

# BOUNDARY BEHAVIOR OF FUNCTIONS IN THE DE BRANGES–ROVNYAK SPACES

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ABSTRACT. This paper deals with the boundary behavior of functions in the de Branges–Rovnyak spaces. First, we give a criterion for the existence of radial limits for the derivatives of functions in the de Branges–Rovnyak spaces. This criterion generalizes a result of Ahern–Clark. Then we prove that the continuity of all functions in a de Branges–Rovnyak space on an open arc  $I$  of the boundary is enough to ensure the analyticity of these functions on  $I$ . We use this property in a question related to Bernstein’s inequality.

## 1. INTRODUCTION

For  $0 < p \leq \infty$ , let  $H^p(\mathbb{D})$  denote the classical Hardy space of analytic functions on the unit disc  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ . As usual, we also treat  $H^p(\mathbb{D})$  as a closed subspace of  $L^p(\mathbb{T}, m)$ , where  $\mathbb{T} := \partial\mathbb{D}$  and  $m$  is the normalized arc length measure on  $\mathbb{T}$ . Let  $b$  be in the unit ball of  $H^\infty(\mathbb{D})$ . Then the canonical factorization of  $b$  is  $b = BF$ , where

$$B(z) = \gamma \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n}z}, \quad (z \in \mathbb{D}),$$

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is the Blaschke product with zeros  $a_n \in \mathbb{D}$  satisfying the Blaschke condition  $\sum_n (1 - |a_n|) < +\infty$ ,  $\gamma$  is a constant of modulus one, and  $F$  is of the form

$$F(z) = \exp \left( - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\sigma(\zeta) \right), \quad (z \in \mathbb{D}),$$

where  $d\sigma = -\log |b| dm + d\mu$  and  $d\mu$  is a positive singular measure on  $\mathbb{T}$ . In the definition of  $B$ , we assume that  $|a_n|/a_n = 1$  whenever  $a_n = 0$ . In this paper, we study some aspects of the de Branges–Rovnyak spaces

$$\mathcal{H}(b) := (Id - T_b T_{\bar{b}})^{1/2} H^2.$$

Here  $T_\varphi$  denotes the Toeplitz operator defined on  $H^2$  by  $T_\varphi(f) = P_+(\varphi f)$ , where  $P_+$  is the (Riesz) orthogonal projection of  $L^2(\mathbb{T})$  onto  $H^2$ . In general,  $\mathcal{H}(b)$  is not closed with respect to the norm of  $H^2(\mathbb{D})$ . However, it is a Hilbert space when equipped with the inner product

$$\langle (Id - T_b T_{\bar{b}})^{1/2} f, (Id - T_b T_{\bar{b}})^{1/2} g \rangle_b = \langle f, g \rangle_2,$$

where  $f$  and  $g$  are chosen so that

$$f, g \perp \ker (Id - T_b T_{\bar{b}})^{1/2}.$$

As a very special case, if  $|b| = 1$  a.e. on  $\mathbb{T}$ , or equivalently when  $b$  is an inner function for the unit disc, then  $Id - T_b T_{\bar{b}}$  is an orthogonal projection and the  $\mathcal{H}(b)$  norm coincides with the  $H^2$  norm. In this case,  $\mathcal{H}(b)$  becomes a closed (ordinary) subspace of  $H^2(\mathbb{D})$ , which coincides with the shift-coinvariant subspace  $K_b := H^2 \ominus bH^2$ .

This paper treats two questions related to the boundary behavior of functions in  $\mathcal{H}(b)$ . The first of these concerns the existence of radial limits for the derivatives of functions in the de Branges–Rovnyak spaces. More precisely, given a non-negative integer  $N$ , we are interested in finding a characterization of points  $\zeta_0 \in \mathbb{T}$  such that every function  $f$  in  $\mathcal{H}(b)$  and its derivatives up to order  $N$  have radial limits at  $\zeta_0$ . Ahern and Clark [1] studied this question when  $b$  is an inner function and they got a characterization in terms of the zeros sets  $(a_n)$  and the measure  $\mu$ . In Section 3, we show that their methods in [1, 2]

can be extended in order to obtain similar results for the general de Branges–Rovnyak spaces  $\mathcal{H}(b)$ , where  $b$  is an arbitrary element of the unit ball of  $H^\infty$ . Let us also mention that Sarason [11, page 58] has obtained another criterion in terms of the measure whose Poisson integral is the real part of  $\frac{\lambda + b}{\lambda - b}$ , with  $\lambda \in \mathbb{T}$ . Recently, Bolotnikov and Kheifets [3] gave a result, in some sense more algebraic, in terms of the Schwarz-Pick matrix.

