# On an extremal problem of Garcia and Ross

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**Abstract**

We show the equivalence of two extremal problems on Hardy spaces, thus answering a question posed by Garcia and Ross. The proof uses a slight generalization of complex symmetric operators.

*Mathematics Subject Classification (2000):* Primary 47A05, 47B35; Secondary 47B99.
*Key words and phrases:* Extremal problem, Hankel operator, complex symmetric operator.

## 1 Introduction

In [4] Garcia and Ross discuss a nonlinear extremal problem for functions in the Hardy space and its relation to a well studied linear extremal problem. Specifically, let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle. For $p > 0$, let $H^p$ denote the classical Hardy space on $\mathbb{D}$ (identified, as usual, with a closed subspace of $L^p = L^p(\mathbb{T})$). For fixed $\psi \in L^\infty$, the following nonlinear extremal problem is considered in [4]:

$$\Gamma(\psi) := \sup_{\substack{f \in H^2 \\ \|f\|_2 = 1}} \left| \frac{1}{2\pi i} \int_{\mathbb{T}} \psi(z) f(z)^2 \, dz \right|. \tag{1}$$

This is closely related to the well known classical linear extremal problem

$$\Lambda(\psi) := \sup_{\substack{F \in H^1 \\ \|F\|_1 = 1}} \left| \frac{1}{2\pi i} \int_{\mathbb{T}} \psi(z) F(z) \, dz \right|; \tag{2}$$

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it is noted in [4] that we always have $\Gamma(\psi) \leq \Lambda(\psi)$, and it is proved that in some particular cases, including the case of rational $\psi$, we have equality. We show in this short note that equality actually holds for all $\psi \in L^\infty$, thus answering an open question stated in [4].

The two problems can be reformulated in terms of operators on a Hilbert space. Denote by $P_+$ the projection in $L^2$ onto $H^2$ and by $P_-$ the projection onto $H^2_- := L^2 \ominus H^2$. The Hankel operator of symbol $\psi$ is $\mathfrak{H}_\psi : H^2 \to H^2_-$, defined by $\mathfrak{H}_\psi f = P_- \psi f$.

By changing the variable $z = e^{it}$ and denoting $\zeta(t) = e^{it}$, we have

$$\Gamma(\psi) = \sup_{\substack{f \in H^2 \\ \|f\|_2 = 1}} \left| \frac{1}{2\pi} \int_0^{2\pi} \zeta(t)\psi(e^{it}) f(e^{it})^2 \, dt \right| = \sup_{\substack{f \in H^2 \\ \|f\|_2 = 1}} |\langle \psi f, \bar{\zeta} \bar{f} \rangle| = \sup_{\substack{f \in H^2 \\ \|f\|_2 = 1}} |\langle \mathfrak{H}_\psi f, \bar{\zeta} \bar{f} \rangle|. \quad (3)$$

On the other hand, any function $F \in H^1$ may be written as $F = fg$ with $f, g \in H^2$ and $\|f\|_2 = \|g\|_2 = \|F\|_1$. Therefore we get

$$\Lambda(\psi) = \sup_{\substack{f, g \in H^2 \\ \|f\|_2 = \|g\|_2 = 1}} \left| \frac{1}{2\pi} \int_0^{2\pi} \zeta(t)\psi(e^{it}) f(e^{it}) g(e^{it}) \, dt \right| = \sup_{\substack{f, g \in H^2 \\ \|f\|_2 = \|g\|_2 = 1}} |\langle \psi f, \bar{\zeta} \bar{g} \rangle| = \|\mathfrak{H}_\psi\|. \quad (4)$$

Both problems (1) and (2) are thus rephrased in terms of Hankel operators. A convenient reference for these, including all results that we shall use below, is [9].

## 2 Complex symmetric operators and their relatives

In [2, 3] the authors introduce the notion of *complex symmetric* operator on a Hilbert space, which has since found several applications; in particular, complex symmetric operators are used in [4] to prove the equivalence, in a particular case, of the two extremal problems. We need an extension of some of these facts to operators acting between two different spaces.

