tag{6}$$

where $P_b=(Id-T_bT_{\overline{b}})^{1/2}$ is the orthogonal projection of H^2 onto $\mathcal{H}(b)=H^2\ominus bH^2$, $Jg=\overline{zg}$, $g\in L^2(\mathbb{T})$ (see [16], Lemma 4.4.4. p. 309).

In the general case, the formula (6) is no longer true. However, we will see that it can be possible to get some similar results for unconditionnal basis of reproducing kernels in their closed linear span. For complete unconditionnal basis, as we will see in the sections 4 and 5, the solution breaks down into two cases depending whether b is extreme or not.

From now on, we denote by $x_n:=\dfrac{k_{\lambda_n}}{\|k_{\lambda_n}\|_2}$ (resp. by $x^b_n:=\dfrac{k^b_{\lambda_n}}{\|k^b_{\lambda_n}\|_b}$) the normalized reproducing kernels of H^2 (resp. of $\mathcal{H}(b)$) associated to a sequence $(\lambda_n)_{n\geqslant 1}$.

3.1 A criterion for unconditionnal basis in its closed linear span.

The next result shows that Carleson's condition is necessary for a sequence of reproducing kernels of $\mathcal{H}(b)$ to be an unconditionnal basis in its closed linear span. The proof is similar to the classical case where b is inner (see [14], Lect. VIII, p. 200) and is left to the reader.

Proposition 3.1 *Assume that $(k^b_{\lambda_n})_{n\geqslant 1}$ forms an unconditionnal basis in its closed linear span. Then $(\lambda_n)_{n\geqslant 1}\in$ (C).*

The next result is the first step in our study of unconditionnal basis property.

Theorem 3.1 *Let $(\lambda_n)_{n\geqslant 1}\subset\mathbb{D}$ and let $b\in H^\infty$, $\|b\|_\infty\leqslant 1$. Assume that*

$$\sup_{n\geqslant 1}|b(\lambda_n)|<1. \tag{7}$$

Then the following statements are equivalent:

(i) $(k^b_{\lambda_n})_{n\geqslant 1}$ *forms an unconditionnal basis of $\mathcal{H}(b)$ (resp. in its closed linear span);*

(ii) a) $(\lambda_n)_{n\geqslant 1}\in(C)$, b) *the operator $Id-T_bT_{\overline{b}}$ is an isomorphism of $\mathcal{H}(B)$ onto $\mathcal{H}(b)$ (resp. onto its range).*

Proof: (i) $\Longrightarrow$ (ii) : Proposition 3.1 implies that $(\lambda_n)_{n\geqslant 1} \in (C)$ and thus $(x_n)_{n\geqslant 1}$ is a Riesz basis of $\mathcal{H}(B)$. Moreover, condition (7) shows that (i) is equivalent to the fact that $((1-|b(\lambda_n)|^2)^{1/2}x_n^b)_{n\geqslant 1}$ forms a Riesz basis. But

$$(Id - T_bT_{\overline{b}})x_n = (1-|\lambda_n|^2)^{1/2}k_{\lambda_n}^b = (1-|b(\lambda_n)|^2)^{1/2}x_n^b.$$

Hence the operator $Id - T_bT_{\overline{b}}$ transforms a Riesz basis of $\mathcal{H}(B)$ onto a Riesz basis of $\mathcal{H}(b)$ (resp. of its closed linear span), so it is an isomorphism of $\mathcal{H}(B)$ onto $\mathcal{H}(b)$ (resp. onto its range).
(ii) $\Longrightarrow$ (i) : from a), we get that $(x_n)_{n\geqslant 1}$ is a Riesz basis of $\mathcal{H}(B)$ and using b), we have that $((Id - T_bT_{\overline{b}})x_n)_{n\geqslant 1}$ is a Riesz basis of $\mathcal{H}(b)$ (resp. of its closed linear span). Hence $(k_{\lambda_n}^b)_{n\geqslant 1}$ forms an unconditionnal basis of $\mathcal{H}(b)$ (resp. in its closed linear span). □

We will now give a criterion for the left invertibily of $(Id - T_bT_{\overline{b}})_{|\mathcal{H}(B)}$.

Lemma 3.1 *Let u be an inner function and let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Then the following statements are equivalent:*

(i) *The operator $Id - T_bT_{\overline{b}}$ is an isomorphism of $\mathcal{H}(u)$ onto its range;*

(ii) *$dist(\overline{u}b, H^\infty) < 1$;*

(iii) $\|P_ub_{|\mathcal{H}(u)}\| < 1$.

Proof: the operator $Id - T_bT_{\overline{b}}$ is an isomorphism of $\mathcal{H}(u)$ onto its range if and only if there exists $c > 0$ such that

$$c\|f\|_2 \leqslant \|(Id - T_bT_{\overline{b}})f\|_b, \qquad (f \in \mathcal{H}(u)).$$

Notice that

$$\|(Id - T_bT_{\overline{b}})f\|_b^2 = \|(Id - T_bT_{\overline{b}})^{1/2}f\|_2^2 = \langle (Id - T_bT_{\overline{b}})f, f\rangle_2 = \|f\|_2^2 - \|T_{\overline{b}}f\|_2^2.$$

Hence the operator $Id - T_bT_{\overline{b}}$ is an isomorphism of $\mathcal{H}(u)$ onto its range if and only if there exists $c > 0$ such that, for all f in $\mathcal{H}(u)$, we have

$$\|T_{\overline{b}}f\|_2^2 \leqslant (1-c^2)\|f\|_2^2,$$

which is equivalent to $\|T_{\overline{b}|\mathcal{H}(u)}\| < 1$. But $T_{\overline{b}|\mathcal{H}(u)} = P_+\overline{b}_{|\mathcal{H}(u)}$ and it is easy to see that $(T_{\overline{b}|\mathcal{H}(u)})^* = P_ub = uP_-\overline{u}b$. It follows that

$$\|T_{\overline{b}|\mathcal{H}(u)}\| = \|uP_-\overline{u}b\| = \|P_-\overline{u}b\| = \|H_{\overline{u}b}\| = \text{dist}(\overline{u}b, H^\infty),$$

which gives the equivalence of the first two statements. Now notice that $P_ub_{|uH^2} = 0$ and so $\|P_ub\| = \|P_ub_{|\mathcal{H}(u)}\|$, which gives the equivalence with the third assertion. □

Using Theorem 3.1 and Lemma 3.1, we get the following criterion which generalizes the classical one (see [11], Theorem 2 and 3 bis. p. 230-232).

Theorem 3.2 *Let $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Assume that*

$$\sup_{n\geqslant 1} |b(\lambda_n)| < 1.$$

Then the following statements are equivalent:

(i) $(k^b_{\lambda_n})_{n\geqslant 1}$ *forms an unconditionnal basis in its closed linear span;*

(ii) a) $(\lambda_n)_{n\geqslant 1} \in (C)$, b) $dist(\overline{B}b, H^\infty) < 1$.

3.2 Applications of Theorem 3.1

We now give some applications of our criterion. The proof of the following facts are similar to the case where b is inner (see [11], Theorem 3.2, 3.5) and are left to the reader.

Corollary 3.1 *Let $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Assume that*

$$\sup_{n\geqslant 1} |b(\lambda_n)| < 1.$$

Then the following statements are equivalent:

(i) *there exits $p \in \mathbb{N}$ sufficiently large such that $(k^{b^p}_{\lambda_n})_{n\geqslant 1}$ forms an unconditionnal basis in its closed linear span;*

(ii) $(\lambda_n)_{n\geqslant 1} \in (C)$.

Corollary 3.2 *Let $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ such that*

$$\lim_{n\to 0} |b(\lambda_n)| = 0.$$

Assume that $(\lambda_n)_{n\geqslant 1} \in (C)$. Then there exists $N \in \mathbb{N}$ sufficiently large such that $(k^b_{\lambda_n})_{n\geqslant N}$ forms an unconditionnal basis in its closed linear span.