Our second theme is related to the analytic continuation of functions in  $\mathcal{H}(b)$  through a given open arc of  $\mathbb{T}$ . In [7], in the case where  $b$  is an inner function, Helson proved that every function in  $K_b$  has an analytic continuation through an open arc  $I$  of  $\mathbb{T}$  if and only if  $b$  has an analytic continuation through  $I$ . Then, in [11, page 42], Sarason extended this result to the de Branges–Rovnyak spaces  $\mathcal{H}(b)$ , when  $b$  is an extreme point of the unit ball of  $H^\infty$ . In the last section, we study the question of continuity on the open arc  $I$  for functions in  $\mathcal{H}(b)$ . In particular, we show that the continuity on some open arc of the boundary of all functions in  $\mathcal{H}(b)$  implies the analyticity on this arc. We apply this remarkable property to discuss a possible generalization of the Bernstein’s inequality obtained by Dyakonov [5] in the model space  $K_b$ .

## 2. PRELIMINARIES

We first recall some basic well-known facts concerning reproducing kernels in  $\mathcal{H}(b)$ . For any  $\lambda \in \mathbb{D}$ , the linear functional  $f \mapsto f(\lambda)$  is bounded on  $H^2(\mathbb{D})$  and thus, by Riesz’ theorem, it is induced by a unique element  $k_\lambda$  of  $H^2(\mathbb{D})$ . On the other hand, by Cauchy’s formula, we have

$$f(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\vartheta})}{1 - \lambda e^{-i\vartheta}} d\vartheta, \quad (f \in H^2(\mathbb{D}), \lambda \in \mathbb{D}),$$

and thus

$$k_\lambda(z) = \frac{1}{1 - \bar{\lambda}z}, \quad (z \in \mathbb{D}).$$

Now, since  $\mathcal{H}(b)$  is contained contractively in  $H^2(\mathbb{D})$ , the restriction to  $\mathcal{H}(\mathbb{D})$  of the evaluation functional at  $\lambda \in \mathbb{D}$  is a bounded linear functional on  $\mathcal{H}(\mathbb{D})$ . Hence, relative to

the inner product in  $\mathcal{H}(b)$ , it is induced by a vector  $k_\lambda^b$  in  $\mathcal{H}(b)$ . In other words, for all  $f \in \mathcal{H}(b)$ , we have

$$f(\lambda) = \langle f, k_\lambda^b \rangle_b.$$

But if  $f = (Id - T_b T_{\bar{b}})^{1/2} f_1 \in \mathcal{H}(b)$ , we have

$$\langle f, (Id - T_b T_{\bar{b}}) k_\lambda \rangle_b = \langle f_1, (Id - T_b T_{\bar{b}})^{1/2} k_\lambda \rangle_2 = \langle f, k_\lambda \rangle_2 = f(\lambda),$$

which implies that

$$k_\lambda^b = (Id - T_b T_{\bar{b}}) k_\lambda.$$

Finally, using the well known result  $T_{\bar{b}} k_w = \overline{b(w)} k_w$ , we obtain

$$k_\lambda^b(z) = \frac{1 - \overline{b(\lambda)} b(z)}{1 - \bar{\lambda} z}, \quad (z \in \mathbb{D}).$$

We know (see [11, page 11]) that  $\mathcal{H}(b)$  is invariant under the backward shift operator  $S^*$  and, in the following, we use extensively the contraction  $X := S^*|_{\mathcal{H}(b)}$ . Its adjoint satisfies the important formula

$$(2.1) \quad X^* h = S h - \langle h, S^* b \rangle_b b,$$

for all  $h \in \mathcal{H}(b)$  (see [11, page 12]).