Suppose then that $\mathcal{X}, \mathcal{Y}$ are two Hilbert spaces. Define $\mathfrak{c} : \mathcal{X} \to \mathcal{Y}$ to be an *antiunitary* operator if it is a conjugate linear surjective map which satisfies $\langle \mathfrak{c}x, \mathfrak{c}x' \rangle = \langle x', x \rangle$ for all $x, x' \in \mathcal{X}$. It is then immediate that $\mathfrak{c}^{-1} : \mathcal{Y} \to \mathcal{X}$ is also an antiunitary operator. A *conjugation* is an antiunitary operator which acts on the same space and is equal to its inverse. If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, we say that $T$ is $\mathfrak{c}$-*symmetric* if $T = \mathfrak{c} T^* \mathfrak{c}$. If $T \in \mathcal{L}(\mathcal{X})$ and there exists a conjugation $C$ such that $T$ is $C$-symmetric, then one says that $T$ is *complex symmetric*; this is the class considered in [2, 3].

In order to go from complex symmetric to $\mathfrak{c}$-symmetric operators, the main tool is the following lemma.

**Lemma 2.1.** *If $\mathfrak{c} : \mathcal{X} \to \mathcal{Y}$ is an antiunitary operator, then there exists a unitary operator $V : \mathcal{X} \to \mathcal{Y}$ (not uniquely defined) such that $C = V^* \mathfrak{c}$ is a conjugation on $\mathcal{X}$. If such a $V$ is* *fixed, then the map $T \mapsto V^*T$ is a bijection between $\mathfrak{c}$-symmetric operators and $C$-symmetric operators.*

*Proof.* Take an orthonormal basis $(e_n)$ in $\mathcal{X}$, and define $V$ to be the unitary operator which maps $e_n$ into $\mathfrak{c}e_n$. Then it is easily seen that $C = V^*\mathfrak{c}$ is precisely the conjugation on $\mathcal{X}$ associated with the basis $(e_n)$.

Now, if $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, then

$$\begin{aligned} T = \mathfrak{c}T^*\mathfrak{c} \quad &\Leftrightarrow \quad V^*T = V^*\mathfrak{c}T^*\mathfrak{c} \quad \Leftrightarrow \quad V^*T = V^*\mathfrak{c}T^*VV^*\mathfrak{c} \\ &\Leftrightarrow \quad V^*T = C(V^*T)^*C, \end{aligned}$$

which proves the second part of the lemma. □

As a consequence, we obtain the result that interests us, namely the analogue of Theorem 1 in [4] (which deals with the complex symmetric case).

**Lemma 2.2.** *Suppose $\mathfrak{c} : \mathcal{X} \to \mathcal{Y}$ is an antiunitary operator and $T : \mathcal{X} \to \mathcal{Y}$ is $\mathfrak{c}$-symmetric. Then:*

*(i) $\|T\| = \sup_{\|x\|=1} |\langle Tx, \mathfrak{c}x\rangle|$.*

*(ii) The supremum in (i) is attained if and only if $T$ attains its norm (or, equivalently, if $\|T\|$ is an eigenvalue for $|T|$.) In this case $Tx = \omega\|T\|\mathfrak{c}x$ for some unimodular constant $\omega$.*

*Proof.* Suppose that $V$ is the unitary operator and $C$ is the conjugation given by Lemma 2.1; thus $T' := V^*T$ is $C$-symmetric. Theorem 1 in [4] says then that $\|T'\| = \sup_{\|x\|=1} |\langle T'x, Cx\rangle|$. Since $\|T\| = \|T'\|$ and

$$\sup_{\|x\|=1} |\langle Tx, \mathfrak{c}x\rangle| = \sup_{\|x\|=1} |\langle V^*Tx, V^*\mathfrak{c}x\rangle| = \sup_{\|x\|=1} |\langle T'x, Cx\rangle|,$$

the first assertion is proved.

For the second, it is immediate by Schwarz's inequality that, if $\|x\| = 1$, then $\|T\| = |\langle Tx, \mathfrak{c}x\rangle|$ if and only if $Tx = \omega\|T\|\mathfrak{c}x$ for some unimodular constant $\omega$. But it is a general fact (for any operator $T$) that $T$ attains its norm if and only if $\|T\|$ is an eigenvalue of $|T|$, given that $\|T\| = \||T|\|$ and $\||T|\| = \sup_{\|x\|=1}\langle |T|x, x\rangle$. □

It might be of independent interest to state, as a corollary, the corresponding version of Theorem 2 in [1], characterizing the spectrum of the modulus of a $\mathfrak{c}$-symmetric operator in terms of what Garcia [1] calls an approximate antilinear eigenvalue problem.