Corollary 3.3 *Let $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Assume that*

$$\lim_{n\to +\infty} |b(\lambda_n)| = 0.$$

Then the following statements are equivalent:

(i) $(k^b_{\lambda_n})_{n\geqslant 1}$ *forms an unconditionnal basis in its closed linear span;*

(ii) $(k^b_{\lambda_n})_{n\geqslant 1}$ *is uniformly minimal.*

In the case where b is inner, S.V. Hruscev, N.K. Nikolski and B.S. Pavlov ([11], Theorem 3.2) show that if $b = BS$ is the canonical factorisation of b, where B is a Blaschke product and S is a singular inner function, and if $S \neq$ const and $\lim_{n\to+\infty} |b(\lambda_n)| = 0$, then the Carleson's condition is sufficient for the sequence $(k^b_{\lambda_n})_{n\geqslant 1}$ to be an unconditionnal basis of its closed linear span. Moreover we have $\dim(\mathcal{H}(b) \ominus \mathrm{Span}(k^b_{\lambda_n} : n \geqslant 1)) = +\infty$. Using Theorem 3.1, we can give an analogue of this result. But before, we will need two lemmas. The first one is an easy generalization of a result for the classical case (see [16], p. 313) and the proof is left to the reader. The second one is also a generalization of a result for the classical case but is more complicated.

Lemma 3.2 *Let $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Assume that*

$$\mathrm{Span}(k^b_{\lambda_n} : n \geqslant 1) \not\subseteq \mathcal{H}(b).$$

Then for all $\mu \neq \lambda_n$, $n \geqslant 1$, we have

a) $k^b_\mu \notin \mathrm{Span}(k^b_{\lambda_n} : n \geqslant 1)$; *in particular $(k^b_{\lambda_n})_{n\geqslant 1}$ is minimal.*

b) $\mathrm{Span}(k^b_{\lambda_n}, k^b_\mu : n \neq p) \not\subseteq \mathcal{H}(b)$, $\forall p \geqslant 1$.

c) $\{k^b_{\lambda_n}, k^b_\mu : n \geqslant 1\}$ *is minimal.*

Lemma 3.3 *Let φ_1 be an inner function, $\varphi_2 \in H^\infty$, $\|\varphi_2\|_\infty \leqslant 1$, and $\varphi = \varphi_1\varphi_2$. Then we have*

$$\mathrm{Ker}(Id - T_{\varphi_1}T_{\overline{\varphi}_1})_{|\mathcal{H}(\varphi)} = \mathcal{H}(\varphi) \cap \varphi_1 H^2 = \varphi_1\mathcal{H}(\varphi_2).$$

Proof: notice that $Id - T_\varphi T_{\overline{\varphi}} \geqslant Id - T_{\varphi_1}T_{\overline{\varphi}_1}$. Indeed, for all $f \in H^2$, we have

$$\begin{aligned}\langle (Id - T_\varphi T_{\overline{\varphi}})f, f\rangle_2 =& \|f\|_2^2 - \|P_+\overline{\varphi}f\|_2^2\\ =& \|f\|_2^2 - \|P_+\overline{\varphi_2}P_+\overline{\varphi_1}f\|_2^2\\ \geqslant& \|f\|_2^2 - \|P_+\overline{\varphi_1}f\|_2^2\\ =& \langle (Id - T_{\varphi_1}T_{\overline{\varphi}_1})f, f\rangle_2.\end{aligned}$$

Using a result of Douglas (see [19], (I-5)), it follows that $\mathcal{H}(\varphi_1) \subset \mathcal{H}(\varphi)$. Hence, we have $\mathcal{H}(\varphi) \in \mathrm{Lat}(Id - T_{\varphi_1}T_{\overline{\varphi}_1})$.

Since φ_1 is inner, we have $\mathrm{Ker}(Id - T_{\varphi_1}T_{\overline{\varphi}_1}) = \varphi_1 H^2$ and it follows that

$$\mathrm{Ker}(Id - T_{\varphi_1}T_{\overline{\varphi}_1})_{|\mathcal{H}(\varphi)} = \mathcal{H}(\varphi) \cap \varphi_1 H^2.$$

Let us show now that $H(\varphi) \cap \varphi_1 H^2 = \varphi_1\mathcal{H}(\varphi_2)$. First let $f \in \mathcal{H}(\varphi) \cap \varphi_1 H^2$. Then there exists $g \in H^2$ such that $f = \varphi_1 g$ and we have

$$T_{\overline{\varphi}_2} g = P_+\overline{\varphi}_2 g = P_+\overline{\varphi}\varphi_1 g = T_{\overline{\varphi}} f.$$

Since $f \in \mathcal{H}(\varphi)$, $T_{\overline{\varphi}} f \in \mathcal{H}(\overline{\varphi})$ (see [19], (II-4)). But $\mathcal{H}(\overline{\varphi}) = \mathcal{H}(\overline{\varphi}_2)$, and so $T_{\overline{\varphi}_2} g \in \mathcal{H}(\overline{\varphi}_2)$. Using once more [19], (II-4), we get that $g \in \mathcal{H}(\varphi_2)$. Reciprocally, let $g \in \mathcal{H}(\varphi_2)$, and $f = \varphi_1 g$. Of course, $f \in \varphi_1 H^2$. On the other hand, we have

$$T_{\overline{\varphi}} f = P_+ \overline{\varphi} \varphi_1 g = P_+ \overline{\varphi}_2 g = T_{\overline{\varphi}_2} g,$$

and an other application of [19], (II-4) show that $f \in \mathcal{H}(\varphi)$. □

As it was mentionned, the next result generalizes Theorem 3.2 in [11].

Theorem 3.3 *Let $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Let u_0 be the inner factor and b_0 the outer factor of b. Assume that u_0 is non constant, that b is not a Blaschke product and furthemore that*

$$\lim_{n\to+\infty} |u_0(\lambda_n)| = 0.$$

Then the following statements are equivalent:

(i) $(k_{\lambda_n}^b)_{n\geqslant 1}$ *forms an unconditionnal basis in its closed linear span;*

(ii) $(\lambda_n)_{n\geqslant 1} \in (C)$.

Moreover in this case, we have $dim(\mathcal{H}(b) \ominus \operatorname{Span}(k_{\lambda_n}^b : n \geqslant 1)) = +\infty$.

Proof: (i) $\Longrightarrow$ (ii): is always true and follows from Proposition 3.1.
(ii) $\Longrightarrow$ (i): write $u_0 = B_0 S_0$, where B_0 is a Blaschke product and S_0 is a singular inner function. Define

$$\varphi_1 = \begin{cases} S_0^{1/2} B_0 & \text{if } b_0 \equiv \text{const} \\ u_0 & \text{if } b_0 \not\equiv \text{const} \end{cases} \quad \text{and} \quad \varphi_2 = \begin{cases} b_0 S_0^{1/2} & \text{if } b_0 \equiv \text{const} \\ b_0 & \text{if } b_0 \not\equiv \text{const}. \end{cases}$$

In the two cases, we have $b = \varphi_1 \varphi_2$, φ_1 is an inner function and $\varphi_2 \in H^\infty$, $\|\varphi_2\|_\infty \leqslant 1$. Moreover, the assumptions on b imply that $\varphi_2 \not\equiv$ const and $\lim_{n\to+\infty} |\varphi_1(\lambda_n)| = \lim_{n\to+\infty} |b(\lambda_n)| = 0$. Consequently, it follows from Corollary 3.2 that there exists $N \in \mathbb{N}$ sufficiently large such that both family $(k_{\lambda_n}^{\varphi_1})_{n\geqslant N}$ and $(k_{\lambda_n}^b)_{n\geqslant N}$ form an unconditionnal bases in their closed linear span. Moreover, we see that the norms $\|k_{\lambda_n}^b\|_b$ and $\|k_{\lambda_n}^{\varphi_1}\|_{\varphi_1}$ are comparable with each other. Now notice that $Id - T_{\varphi_1} T_{\overline{\varphi}_1} = (Id - T_{\varphi_1} T_{\overline{\varphi}_1})_{|\mathcal{H}(b)} (Id - T_b T_{\overline{b}})$. Indeed, we have, for all $f \in H^2$

$$(Id - T_{\varphi_1} T_{\overline{\varphi}_1})_{|\mathcal{H}(b)} (Id - T_b T_{\overline{b}}) f = (Id - T_{\varphi_1} T_{\overline{\varphi}_1}) f - T_b T_{\overline{b}} f + T_{\varphi_1} T_{\overline{\varphi}_1} T_b T_{\overline{b}} f,$$

and

$$\begin{aligned}T_{\varphi_1}T_{\overline{\varphi}_1}T_bT_{\overline{b}}f =&\varphi_1P_+\overline{\varphi}_1P_+bP_+\overline{b}f\\ =&\varphi_1P_+\overline{\varphi}_1bP_+\overline{b}f\\ =&\begin{cases}\varphi_1P_+\overline{S_0^{1/2}B_0}S_0B_0P_+\overline{b}f & \text{if } b_0\equiv \text{cte}\\ \varphi_1P_+\overline{u_0}b_0u_0P_+\overline{b}f & \text{if } b_0\not\equiv \text{cte}\end{cases}\\ =&\begin{cases}\varphi_1S_0^{1/2}P_+\overline{b}f & \text{if } b_0\equiv \text{cte}\\ \varphi_1b_0P_+\overline{b}f & \text{if } b_0\not\equiv \text{cte}\end{cases}\\ =&T_bT_{\overline{b}}f.\end{aligned}$$