We end this section by recalling the definition of the spectrum of a function  $b$  in the unit ball of  $H^\infty(\mathbb{D})$  (see [9, page 103]). A point  $\lambda \in \overline{\mathbb{D}}$  is said to be regular (for  $b$ ) if either  $\lambda \in \mathbb{D}$  and  $b(\lambda) \neq 0$ , or  $\lambda \in \mathbb{T}$  and  $b$  admits an analytic continuation across a neighbourhood  $V_\lambda = \{z : |z - \lambda| < \varepsilon\}$  of  $\lambda$  with  $|b| = 1$  on  $V_\lambda \cap \mathbb{T}$ . The spectrum of  $b$ , denoted by  $\sigma(b)$ , is then defined as the complement in  $\overline{\mathbb{D}}$  of all regular points of  $b$ .

### 3. EXISTENCE OF DERIVATIVES FOR FUNCTIONS OF DE BRANGES–ROVNYAK SPACES

We first begin with a lemma which is essentially due to Ahern-Clark [1, Lemma 2.1].

**Lemma 3.1.** *Let  $S_1, \dots, S_p$  be bounded commuting operators of norm less or equal to 1 on a Hilbert space  $X$ . Let  $(\lambda_1, \dots, \lambda_p) \in \mathbb{T}^p$  such that  $Id - \lambda_j S_j$  is one to one. Furthermore, let  $(\lambda_1^{(n)}, \dots, \lambda_p^{(n)}) \in \mathbb{D}^p$  tends nontangentially to  $(\lambda_1, \dots, \lambda_p)$  as  $n \rightarrow +\infty$ . Then, for any  $y \in X$ , the sequence  $w_n := (Id - \lambda_1^{(n)} S_1)^{-1} \dots (Id - \lambda_p^{(n)} S_p)^{-1} y$  is uniformly bounded if and only if  $y$  belongs to the range of the operator  $(Id - \lambda_1 S_1) \dots (Id - \lambda_p S_p)$ , in which case,  $w_n$  tends weakly to  $w_0 := (Id - \lambda_1 S_1)^{-1} \dots (Id - \lambda_p S_p)^{-1} y$ .*

**Proof:** If  $\|S_j\| < 1$ , then the operator  $Id - \lambda_j S_j$  is invertible and  $(Id - \lambda_j^{(n)} S_j)^{-1}$  tends to  $(Id - \lambda_j S_j)^{-1}$  in operator norm, as  $n \rightarrow +\infty$ . Therefore, we see that we can assume that all operators  $S_j$  are of norm equal to 1. This case is precisely the result of Ahern-Clark. □

The following result gives a criterion for the existence of the derivatives for functions of  $\mathcal{H}(b)$  and it generalizes the Ahern-Clark result.

**Theorem 3.2.** *Let  $b$  be a point in the unit ball of  $H^\infty(\mathbb{D})$  and let*

$$(3.1) \quad b(z) = \gamma \prod_n \left( \frac{|a_n| a_n - z}{a_n (1 - \bar{a}_n z)} \right) \exp \left( - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right) \exp \left( \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log |b(\zeta)| dm(\zeta) \right)$$

*be its canonical factorization. Let  $\zeta_0 \in \mathbb{T}$  and let  $N$  be a non-negative integer. Then the following are equivalent.*

- (i) *for every function  $f \in \mathcal{H}(b)$ ,  $f(z), f'(z), \dots, f^{(N)}(z)$  have finite limits as  $z$  tends radially to  $\zeta_0$ ;*
- (ii) *for every function  $f \in \mathcal{H}(b)$ ,  $|f^{(N)}(z)|$  remains bounded as  $z$  tends radially to  $\zeta_0$ ;*
- (iii)  *$\|\partial^N k_z^b / \partial \bar{z}^N\|_b$  is bounded as  $z$  tends radially to  $\zeta_0$ ;*
- (iv)  *$X^{*N} k_0^b$  belongs to the range of  $(Id - \bar{\zeta}_0 X^*)^{N+1}$ ;*
- (v) *we have*

$$\sum_n \frac{1 - |a_n|^2}{|\zeta_0 - a_n|^{2N+2}} + \int_0^{2\pi} \frac{d\mu(e^{it})}{|\zeta_0 - e^{it}|^{2N+2}} + \int_0^{2\pi} \frac{|\log |b(e^{it})||}{|\zeta_0 - e^{it}|^{2N+2}} dm(e^{it}) < +\infty.$$

**Proof:**

(i)  $\implies$  (ii): it is obvious.