**Proposition 2.3.** Let $T$ be a bounded $\mathfrak{c}$-symmetric operator and $\lambda \geq 0$. Then

(i) $\lambda$ belongs to the spectrum of $|T|$ if and only if there exists a sequence of unit vectors $(f_n)_n$ such that $\lim_{n\to\infty} \|(T - \lambda\mathfrak{c})f_n\| = 0$.

(ii) $\lambda$ is a singular value of $T$ if and only if $Tf = \lambda\mathfrak{c}f$ has a nonzero solution $f$.

# 3 Main result

We can now prove the equivalence of the two problems (1) and (2) in the general case.

**Theorem 3.1.** *For any* $\psi \in L^\infty$ *we have* $\Gamma(\psi) = \Lambda(\psi)$.

*Proof.* We intend to apply Lemma 2.2 to the following situation: $\mathcal{X} = H^2$, $\mathcal{Y} = H^2_-$, $T = \mathfrak{H}_\psi$ and $\mathfrak{c} : H^2 \to H^2_-$ defined by $\mathfrak{c}f = \bar{\zeta}\bar{f}$. It is easy to see that $\mathfrak{c}$ is antiunitary. Note that $\mathfrak{c}^{-1} : H^2_- \to H^2$ is given formally by the same formula as $\mathfrak{c}$. To be more accurate, we will define $\mathfrak{C} : L^2 \to L^2$ by $\mathfrak{C}f = \bar{\zeta}\bar{f}$. Then $\mathfrak{c} = \mathfrak{C}|H^2 = P_-\mathfrak{C}|H^2$ and $\mathfrak{c}^{-1} = \mathfrak{C}|H^2_- = P_+\mathfrak{C}|H^2_-$. Moreover, we have $\mathfrak{C}P_+ = P_-\mathfrak{C}$.

Then $\mathfrak{H}_\psi$ is $\mathfrak{c}$-symmetric: $\mathfrak{H}^*_\psi : H^2_- \to H^2$ acts by the formula $\mathfrak{H}^*_\psi g = P_+\bar{\psi}g$, so

$$\begin{aligned}(\mathfrak{c}\mathfrak{H}^*_\psi\mathfrak{c})(f) &= (\mathfrak{c}\mathfrak{H}^*_\psi)(\bar{\zeta}\bar{f}) = \mathfrak{c}(P_+\bar{\psi}\bar{\zeta}\bar{f}) = \mathfrak{C}P_+(\bar{\psi}\bar{\zeta}\bar{f})\\ &= P_-\mathfrak{C}(\bar{\psi}\bar{\zeta}\bar{f}) = P_-(\bar{\zeta}\psi\zeta f) = P_-(\psi f) = \mathfrak{H}_\psi f.\end{aligned}$$

We may apply Lemma 2.2 (i), which gives:

$$\|\mathfrak{H}_\psi\| = \sup_{\|f\|=1} |\langle \mathfrak{H}_\psi f, \mathfrak{c}f\rangle| = \sup_{\|f\|=1} |\langle P_-(\psi f), \bar{\zeta}\bar{f}\rangle|.$$

Since $\mathfrak{c}f = \bar{\zeta}\bar{f} \in H^2_-$, there is no need of $P_-$ in the last scalar product, and therefore, by (3),

$$\|\mathfrak{H}_\psi\| = \sup_{\|f\|=1} |\langle \psi f, \bar{\zeta}\bar{f}\rangle| = \Gamma(\psi).$$

Since $\|\mathfrak{H}_\psi\| = \Lambda(\psi)$ by (4), the theorem is proved. □

Also, from the second part of Lemma 2.2 it follows that the existence of an extremal function (a function that realizes $\Gamma(\psi)$) is equivalent to the fact that the Hankel operator attains its norm. This happens, for instance, if $\mathfrak{H}_\psi$ is compact, which is equivalent, via Hartman's theorem [6], to $\psi \in H^\infty + C(\mathbb{T})$, where $C(\mathbb{T})$ denotes the algebra of continuous functions on $\mathbb{T}$.