Hence $Id-T_{\varphi_1}T_{\overline{\varphi}_1}=(Id-T_{\varphi_1}T_{\overline{\varphi}_1})_{|\mathcal{H}(b)}(Id-T_bT_{\overline{b}})$. This implies that

$$k_{\lambda_n}^{\varphi_1}=(Id-T_{\varphi_1}T_{\overline{\varphi}_1})k_{\lambda_n}=(Id-T_{\varphi_1}T_{\overline{\varphi}_1})(Id-T_bT_{\overline{b}})k_{\lambda_n}=(Id-T_{\varphi_1}T_{\overline{\varphi}_1})_{|\mathcal{H}(b)}k_{\lambda_n}^b.$$

Therefore, $(Id-T_{\varphi_1}T_{\overline{\varphi}_1})_{|\mathcal{H}(b)}$ is an isomorphism from $\text{Span}(k_{\lambda_n}^b : n\geqslant 1)$ onto $\text{Span}(k_{\lambda_n}^{\varphi_1} : n\geqslant 1)\subset\mathcal{H}(\varphi_1)$. Using Lemma 3.3, we get

$$\dim(\mathcal{H}(b)\ominus\text{Span}(k_{\lambda_n}^b : n\geqslant N))\geqslant\dim\text{Ker}\,(Id-T_{\varphi_1}T_{\overline{\varphi}_1})_{|\mathcal{H}(b)}=\dim(\varphi_1\mathcal{H}(\varphi_2)).$$

But $\varphi_2\not\equiv$ const and thus

$$\dim(\mathcal{H}(b)\ominus\text{Span}(k_{\lambda_n}^b : n\geqslant N))=+\infty.$$

Applying repeatedly Lemma 3.2, it follows that $(k_{\lambda_n}^b)_{n\geqslant 1}$ is an unconditionnal basis in its closed linear span, which has infinite codimension. □

Theorem 3.2 gives also a criterion for a sequence $(k_{\lambda_n})_{n\geqslant 1}$ to be an unconditionnal basis in the closed subspace of $H^2(\mu)$ it generates.

Theorem 3.4 *Let $(\lambda_n)_{n\geqslant 1}\subset\mathbb{D}$ and let μ be a positive Borel measure on $\mathbb{T}$. Let $b\in H^\infty$, $\|b\|_\infty\leqslant 1$ such that*

$$\frac{1-|b(z)|^2}{|1-b(z)|^2}=\int_{\mathbb{T}}\frac{1-|z|^2}{|1-ze^{-i\theta}|^2}\,d\mu(e^{i\theta}).$$

Assume that

$$\sup_{n\geqslant 1}|b(\lambda_n)|<1.$$

The following statements are equivalent:

(i) $(k_{\lambda_n})_{n\geqslant 1}$ *is an unconditionnal basis in the closed subspace of $H^2(\mu)$ it generates;*

(ii) $(k_{\lambda_n}^b)_{n\geqslant 1}$ *is an unconditionnal basis in its closed linear span;*

(iii) a) $(\lambda_n)_{n\geqslant 1}\in(C)$ b) *$dist(\overline{B}b,H^\infty)<1$.*

Proof: the equivalence of (ii) and (iii) follows from Theorem 3.2. To show that (i) $\iff$ (ii), consider the linear map $V_b : L^2(\mu) \to \mathrm{Hol}(\mathbb{D})$ defined by $V_b q(z) = (1 - b(z))K_\mu q(z)$, $q \in L^2(\mu)$, $z \in \mathbb{D}$. We know that V_b is an isometry from $H^2(\mu)$ onto $\mathcal{H}(b)$ and $V_b k_{\lambda_n} = (1 - \overline{b(\lambda_n)})^{-1} k^b_{\lambda_n}$ (see [19], (III-7)). Hence

$$V_b\left(\frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_{L^2(\mu)}}\right) = (1 - \overline{b(\lambda_n)})^{-1} \frac{\|k^b_{\lambda_n}\|_b}{\|k_{\lambda_n}\|_{L^2(\mu)}} x^b_n = \alpha_n x^b_n,$$

with $\alpha_n = (1 - \overline{b(\lambda_n)})^{-1} \dfrac{\|k^b_{\lambda_n}\|_b}{\|k_{\lambda_n}\|_{L^2(\mu)}}$. Notice that $|\alpha_n| = 1$, $n \geqslant 1$ and it follows that $(k_{\lambda_n})_{n \geqslant 1}$ is an unconditionnal basis in the closed subspace of $H^2(\mu)$ it generates if and only if $(\alpha_n x^b_n)_{n\geqslant 1}$ is a Riesz basis in its closed linear span, which is equivalent to $(k^b_{\lambda_n})_{n \geqslant 1}$ is an unconditionnal basis in its closed linear span. □

4 The extreme case

In this section, we want to caracterize sequences $(k^b_{\lambda_n})_{n\geqslant 1}$ which form an unconditionnal basis of $\mathcal{H}(b)$. So thanks to Theorem 3.1, this problem can be reduced to the fact that $Id - T_b T_{\overline{b}}$ is an isomorphism of $\mathcal{H}(B)$ onto $\mathcal{H}(b)$. Recall that in the classical case where b is inner, thanks to formula (6), we can reformulate this property in terms of the invertibility of $T_{\overline{B}b}$ and then get a criterion in terms of $\mathrm{dist}(\overline{B}b, H^\infty)$ and $\mathrm{dist}(\overline{b}B, H^\infty)$. In the general case, formula (6) is no longer true but nevertheless we can obtain a similar criterion. First, we will give two lemmas.

Lemma 4.1 *Let $b \in H^\infty$ and $\lambda \in \mathbb{D}$. Then we have*

$$Id - T_{\overline{z}b} T_{z\overline{b}} = Id - T_b T_{\overline{b}} - S^* b \otimes S^* b \tag{8}$$

and

$$(Id - T_b T_{\overline{b}})(Id - \lambda S^*) = (Id - \lambda S^*)(Id - T_b T_{\overline{b}}) - \lambda S^* b \otimes b. \tag{9}$$

Proof: notice that $T_{\overline{z}b} = S^* T_b$; hence we have $Id - T_{\overline{z}b} T_{z\overline{b}} = Id - S^* T_b T_{\overline{b}} S$. But $SS^* = Id - \mathbb{1} \otimes \mathbb{1}$, which implies that

$$\begin{aligned} Id - T_{\overline{z}b} T_{z\overline{b}} =& Id - S^* T_b (SS^* + \mathbb{1} \otimes \mathbb{1}) T_{\overline{b}} S \\ =& Id - T_b T_{\overline{b}} - S^* b \otimes S^* b. \end{aligned}$$

For the formula (9), write

$$\begin{aligned} (Id - T_b T_{\overline{b}})(Id - \lambda S^*) - (Id - \lambda S^*)(Id - T_b T_{\overline{b}}) =& \lambda (S^* T_b S T_{\overline{b}} S^* - S^* T_b T_{\overline{b}}) \\ =& \lambda S^* T_b (S T_{\overline{b}} S^* - T_{\overline{b}}) \\ =& \lambda S^* T_b (SS^* - Id) T_{\overline{b}} \\ =& -\lambda S^* b \otimes b. \end{aligned}$$