(ii)  $\implies$  (iii): for a point  $z$  in  $\mathbb{D}$ , the function  $\frac{\partial^N k_z^b}{\partial \bar{z}^N}$  is easily seen to be the kernel function in  $\mathcal{H}(b)$  for the functional of evaluation of the  $N$ th derivative at  $z$ :

$$(3.2) \quad f^{(N)}(z) = \langle f, \frac{\partial^N k_z^b}{\partial \bar{z}^N} \rangle_b, \quad \forall f \in \mathcal{H}(b).$$

Therefore, the implication (ii)  $\implies$  (iii) follows from the principle of uniform boundedness.

The equivalence of (i) and (iii) is not new and can be found in [11, page 58].

(iii)  $\implies$  (iv): using the fact that  $k_z^b = (Id - \bar{z}X^*)^{-1}k_0^b$  (see [11, page 42]), we easily get

$$(3.3) \quad \frac{\partial^N k_z^b}{\partial \bar{z}^N} = N!(Id - \bar{z}X^*)^{-(N+1)}X^{*N}k_0^b.$$

We know from [6, Lemma 2.2] that  $\sigma_p(X^*) \subset \mathbb{D}$  and thus the operator  $Id - \bar{\zeta}_0 X^*$  is one-to-one. By assumption,  $(Id - \bar{z}_n X^*)^{-(N+1)}X^{*N}k_0^b$  is uniformly bounded for any sequence  $z_n \in \mathbb{D}$  tending radially to  $\zeta_0$ . Hence, by Lemma 3.1,  $X^{*N}k_0^b$  belongs to the range of  $(Id - \bar{\zeta}_0 X^*)^{N+1}$ .

(iv)  $\implies$  (i): using once more Lemma 3.1 with  $p = N + 1$ ,  $S_1 = \dots = S_p = X^*$ ,  $\lambda_1 = \dots = \lambda_p = \bar{\zeta}_0$  and  $y = X^{*N}k_0^b$ , we see that (iv) implies that  $(Id - \bar{z}_n X^*)^{-(N+1)}X^{*N}k_0^b$  tends weakly to  $(Id - \bar{\zeta}_0 X^*)^{-(N+1)}X^{*N}k_0^b$ , for any sequence  $z_n \in \mathbb{D}$  tending radially to  $\zeta_0$ . Hence (3.2) and (3.3) imply that, for every function  $f$  in  $\mathcal{H}(b)$ ,  $f^{(N)}(z)$  has a finite limit as  $z$  tends radially to  $\zeta_0$ . Now of course, for every  $0 \leq j \leq N$ , (iv) ensures that  $X^{*j}k_0^b$  belongs to the range of  $(Id - \bar{\zeta}_0 X^*)^{j+1}$  and similar arguments show that, for every function  $f$  in  $\mathcal{H}(b)$ ,  $f^{(j)}(z)$  has a finite limit as  $z$  tends radially to  $\zeta_0$ .

(v)  $\implies$  (iii): without loss of generality we assume that  $\zeta_0 = 1$ . Using Leibnitz' rule, by straightforward computations we obtain

$$(3.4) \quad k_{\omega, N}^b(z) := \frac{\partial^N k_\omega^b}{\partial \bar{\omega}^N}(z) = \frac{h_{\omega, N}^b(z)}{(1 - \bar{\omega}z)^{N+1}},$$

with

$$(3.5) \quad h_{\omega, N}^b(z) = N!z^N - b(z) \sum_{j=0}^N \binom{N}{j} \overline{b^{(j)}(\omega)} (N-j)! z^{N-j} (1 - \bar{\omega}z)^j.$$

Hence, by (3.2), we have

$$\left\| \frac{\partial^N k_{\omega}^b}{\partial \bar{\omega}^N} \right\|_b^2 = (k_{\omega, N}^b)^{(N)}(\omega),$$

and thus, we need to prove that  $(k_{r, N}^b)^{(N)}(r)$  is bounded as  $r \rightarrow 1^-$ .