Note that in [4] the solution to the extremal problem is related to truncated Toeplitz operators. These are operators on $K_\Theta = H^2 \ominus \Theta H^2$ defined, for $\phi \in H^\infty$, by the formula

$$A^\Theta_\phi(f) = P_\Theta \phi f, \quad f \in K_\Theta,$$

where $P_\Theta$ is the orthogonal projection onto $K_\Theta$. More precisely, it is shown in [4] that, if there is an inner function $\Theta$ such that $\psi\Theta \in H^\infty$, then

$$\Lambda(\psi) = \Gamma(\psi) = \|A^\Theta_{\psi\Theta}\|.$$

The relation with Theorem 3.1 above is made by the following observation. Consider the orthogonal decompositions $H^2 = K_\Theta \oplus \Theta H^2$ and $H^2_- = \bar\Theta K_\Theta \oplus \bar\Theta H^2_-$. With respect to them, the only nonzero entry of the matrix of $\mathfrak{H}_\psi$ is in the upper left corner, and it is equal to $A_{\psi\Theta} : K_\Theta \to K_\Theta$ followed by multiplication with $\bar\Theta$. Consequently, in this case most of the results for the Hankel operators can be translated in terms of the truncated Toeplitz operator. Moreover, this is an *analytic* truncated Toeplitz operator, that is, one whose symbol is in $H^\infty$. Their theory is significantly simpler that in the case of general truncated Toeplitz operators, since we may apply Sarason's interpolation arguments.

# 4 Final remarks

This section has no claim of novelty; its purpose is to put some other results in [4] in a more general context.

**4.1** First, note that it is immediate that $\Gamma(\psi) \le \|\psi\|_\infty$. Obviously $\Gamma(\psi)$ depends only on the antianalytic part of $\psi$. Using the equivalence of (1) and (2), and Nehari's theorem [8], it follows that for each $\psi \in L^\infty$ there exists $\hat\psi$ such that $\psi - \hat\psi \in H^\infty$ and $\|\hat\psi\|_\infty = \Gamma(\psi)$. In the context of truncated Toeplitz operators used in [4], $\hat\psi$ corresponds to what is called therein a *norm attaining symbol.*

**4.2** In case an extremal function exists (equivalently, when the Hankel operator attains its norm) one can say more. With the previous notations, suppose $g \in H^2$ is an extremal function with $\|g\|_2 = 1$; thus $\|\mathfrak{H}_{\hat\psi} g\|_2 = \|\mathfrak{H}_{\hat\psi}\|$. The sequence of inequalities

$$\|\hat\psi\|_\infty = \|\mathfrak{H}_{\hat\psi}\| = \|\mathfrak{H}_{\hat\psi} g\|_2 = \|P_-(\hat\psi g)\|_2 \le \|\hat\psi g\|_2 \le \|\hat\psi\|_\infty \|g\|_2 = \|\hat\psi\|_\infty$$

imply that $\|P_-(\hat\psi g)\|_2 = \|\hat\psi g\|_2 = \|\hat\psi\|_\infty$. It follows then, first that $\hat\psi$ has constant modulus, and secondly that $\hat\psi g \in H^2_-$ and thus $\hat\psi g_o \in H^2_-$, where $g_o$ is the outer part of $g$. Then $\|\mathfrak{H}_{\hat\psi} g_o\|_2 = \|\hat\psi g_o\|_2 = \|\hat\psi\|_\infty = \|\mathfrak{H}_{\hat\psi}\|$. Therefore, in case an extremal function exists, one can chose it outer, and thus not having zeros in $\mathbb{D}$. This recaptures the result of [7] quoted in [4], which says that if the symbol is continuous then there exists an extremal function which is nonzero on $\mathbb{D}$ (as noted above, a Hankel operator with continuous symbol is compact and thus attains its norm).

**4.3** In case there exists an inner function $\Theta$ such that $\phi = \hat{\psi}\Theta \in H^\infty$, then the result can be strengthened. With the above notations, we have then $\bar{\Theta}\phi g_o \in H^2_-$, which implies $\phi g_o \in K_\Theta$; thus $\phi$ is a scalar multiple of the inner part of a function in $K_\Theta$. This is essentially noticed in the remarks after [4, Theorem 2].

**4.4** Finally, let us note that, in case there exists no extremal function, norm attaining symbols might not have constant modulus. An example appears in [5, Ch. IV, Example 4.2]. Namely, suppose $\Theta$ is an inner function that does not extend analytically across the unit circle in the neighborhood of 1, while $f$ is a nonconstant invertible outer function with $\|f\|_\infty = 1$ that has modulus 1 on an arc of $\mathbb{T}$ around 1. Then the only norm attaining symbol for the Hankel operator $\mathfrak{H}_{\bar{\Theta} f}$ is $\bar{\Theta} f$, which has not constant modulus.

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