□

Lemma 4.2 *Let b be an extreme point of the unit ball of H^∞. Then*

$$\text{Span}\left(\frac{b-b(\lambda)}{z-\lambda} : \lambda \in \mathbb{D}\right) = \mathcal{H}(b).$$

Proof: let $f \in \mathcal{H}(b) \ominus \text{Span}\left(\frac{b-b(\lambda)}{z-\lambda} : \lambda \in \mathbb{D}\right)$. Using the equality

$$\frac{b-b(\lambda)}{z-\lambda} = (1-\lambda S^*)^{-1}S^*b = \sum_{n\geqslant 0} \lambda^n S^{*n+1}b,$$

we get $\langle f, S^{*n+1}b\rangle_b = 0$, $\forall n \geqslant 0$. It follows from the relation between $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$ (see [19], (II-4)), that

$$\langle f, S^{*n+1}b\rangle_b = \langle f, S^{*n+1}\rangle_2 + \langle T_{\overline{b}}f, T_{\overline{b}}S^{*n+1}b\rangle_{\overline{b}} = \langle f\overline{b}, \overline{z}^{n+1}\rangle_2 + \langle T_{\overline{b}}f, T_{\overline{b}}S^{*n+1}b\rangle_{\overline{b}}.$$

Now recall that if ρ is the function $1-|b|^2$ on $\mathbb{T}$, then the operator $K_\rho : H^2(\rho) \to \mathcal{H}(\overline{b})$ defined by $K_\rho g := P_+(g\rho)$ is an isometry of $H^2(\rho)$ onto $\mathcal{H}(\overline{b})$. Moreover we have $K_\rho J_\rho = T_\rho$, where J_ρ is the canonical injection from H^2 into $L^2(\rho)$ and

$$K_\rho Z_\rho^* = S^* K_\rho,$$

(see [19], (III-2) and (III-3)). Since $f \in \mathcal{H}(b)$, $T_{\overline{b}}f \in \mathcal{H}(\overline{b})$ and there exists $g \in H^2(\rho)$ such that $T_{\overline{b}}f = K_\rho g = P_+(g\rho)$. Moreover notice that $T_{\overline{b}}S^*b = S^*T_{\overline{b}}b = S^*(\mathbb{1} - (Id - T_{\overline{b}}T_b)\mathbb{1}) = -S^*T_\rho\mathbb{1} = -K_\rho Z_\rho^*\mathbb{1}$ and by induction

$$T_{\overline{b}}S^{*n+1} = -K_\rho Z_\rho^{*n+1}\mathbb{1}.$$

It follows that

$$\begin{aligned} \langle T_{\overline{b}}f, T_{\overline{b}}S^{*n+1}b\rangle_{\overline{b}} &= -\langle K_\rho g, K_\rho Z_\rho^{*n+1}\rangle_{\overline{b}} \\ &= -\langle g, Z_\rho^{*n+1}\mathbb{1}\rangle_\rho \\ &= -\langle Z_\rho^{n+1}g, \mathbb{1}\rangle_\rho \\ &= -\langle \rho g, \overline{z}^{n+1}\rangle_2. \end{aligned}$$

Finally, we get

$$\langle f, S^{*n+1}b\rangle_b = \langle f\overline{b}, \overline{z}^{n+1}\rangle_2 - \langle \rho g, \overline{z}^{n+1}\rangle_2,$$

which implies that $\langle f\overline{b} - \rho g, \overline{z}^{n+1}\rangle_2 = 0$, $\forall n \geqslant 0$. That means that $f\overline{b} - \rho g \in H^2$. But since $T_{\overline{b}}f = P_+(\rho g)$, we also have $f\overline{b} - \rho g \in H^2_-$, and thus $f\overline{b} = \rho g$. Notice now that $|\rho g|$ is not log-integrable. Indeed, we have

$$\log|\rho g| \leqslant \log^+|\rho g|^{1/2} + \frac{1}{2}\log\rho,$$

and the first term on the right side is integrable, whereas the second term has integral $-\infty$ because b is extreme. That implies that $\log|fb| = \log|\rho g| \notin L^1$. But $fb \in H^2$, thus $fb \equiv 0$, that is $f \equiv 0$, which ends the proof. □

Theorem 4.1 *Let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$, and let u be an inner function. Assume that b is an extreme point of the unit ball of H^∞ and that $(Id - T_bT_{\overline{b}})_{|\mathcal{H}(u)}$ is left invertible. Then the following statements are equivalent:*

(i) $Id - T_bT_{\overline{b}}$ *is an isomorphism of* $\mathcal{H}(u)$ *onto* $\mathcal{H}(b)$*;*

(ii) $\mathrm{Ker}\,(Id - T_bT_{\overline{b}})_{|\mathcal{H}(zu)} \neq \{0\}$*;*

(iii) $(Id - T_bT_{\overline{b}})_{|\mathcal{H}(zu)}$ *is not left invertible;*

(iv) $S^*b \in (Id - T_bT_{\overline{b}})\mathcal{H}(u)$*;*

(v) $dist(\overline{zu}b, H^\infty) = 1$*;*

Proof: notice that (ii) $\Longrightarrow$ (iii) is trivial and (iii) $\Longleftrightarrow$ (v) follows from Lemma 3.1.
(i) $\Longrightarrow$ (ii): there exists $f \in \mathcal{H}(u)$ such that $(Id - T_bT_{\overline{b}})u = (Id - T_bT_{\overline{b}})f$. Define $g := f - u$. It is easy to see that $g \in \mathcal{H}(zu) = H^2 \ominus zuH^2$ and thus $g \in \mathrm{Ker}((Id - T_bT_{\overline{b}})_{|\mathcal{H}(zu)})$. Moreover, $g \not\equiv 0$ (because otherwise $u = f \in \mathcal{H}(u)$), which proves that $(Id - T_bT_{\overline{b}})_{|\mathcal{H}(zu)}$ is not injective.
(iii) $\Longrightarrow$ (iv): using the fact that ${S^*}_{|\mathcal{H}(b)}$ is a contraction, we get

$$\inf_{\substack{f \in \mathcal{H}(zu) \\ \|f\|_2 = 1}} \|S^*(Id - T_bT_{\overline{b}})f\|_b = 0.$$

Writing now $f = SS^*f + f(0)$, we have

$$S^*(Id - T_bT_{\overline{b}})f = S^*(Id - T_bT_{\overline{b}})SS^*f + f(0)S^*k_0^b.$$

But $S^*(Id - T_bT_{\overline{b}})S = Id - T_{\overline{z}b}T_{z\overline{b}}$ and $S^*k_0^b = S^*(1 - \overline{b(0)}b) = -\overline{b(0)}S^*b$, which gives

$$S^*(Id - T_bT_{\overline{b}})f = (Id - T_{\overline{z}b}T_{z\overline{b}})S^*f - f(0)\overline{b(0)}S^*b.$$

Now it follows from (8) that

$$S^*(Id - T_bT_{\overline{b}})f = (Id - T_bT_{\overline{b}})S^*f - \left(\langle S^*f, S^*b\rangle_2 + f(0)\overline{b(0)}\right) S^*b,$$

which implies

$$\inf_{\substack{f \in \mathcal{H}(zu) \\ \|f\|_2 = 1}} \left\|(Id - T_bT_{\overline{b}})S^*f - \left(\langle S^*f, S^*b\rangle_2 + f(0)\overline{b(0)}\right) S^*b\right\|_b = 0.$$

Thus there exists a sequence $(f_n)_{n\geqslant 1} \subset \mathcal{H}(zu)$, $\|f_n\|_2 = 1$, such that

$$\lim_{n\to+\infty}((Id - T_bT_{\overline{b}})S^*f_n - \left(\langle S^*f_n, S^*b\rangle_2 + f_n(0)\overline{b(0)}\right) S^*b) = 0.$$

Notice that the sequence of complex numbers $a_n := \langle S^* f_n, S^* b\rangle_2 + f_n(0)\overline{b(0)}$ is bounded. Hence we can find a subsequence $(a_{n_p})_{p\geqslant 1}$ which converges, say to c. So we have $\lim_{p\to+\infty}(Id - T_b T_{\overline{b}})S^* f_{n_p} = cS^* b$. Since $f_{n_p} \in \mathcal{H}(zu)$, we have $S^* f_{n_p} \in \mathcal{H}(u)$ and thus $cS^* b \in \overline{(Id - T_b T_{\overline{b}})\mathcal{H}(u)}$. Using the fact that $(Id - T_b T_{\overline{b}})_{|\mathcal{H}(u)}$ is left invertible, we get that $cS^* b \in (Id - T_b T_{\overline{b}})\mathcal{H}(u)$. Moreover, we have

$$\|S^* f_{n_p}\|_2^2 = \|f_{n_p}\|_2^2 - |f_{n_p}(0)|^2 = 1 - |f_{n_p}(0)|^2. \tag{10}$$