But the condition (v) clearly implies that

$$\sum_n \frac{1 - |a_n|^2}{|\zeta_0 - a_n|^j} + \int_0^{2\pi} \frac{d\mu(e^{it})}{|\zeta_0 - e^{it}|^j} + \int_0^{2\pi} \frac{|\log |b(e^{it})||}{|\zeta_0 - e^{it}|^j} dm(e^{it}) < +\infty,$$

for  $0 \leq j \leq 2N + 2$  and then it follows from [2, Lemma 4] that

$$\lim_{r \rightarrow 1^-} b^{(j)}(r) \quad \text{and} \quad \lim_{R \rightarrow 1^+} b^{(j)}(R)$$

exist and are equal. Here we extend the function  $b$  outside the unit disk by the formula (3.1), which represents an analytic function for  $|z| > 1$ ,  $z \neq 1/\bar{a}_n$ . We denote this function also by  $b$  and it is easily verified that it satisfies

$$(3.6) \quad b(z) = \frac{1}{\overline{b(1/\bar{z})}}, \quad \forall z \in \mathbb{C}.$$

Therefore, there exists  $R_0 > 1$  such that  $b$  has  $2N + 1$  continuous derivatives on  $[0, R_0]$ . Now take  $R_0^{-1} < r < 1$ . Noting that  $b$  can have only a finite number of real zeros, we can assume that the interval  $(R_0^{-1}, 1)$  is free of zeros. Then straightforward computations using (3.5) and (3.6) show that  $h_{r, N}^b$  and its first  $N$  derivatives must vanish at  $z = 1/r$ . Therefore we can write, for  $s \in (0, 1)$ ,

$$\begin{aligned} h_{r, N}^b(s) &= \int_0^1 \frac{d}{dt} h_{r, N}^b \left( \frac{1}{r} + t \left( s - \frac{1}{r} \right) \right) dt \\ &= \left( s - \frac{1}{r} \right) \int_0^1 (h_{r, N}^b)' \left( \frac{1}{r} + t \left( s - \frac{1}{r} \right) \right) dt \\ &= \left( s - \frac{1}{r} \right)^2 \int_0^1 \int_0^1 (h_{r, N}^b)'' \left( \frac{1}{r} + tu \left( s - \frac{1}{r} \right) \right) t du dt. \end{aligned}$$

Continuing this procedure, we get

$$h_{r,N}^b(s) = \left(s - \frac{1}{r}\right)^{N+1} \int_0^1 \int_0^1 \cdots \int_0^1 (h_{r,N}^b)^{(N+1)} \left(\frac{1}{r} + t_1 t_2 \cdots t_{N+1} \left(s - \frac{1}{r}\right)\right) m(t) dt_1 \cdots dt_{N+1},$$

where  $m(t)$  is a monomial in  $t_1, \dots, t_{N+1}$ . Hence, using (3.4), we obtain

$$k_{r,N}^b(s) = \frac{1}{r^{N+1}} \int_0^1 \int_0^1 \cdots \int_0^1 (h_{r,N}^b)^{(N+1)} \left(\frac{1}{r} + t_1 t_2 \cdots t_{N+1} \left(s - \frac{1}{r}\right)\right) m(t) dt_1 \cdots dt_{N+1}.$$

But, thanks to properties of  $b$ , we can differentiate under the integral sign to get

$$(k_{r,N}^b)^{(N)}(s) = \frac{1}{r^{N+1}} \int_0^1 \int_0^1 \cdots \int_0^1 (h_{r,N}^b)^{(2N+1)} \left(\frac{1}{r} + t_1 t_2 \cdots t_{N+1} \left(s - \frac{1}{r}\right)\right) v(t) dt_1 \cdots dt_{N+1},$$

where  $v(t)$  is a monomial in  $t_1, \dots, t_{N+1}$ . Since  $(h_{r,N}^b)^{(2N+1)}$  is bounded on  $(0, R_0)$ , we deduce that  $|(k_{r,N}^b)^{(N)}(r)| \leq \frac{1}{r^{N+1}} \|(h_{r,N}^b)^{(2N+1)}\|_\infty$ , which is bounded as  $r \rightarrow 1^-$ .