<u>First case:</u> $\delta := \sup_{p\geqslant 1} |f_{n_p}(0)| < 1$. Using the left invertibility of $(Id - T_b T_{\overline{b}})_{|\mathcal{H}(u)}$, there exists $k > 0$ such that

$$\|(Id - T_b T_{\overline{b}})f\|_b \geqslant k\|f\|_2 \qquad \forall f \in \mathcal{H}(u).$$

It now follows, using (10) that

$$|c|^2 \|S^* b\|_b^2 = \lim_{p\to+\infty} \|(Id - T_b T_{\overline{b}})S^* f_{n_p}\|_b^2 \geqslant k^2 \limsup_{p\to+\infty} \|S^* f_{n_p}\|_2^2 \geqslant k^2(1-\delta^2) > 0,$$

which implies that $c \neq 0$ and thus $S^* b \in (Id - T_b T_{\overline{b}})\mathcal{H}(u)$.
<u>Second case:</u> $\sup_{p\geqslant 1} |f_{n_p}(0)| = 1$ and $b(0) \neq 0$. We can assume that the sequence $(f_{n_p}(0))_{p\geqslant 1}$ is convergent, say to λ. Since $|\lambda| = 1$, we have, using (10) $\lim_{p\to+\infty} \|S^* f_{n_p}\|_2 = 0$, which implies, in particular that

$$\lim_{p\to+\infty} \langle S^* f_{n_p}, S^* b\rangle_2 = 0.$$

It now follows that $\lim_{p\to+\infty} a_{n_p} = \lambda\overline{b(0)}$. Thus $c = \lambda\overline{b(0)} \neq 0$ and $S^* b \in (Id - T_b T_{\overline{b}})\mathcal{H}(u)$.
<u>Third case:</u> $b(0) = 0$. Then $b_1 := \overline{z}b \in H^\infty$ and applying Lemma 3.1, we get that $(Id - T_{b_1} T_{\overline{b_1}})_{|\mathcal{H}(u)}$ is not left invertible. Hence

$$\inf_{\substack{f\in\mathcal{H}(u)\\ \|f\|_2=1}} \|(Id - T_{b_1} T_{\overline{b_1}})f\|_{b_1} = 0.$$

But $\mathcal{H}(b_1) \subset \mathcal{H}(b)$, and closed graph Theorem gives

$$\inf_{\substack{f\in\mathcal{H}(u)\\ \|f\|_2=1}} \|(Id - T_{b_1} T_{\overline{b_1}})f\|_b = 0.$$

Using now (8), we have

$$\inf_{\substack{f\in\mathcal{H}(u)\\ \|f\|_2=1}} \left\|(Id - T_b T_{\overline{b}})f - \langle f, S^* b\rangle_2 S^* b\right\|_b = 0.$$

Since $(Id - T_b T_{\overline{b}})_{|\mathcal{H}(u)}$ is left invertible, we get as above that $S^* b \in (Id - T_b T_{\overline{b}})\mathcal{H}(u)$.

(iv) $\Longrightarrow$ (i): let $\lambda \in \mathbb{D}$ and $f \in \mathcal{H}(u)$ such that $S^*b = (Id - T_bT_{\overline{b}})f$. Then we have

$$\frac{b-b(\lambda)}{z-\lambda} = (Id-\lambda S^*)^{-1}S^*b = (Id-\lambda S^*)^{-1}(Id-T_bT_{\overline{b}})f.$$

But thanks to (9), we have

$$(Id-\lambda S^*)^{-1}(Id-T_bT_{\overline{b}}) = (Id-T_bT_{\overline{b}})(Id-\lambda S^*)^{-1} - \lambda(Id-\lambda S^*)^{-1}S^*b \otimes (Id-\overline{\lambda}S)^{-1}b,$$

which gives

$$\begin{aligned}\frac{b-b(\lambda)}{z-\lambda} =& (Id-T_bT_{\overline{b}})(Id-\lambda S^*)^{-1}f - \lambda\langle f, (Id-\overline{\lambda}S)^{-1}b\rangle_2 (Id-\lambda S^*)^{-1}S^*b \\ =& (Id-T_bT_{\overline{b}})(Id-\lambda S^*)^{-1}f - \lambda\langle f, (Id-\overline{\lambda}S)^{-1}b\rangle_2 \frac{b-b(\lambda)}{z-\lambda}.\end{aligned}$$

Thus

$$\big(1+\lambda\langle f, (Id-\overline{\lambda}S)^{-1}b\rangle_2\big)\,\frac{b-b(\lambda)}{z-\lambda} = (Id-T_bT_{\overline{b}})(Id-\lambda S^*)^{-1}f.$$

Notice that $(Id-\lambda S^*)^{-1}f \in \mathcal{H}(u)$. Moreover if $c := 1+\lambda\langle f, (Id-\overline{\lambda}S)^{-1}b\rangle_2 = 0$, then $(Id-\lambda S^*)^{-1}f \in \mathcal{H}(u) \cap \mathrm{Ker}(Id-T_bT_{\overline{b}})$, which implies by left invertibility of $(Id-T_bT_{\overline{b}})_{|\mathcal{H}(u)}$ that $f=0$, which is absurd. Thus $c \neq 0$ and

$$\frac{b-b(\lambda)}{z-\lambda} \in (Id-T_bT_{\overline{b}})\mathcal{H}(u).$$

Using Lemma 4.2, we get that $\mathcal{H}(b) = (Id-T_bT_{\overline{b}})\mathcal{H}(u)$, which proves that $Id - T_bT_{\overline{b}}$ is an isomorphism of $\mathcal{H}(u)$ onto $\mathcal{H}(b)$. □

We can now give our criterion for unconditionnal basis in $\mathcal{H}(b)$.

Theorem 4.2 *Let $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Assume that b is an extreme point of the unit ball of H^∞ and*

$$\sup_{n\geqslant 1} |b(\lambda_n)| < 1.$$

Then the following statements are equivalent:

(i) *$(k_{\lambda_n})_{n\geqslant 1}$ is an unconditionnal basis of $\mathcal{H}(b)$;*

(ii) a) $(\lambda_n)_{n\geqslant 1} \in (C)$. b) $dist(\overline{B}b, H^\infty) < 1$. c) $dist(\overline{zB}b, H^\infty) = 1$.

Proof: it suffices to combine Theorem 3.1, Lemma 3.1 and Theorem 4.1. □

To finish this section, we would like to give a generalization of Theorem 9 in [11], which underlines the link between spectral properties of the model operator and geometric properties of reproducing kernels.

First of all, recall that when b is extreme then

$$\sigma_p(X) = \{\overline{\lambda} \in \mathbb{D} : b(\lambda) = 0\}, \quad \mathrm{Ker}(X - \overline{\lambda}) = \mathbb{C}k_\lambda,$$

and

$$\sigma_p(X^*) = \{\lambda \in \mathbb{D} : b(\lambda) = 0\}, \quad \mathrm{Ker}(X^* - \lambda) = \mathbb{C}\frac{b}{z-\lambda},$$

(see [19] for the result for X and Proposition 2.1 for X^*).

Assume that b has an infinite sequence $(\lambda_n)_{n\geqslant 1}$ of zeros and let B be the Blaschke product associated to $(\lambda_n)_{n\geqslant 1}$ and let $b_1 = \overline{B}b$. Then the following result gives a criterion for the sequence of eigenvectors of X and X^* forms an unconditionnal basis of $\mathcal{H}(b)$.

Theorem 4.3 *With the previous notations, the following statements are equivalent:*

(i) $(k_{\lambda_n})_{n\geqslant 1} \cup \left(\frac{b}{z-\lambda_n}\right)_{n\geqslant 1}$ *forms an unconditionnal basis of* $\mathcal{H}(b)$*;*

(ii) $\sup_{n\geqslant 1} |b_1(\lambda_n)| < 1$ *and* $(k^{b_1}_{\lambda_n})_{n\geqslant 1}$ *forms an an unconditionnal basis of* $\mathcal{H}(b_1)$*;*

(iii) $(\lambda_n)_{n\geqslant 1} \in (C)$*,* $\mathit{dist}(\overline{B}b_1, H^\infty) < 1$*,* $\mathit{dist}(\overline{zB}b_1, H^\infty) = 1$*.*

Proof: (ii) $\Longleftrightarrow$ (iii): notice that if b is extreme then b_1 is also extreme. Moreover if $\mathrm{dist}(\overline{B}b_1, H^\infty) < 1$, then there exists $h \in H^\infty$ such that $\|b_1 - Bh\|_\infty < 1$, and we have

$$\sup_{n\geqslant 1} |b_1(\lambda_n)| = \sup_{n\geqslant 1} |(b_1 - Bh)(\lambda_n)| \leqslant \|b_1 - Bh\|_\infty < 1.$$

Now it suffices to apply Theorem 4.2.
For (i) $\Longleftrightarrow$ (ii), we will need the following lemmas.

Lemma 4.3 *With the previous notations, we have*

a) $\mathcal{H}(b) = \mathcal{H}(B) \oplus^\perp B\mathcal{H}(b_1)$.

b) $\mathcal{H}(b) = \mathcal{H}(b_1) \oplus^\perp b_1\mathcal{H}(B)$.

Moreover, T_B *(resp.* T_{b_1}*) acts as an isometry of* $\mathcal{H}(b_1)$ *(resp. of* $\mathcal{H}(B)$*) into* $\mathcal{H}(b)$*.*

Lemma 4.4 *With the previous notations, the sequence* $(k^{b_1}_{\lambda_n})_{n\geqslant 1}$ *is complete in* $\mathcal{H}(b_1)$ *if and only if*

$$\mathcal{H}(b) = \overline{\mathcal{H}(B) + b_1\mathcal{H}(B)}^{\mathcal{H}(b)}.$$

Lemma 4.5 *Let H be an Hilbert space and X, Y be two closed subspaces of H. Assume that $(x_n)_{n\geqslant 1}$ (resp. $(y_n)_{n\geqslant 1}$) is an unconditionnal basis of X (resp. Y). Then $(x_n)_{n\geqslant 1} \cup (y_n)_{n\geqslant 1}$ is an unconditionnal basis of H if and only if*

$$\overline{X+Y} = H \quad \textit{and} \quad \langle X, Y\rangle > 0.$$

(i) $\Longrightarrow$ (ii): recall (see [14], Lect. IV) that

$$\mathrm{Span}(k_{\lambda_n} : n \geqslant 1) = \mathcal{H}(B) = \mathrm{Span}\left(\frac{B}{z-\lambda_n} : n \geqslant 1\right).$$

Then it follows from Lemma 4.3 that

$$\mathrm{Span}\left(\frac{b}{z-\lambda_n} : n \geqslant 1\right) = b_1\mathcal{H}(B).$$

Now Lemma 4.5 implies that $\langle \mathcal{H}(B), b_1\mathcal{H}(B)\rangle > 0$ and $\mathcal{H}(b) = \overline{\mathcal{H}(B) + b_1\mathcal{H}(B)}$. Thus, using Lemma 4.4, we get that $(k_{\lambda_n}^{b_1})_{n\geqslant 1}$ is complete in $\mathcal{H}(b_1)$. Since $(k_{\lambda_n})_{n\geqslant 1}$ is an unconditionnal basis of $\mathcal{H}(B)$, it remains, thanks to Theorem 3.1, to show that $(Id - T_{b_1}T_{\overline{b}_1})_{|\mathcal{H}(B)}$ is an isomorphism onto its range. But it follows from Lemma 4.3 that $\mathcal{H}(b_1)$ is a closed subspace of $\mathcal{H}(b)$ and then we can consider $P_{\mathcal{H}(b_1)}$ the orthogonal projection of $\mathcal{H}(b)$ onto $\mathcal{H}(b_1)$. Now notice that

$$k_{\lambda_n} = k_{\lambda_n}^b = \frac{1-\overline{b_1(\lambda_n)}b_1}{1-\overline{\lambda_n}z} + \overline{b_1(\lambda_n)}\frac{b_1}{1-\overline{\lambda_n}z} = (Id - T_{b_1}T_{\overline{b}_1})k_{\lambda_n} + \overline{b_1(\lambda_n)}b_1 k_{\lambda_n},$$

which implies, using the fact that $b_1\mathcal{H}(B) = (\mathcal{H}(b_1))^{\perp}$,

$$P_{\mathcal{H}(b_1)}k_{\lambda_n} = (Id - T_{b_1}T_{\overline{b}_1})k_{\lambda_n}.$$

Consequently we have $P_{\mathcal{H}(b_1)|\mathcal{H}(B)} = (Id - T_{b_1}T_{\overline{b}_1})_{|\mathcal{H}(B)}$. Since $\langle \mathcal{H}(B), (\mathcal{H}(b_1))^{\perp}\rangle > 0$, it follows, from [14], (Lemma on Close Subspaces, Lect. VIII, p. 201), that $P_{\mathcal{H}(b_1)}$, and thus $Id - T_{b_1}T_{\overline{b}_1}$, is an isomorphism of $\mathcal{H}(B)$ onto its range.
(ii) $\Longrightarrow$ (i): using Proposition 3.1, we have $(\lambda_n)_{n\geqslant 1} \in (C)$. It follows that $(k_{\lambda_n})_{n\geqslant 1}$ and $\left(\frac{B}{z-\lambda_n}\right)_{n\geqslant 1}$ form an unconditionnal basis of $\mathcal{H}(B)$ (see [14], Lecture VI). Thanks to Lemma 4.3, we get that $\left(\frac{b}{z-\lambda_n}\right)_{n\geqslant 1}$ forms an unconditionnal basis of $b_1\mathcal{H}(B)$. Using Lemma 4.5, it remains to show that

$$\langle \mathcal{H}(B), b_1\mathcal{H}(B)\rangle > 0 \quad \text{and} \quad \mathcal{H}(b) = \overline{\mathcal{H}(B) + b_1\mathcal{H}(B)}.$$

But we know that $(Id - T_{b_1}T_{\overline{b}_1})_{|\mathcal{H}(B)}$ is an isomorphism onto its range, which implies that $P_{\mathcal{H}(b_1)|\mathcal{H}(B)}$ is also an isomorphism onto its range. Now using once more Lemma on close subspaces from [14], we get that

$$\langle \mathcal{H}(B), \mathcal{H}(b_1)^{\perp}\rangle > 0 \quad \text{and} \quad \mathcal{H}(b) = \overline{\mathcal{H}(B) + \mathcal{H}(b_1)^{\perp}},.$$

which ends the proof because $\mathcal{H}(b_1)^\perp = b_1\mathcal{H}(B)$. □

Proof of Lemma 4.3: a): follows from [19] (see (II-6)).
b): let $A := T_b$, $A_1 := T_{b_1}$ and $A_2 := T_B$. Using [19] (I-10), we have

$$\mathcal{H}(b) = \mathcal{H}(A) = \mathcal{H}(A_1) + A_2\mathcal{H}(A_1) = \mathcal{H}(b_1) + b\mathcal{H}(B).$$

Moreover, we have

$$\mathcal{H}(b) = \mathcal{H}(b_1) \oplus^\perp b_1\mathcal{H}(B) \Longleftrightarrow \mathcal{H}(b_1) \cap b_1\mathcal{H}(B) = \{0\}.$$

But $\mathcal{H}(b_1) \cap b_1\mathcal{H}(B) \subset \mathcal{H}(b_1) \cap b_1H^2 = \mathcal{H}(b_1) \cap \mathcal{M}(b_1) = T_{b_1}\mathcal{H}(\overline{b}_1)$ (see [19], (II-5)). Now let $f \in \mathcal{H}(b_1) \cap b_1\mathcal{H}(B)$. Then there exists $h \in \mathcal{H}(B)$ and $g \in \mathcal{H}(\overline{b}_1)$ such that $f = b_1 g = b_1 h$. Thus $g = h$. Since $h \in \mathcal{H}(B)$, h is not a cyclic vector of S^* (see [8]). It is known that when b is extreme, the nonzero functions in $\mathcal{H}(\overline{b})$ are cyclic vectors of S^* (see [19], (V-2)). Thus $g \equiv 0$ and $f \equiv 0$. The fact that T_{b_1} acts as an isometry of $\mathcal{H}(B)$ into $\mathcal{H}(b)$ follows from [19], (I-11). □

Proof of Lemma 4.4: recall that if M and N are two closed subspaces of an Hilbert space H, then $H = \overline{M + N^\perp}$ if and only if $M^\perp \cap N = \{0\}$ (see [14], Lemma on Close Subspaces, Lect. VIII, p. 201). Moreover, thanks to Lemma 4.3, if $M = b_1\mathcal{H}(B)$ and $N = B\mathcal{H}(b_1)$, we have $M^\perp = \mathcal{H}(b_1)$ and $N^\perp = \mathcal{H}(B)$. Thus

$$\mathcal{H}(b) = \overline{\mathcal{H}(B) + b_1\mathcal{H}(B)}^{\mathcal{H}(b)} \Longleftrightarrow \mathcal{H}(b_1) \cap B\mathcal{H}(b_1) = \{0\}.$$

On the other hand, if $f \in \mathcal{H}(b_1)$ and $f(\lambda_n) = 0$, $n \geqslant 1$, then $\frac{f}{B} \in \mathcal{H}(b_1)$. Thus

$$\mathcal{H}(b_1) \ominus \mathrm{Span}(k_{\lambda_n}^{b_1} : n \geqslant 1) = \mathcal{H}(b_1) \cap B\mathcal{H}(b_1),$$

which gives the result. □

Proof of Lemma 4.5: it suffices to use the link between the angle and the skew projections (see [15] or [14]).

□

5 The nonextreme case

In this section, we discuss the nonextreme case. As we shall see, contrary to the extreme case, there cannot exist basis of reproducing kernels in $\mathcal{H}(b)$.

First, recall that if $(k_{\lambda_n}^b)_{n\geqslant 1}$ is not complete in $\mathcal{H}(b)$ then it is minimal (see Lemma 3.2). The following result shows that the converse is also true in the nonextreme case. The key point is the fact that $\mathcal{H}(b)$ is invariant under the shift.

Proposition 5.1 *Let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ and $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$. Assume that b is nonextreme. The following statements are equivalent:*

(i) $(k_{\lambda_n})_{n\geqslant 1}$ *is minimal;*

(ii) $(k_{\lambda_n})_{n\geqslant 1}$ *is not complete in* $\mathcal{H}(b)$.

Moreover, in this case, we have:

$$dim(\mathcal{H}(b) \ominus \text{Span}(k^b_{\lambda_n} : n \geqslant 1)) = +\infty.$$

Proof: thanks to Lemma 3.2, it suffices to prove that if $(k_{\lambda_n})_{n\geqslant 1}$ is minimal, then $\dim(\mathcal{H}(b) \ominus \text{Span}(k^b_{\lambda_n} : n \geqslant 1)) = +\infty$. Suppose that $\dim(\mathcal{H}(b) \ominus \text{Span}(k^b_{\lambda_n} : n \geqslant 1)) = N < +\infty$. Then it implies the existence of a sequence of reproducing kernels which is minimal and complete in $\mathcal{H}(b)$. Indeed, we can assume that $N \geqslant 1$. Applying repeatidely Lemma 3.2, we get that if $(\mu_i)_{1\leqslant i\leqslant N} \subset \mathbb{D}$, with $\mu_i \neq \mu_j$, $i \neq j$ and $\mu_i \neq \lambda_n$, then $(k^b_{\lambda_n}, k^b_{\mu_i})_{\substack{n\geqslant 1\\ N\geqslant i\geqslant 1}}$ is minimal and complete in $\mathcal{H}(b)$. In particular, it implies the existence of a function $h \in \mathcal{H}(b)$ such that $h(\lambda_1) = 0$ and $h(\lambda_n) = h(\mu_i) = 0$, $n \geqslant 2$, $1 \leqslant i \leqslant N$. Now consider $f := (z - \lambda_1)h$. Since $S\mathcal{H}(b) \subset \mathcal{H}(b)$ in the nonextreme case (see [19], (IV-5)), we have

$$f \in \mathcal{H}(b) \ominus \text{Span}(k^b_{\lambda_n}, k^b_{\mu_i} : n \geqslant 1, N \geqslant i \geqslant 1).$$

Since $h \not\equiv 0$, we have $f \not\equiv 0$, which contredicts the completeness of $(k^b_{\lambda_n}, k^b_{\mu_i})_{\substack{n\geqslant 1\\ N\geqslant i\geqslant 1}}$.
□

Corollary 5.1 *Let* $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ *and* $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$. *Assume that b is nonextreme and that*

$$\sup_{n\geqslant 1} |b(\lambda_n)| < 1.$$

Then the following statements are equivalent:

(i) $(k^b_{\lambda_n})_{n\geqslant 1}$ *forms an unconditionnal basis in its closed linear span;*

(ii) a) $(\lambda_n)_{n\geqslant 1} \in (C)$, b) $dist(\overline{B}b, H^\infty) < 1$.

Moreover in this case, we have

$$dim(\mathcal{H}(b) \ominus \text{Span}(k^b_{\lambda_n} : n \geqslant 1)) = +\infty.$$

Proof: it suffices to combine Proposition 5.1 and Theorem 3.2. □

We can precise a little more Proposition 5.1 and get a caracterization of completeness (and thus of minimality).

Proposition 5.2 *Let* $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ *and* $(\lambda_n)_{n\geqslant 1} \in \mathbb{D}$. *Assume that b is nonextreme.*

(a) *If b is pseudocontinuable, then the following statements are equivalent:*

(i) *the sequence* $(k^b_{\lambda_n})_{n\geqslant 1}$ *is complete in* $\mathcal{H}(b)$*;*

(ii) $\displaystyle\sum_{n\geqslant 1}(1-|\lambda_n|)=+\infty.$

(b) *If* b *is not pseudocontinuable, then the following statements are equivalent:*

(i) *the sequence* $(k^b_{\lambda_n})_{n\geqslant 1}$ *is complete in* $\mathcal{H}(b)$*;*

(ii) $S^*b\in \mathrm{Span}(k^b_{\lambda_n} : n\geqslant 1)$.

Recall that a function f in H^2 is pseudocontinuable (across $\mathbb{T}$) if there exist functions $f_1, f_2 \in H^\infty$ such that $f=\overline{f_1/f_2}$ a.e. on $\mathbb{T}$. R. Douglas, H. Shapiro and A. Shields show that a function $f\in H^2$ is pseudocontinuable if and only if f is not S^*-cyclic (see [8]).

Proof: (a): assume that b is nonextreme and pseudocontinuable.

(ii) $\Longrightarrow$ (i) : follows from the fact that $\mathcal{H}(b)\subset H^2$.

(i) $\Longrightarrow$ (ii) : assume that $(k^b_{\lambda_n})_{n\geqslant 1}$ is complete in $\mathcal{H}(b)$ and that $(\lambda_n)_{n\geqslant 1}$ is a Blaschke sequence. Denote by B the Blaschke product associated to $(\lambda_n)_{n\geqslant 1}$. Since b is pseudocontinuable, there exists a nonconstant inner function u such that $b\in\mathcal{H}(u)$. Then it follows that $b=\overline{z}\overline{h}u$, where $h\in H^2$. We will show that $k^b_{\lambda_n}\in\mathcal{H}(uB)$, $n\geqslant 1$. For all polymial p,, we have

$$\begin{aligned}\langle k^b_{\lambda_n}, Bup\rangle_2 =&\langle k_{\lambda_n}, Bup\rangle_2-\overline{b(\lambda_n)}\langle bk_{\lambda_n}, Bup\rangle_2\\ =&-\overline{b(\lambda_n)}\langle \overline{z}\overline{h}uk_{\lambda_n}, Bup\rangle_2\\ =&-\overline{b(\lambda_n)}\langle k_{\lambda_n}, zhBp\rangle_2=0.\end{aligned}$$

Hence, using the density of polynomials in H^2, we get that $k^b_{\lambda_n}\in\mathcal{H}(uB)$, $n\geqslant 1$. Thus, we have

$$\mathrm{Span}_{\mathcal{H}(b)}(k^b_{\lambda_n} : n\geqslant 1)\subset\overline{\mathcal{H}(uB)}^{\mathcal{H}(b)}\subset\mathcal{H}(uB),$$

because $\mathcal{H}(b)$ is contained contractively in H^2. Since the sequence $(k^b_{\lambda_n})_{n\geqslant 1}$ is complete in $\mathcal{H}(b)$, we get that $\mathcal{H}(b)\subset\mathcal{H}(uB)$. But since b is nonextreme, the polynomials belong to $\mathcal{H}(b)$ (see [19], (IV-2)) and thus to $\mathcal{H}(uB)$. It follows that $H^2\subset\mathcal{H}(uB)$, which is absurd.

(b): assume that b is nonextreme and not pseudocontinuable.

(i) $\Longrightarrow$ (ii): is trivial.

(ii) $\Longrightarrow$ (i) : using the equality $Xk^b_{\lambda_n}=\overline{\lambda_n}k^b_{\lambda_n}-\overline{b(\lambda_n)}S^*b$ (see [19], (II-9)), we get that $\mathrm{Span}(k^b_{\lambda_n} : n\geqslant 1)$ is invariant under X. But we know that invariant subspaces of the operator X, when b is nonextreme, are just the intersections of $\mathcal{H}(b)$ with the invariant subspaces of S^* (see [18]). Hence there is an inner function u such that

$$\mathrm{Span}(k^b_{\lambda_n} : n\geqslant 1)=\mathcal{H}(b)\cap\mathcal{H}(u).$$

But then the fact that $S^*b \in \mathcal{H}(u)$ implies that $b \in \mathcal{H}(uz)$, which is absurd unless $u \equiv 0$ (because b is not pseudocontinuable). Hence $\mathrm{Span}(k_{\lambda_n}^b : n \geqslant 1) = \mathcal{H}(b)$.
$\square$

Remark 5.1 *For the extreme case, an analogue of this result is far from being known, even in the particular case where* $b(z) = \exp(-a\frac{1+z}{1-z})$, $a > 0$.

If $(k_{\lambda_n})_{n\geqslant 1}$ is a minimal sequence, then it is well-known that there exists a summable method V such that $(k_{\lambda_n})_{n\geqslant 1}$ is a V-basis of $\mathcal{H}(B)$ (see [14], Lect. VIII, p. 194). If we make assumption on multipliers of $\mathcal{H}(b)$, we can give an analogue of this result.

First of all, recall that we say that a function $\varphi \in H^\infty$ is a multiplier of $\mathcal{H}(b)$ if $\mathcal{H}(b)$ is invariant under T_φ. From the closed graph Theorem, it follows that T_φ is a bounded operator of $\mathcal{H}(b)$. We denote in this case, $M_\varphi := T_{\varphi|\mathcal{H}(b)}$.

Many authors study multipliers of $\mathcal{H}(b)$ (see for instance [12], [13] or [5]). In particular, it is proved in [12] that if b is extreme, then $\mathcal{H}(b)$ does not have inner multipliers.

Theorem 5.1 *Let* $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ *and* $(\lambda_n)_{n\geqslant 1} \in \mathbb{D}$. *Assume that b is nonextreme and that B is a multiplier of* $\mathcal{H}(b)$. *Then the sequence* $(k_{\lambda_n}^b)_{n\geqslant 1}$ *is minimal. Moreover, if* $(\varphi_n)_{n\geqslant 1}$ *is the unique biorthogonal of* $(k_{\lambda_n}^b)_{n\geqslant 1}$, *with* $\varphi_n \in \mathrm{Span}(k_{\lambda_n}^b : n \geqslant 1)$, *then for all function* $f \in \mathrm{Span}(k_{\lambda_n}^b : n \geqslant 1)$, *we have*

$$f = \sum_{n\geqslant 1} \overline{B^p(\lambda_n)} \langle f, \varphi_n \rangle_b k_{\lambda_n}^b,$$

where $B^{(p)} = \prod_{n\geqslant p} b_{\lambda_n}$. *In particular, we have*

$$\mathrm{Span}(\varphi_n : n \geqslant 1) = \mathrm{Span}(k_{\lambda_n}^b : n \geqslant 1).$$

Proof: recall that when b is nonextreme, the polynomials belong to $\mathcal{H}(b)$ (see [19], (IV-2)), thus $\mathbb{1} \in \mathcal{H}(b)$. Since B is a multiplier of $\mathcal{H}(b)$, we get that $B \in \mathcal{H}(b)$. It follows that

$$B_n = \frac{B}{b_{\lambda_n}} = P_+(\overline{b_{\lambda_n}} B) = T_{\overline{b_{\lambda_n}}} B \in \mathcal{H}(b),$$

because $\mathcal{H}(b)$ is invariant under $T_{\overline{\varphi}}$, for all $\varphi \in H^\infty$. Moreover, we have

$$\langle \frac{B_n}{B_n(\lambda_n)}, k_{\lambda_p}^b \rangle_b = \frac{B_n(\lambda_p)}{B_n(\lambda_n)} = \delta_{n,p},$$

which implies that $\left(\frac{B_n}{B_n(\lambda_n)}\right)_{n\geqslant 1}$ is a biorthogonal of $(k_{\lambda_n}^b)_{n\geqslant 1}$. Thus the sequence $(k_{\lambda_n}^b)_{n\geqslant 1}$ is minimal.

Now we will show that $B^{(p)}$ is multiplier of $\mathcal{H}(b)$ and that $\|M_{B^{(p)}}\| \leqslant \|M_B\|$, for all $p \geqslant 1$.

First notice that if $\widetilde{B}^{(p)} := \prod_{n<p} b_{\lambda_n}$, then for all $f \in \mathcal{H}(b)$, we have

$$B^{(p)} f = P_+(\overline{\widetilde{B}^{(p)}} B f) = T_{\overline{\widetilde{B}^{(p)}}}(Bf).$$

Since B is a multiplier, we have $Bf \in \mathcal{H}(b)$ and thus $B^{(p)} f \in \mathcal{H}(b)$. Moreover, we have

$$\|B^{(p)} f\|_b \leqslant \|\widetilde{B}^{(p)}\|_\infty \|M_B\| \|f\|_b = \|M_B\| \|f\|_b,$$

because the norm of $T_{\overline{\widetilde{B}^{(p)}}}$ as an operator of $\mathcal{H}(b)$ does not exceed $\|\widetilde{B}^{(p)}\|_\infty$ (see [19], (II-7)). That proves that $B^{(p)}$ is multiplier of $\mathcal{H}(b)$ and $\|M_{B^{(p)}}\| \leqslant \|M_B\|$, for all $p \geqslant 1$. On the other hand, since $B^{(p)}$ is multiplier of $\mathcal{H}(b)$, we have

$$M^*_{B^{(p)}}(k^b_{\lambda_n}) = \overline{B^{(p)}(\lambda_n)} k^b_{\lambda_n},$$

(see [19], (II-10)) and thus $\lim_{p\to+\infty} M^*_{B^{(p)}}(k^b_{\lambda_n}) = k^b_{\lambda_n}$, $n \geqslant 1$. Since $\|M^*_{B^{(p)}}\| = \|M_{B^{(p)}}\| \leqslant \|M_B\|$, Banach-Steinhauss Theorem implies that, for all $f \in \mathrm{Span}(k^b_{\lambda_n} : n \geqslant 1)$, we have $\lim_{p\to+\infty} M^*_{B(p)} f = f$. But now it is easy to see that, for all $f \in \mathrm{Lin}(k^b_{\lambda_n} : n \geqslant 1)$, we have

$$M^*_{B^{(p)}} f = \sum_{n=1}^{p-1} \overline{B^{(p)}(\lambda_n)} \langle f, \varphi_n \rangle_b k^b_{\lambda_n}.$$

By density, we get this equality for all $f \in \mathrm{Span}(k^b_{\lambda_n} : n \geqslant 1)$, which implies, letting p tends to ∞ that

$$f = \sum_{n\geqslant 1} \overline{B^p(\lambda_n)} \langle f, \varphi_n \rangle_b k^b_{\lambda_n}.$$

□

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