(iii)  $\implies$  (v): here we also assume that  $\zeta_0 = 1$ . According to [1, Lemma 4.2] we can take a sequence  $(B_j)_{j \geq 1}$  of Blaschke products converging uniformly to  $b$  on compact subsets of  $\mathbb{D}$  and such that

$$\sum_k \frac{1 - |a_{j,k}|^2}{|1 - r a_{j,k}|^{2N+2}} \xrightarrow{j \rightarrow +\infty} \sum_k \frac{1 - |a_k|^2}{|1 - r a_k|^{2N+2}} + \int_0^{2\pi} \frac{d\mu(e^{it})}{|e^{it} - r|^{2N+2}} + \int_0^{2\pi} \frac{|\log |b(e^{it})||}{|e^{it} - r|^{2N+2}} dm(e^{it}),$$

where  $(a_{j,k})_{k \geq 1}$  is the sequence of zeros of  $B_j$ . As before, let  $k_{\omega,N}^b := \frac{\partial^N k_\omega^b}{\partial \bar{\omega}^N}$  and let  $k_{\omega,N}^{B_j} := \frac{\partial^N k_\omega^{B_j}}{\partial \bar{\omega}^N}$ . Hence, we have

$$(3.7) \quad k_{\omega,N}^{B_j}(z) = \frac{N! z^N - B_j(z) \sum_{p=0}^N \binom{N}{p} \overline{B_j^{(p)}(\omega)} (N-p)! z^{N-p} (1 - \bar{\omega} z)^p}{(1 - \bar{\omega} z)^{N+1}}$$

and thus  $k_{\omega,N}^{B_j}$  tends to  $k_{\omega,N}^b$  uniformly on compact subsets of  $\mathbb{D}$ . Therefore,

$$\lim_{j \rightarrow +\infty} (k_{\omega,N}^{B_j})^{(N)}(\omega) = (k_{\omega,N}^b)^{(N)}(\omega).$$

But,

$$\left\| \frac{\partial^N k_\omega^b}{\partial \bar{\omega}^N} \right\|_b^2 = (k_{\omega,N}^b)^{(N)}(\omega), \quad \text{and} \quad \left\| \frac{\partial^N k_\omega^{B_j}}{\partial \bar{\omega}^N} \right\|_2^2 = (k_{\omega,N}^{B_j})^{(N)}(\omega),$$

and condition (iii) implies that there exists  $C_1 > 0$  such that, for all  $0 < r < 1$ , we have  $|(k_{r,N}^b)^{(N)}(r)| \leq C_1$ . Therefore, for all  $0 < r < 1$ , there exists  $j_r \in \mathbb{N}$ , such that for  $j \geq j_r$ , we have

$$\left\| \frac{\partial^N k_r^{B_j}}{\partial r^N} \right\|_2^2 = |(k_{r,N}^{B_j})^{(N)}(r)| \leq C_1 + 1.$$

Moreover, using (3.7), we see that

$$(1 - rz)^{N+1} \frac{\partial^N k_r^{B_j}}{\partial r^N}(z) = N!z^N - B_j(z)g_j(z),$$

where  $g_j \in H^2$ . Hence, it follows from [1, Theorem 3.1] that there is a constant  $K$  (independent of  $r$ ) such that

$$\sum_k \frac{1 - |a_{j,k}|^2}{|1 - ra_{j,k}|^{2N+2}} \leq K, \quad (j \geq j_r),$$

Letting  $j \rightarrow +\infty$ , we obtain

$$\sum_k \frac{1 - |a_k|^2}{|1 - ra_k|^{2N+2}} + \int_0^{2\pi} \frac{d\mu(e^{it})}{|e^{it} - r|^{2N+2}} + \int_0^{2\pi} \frac{|\log |b(e^{it})||}{|e^{it} - r|^{2N+2}} dm(e^{it}) \leq K$$

for all  $r \in (0, 1)$ . Now we let  $r \rightarrow 1^-$ , we get the desired condition (v). □

#### 4. CONTINUITY AND ANALYTIC CONTINUATION FOR FUNCTIONS OF THE DE BRANGES–ROVNYAK SPACES

In this section, we study the continuity and analyticity of functions in the de Branges–Rovnyak spaces  $\mathcal{H}(b)$  on an open arc of  $\mathbb{T}$ . As we will see the theory bifurcates into two opposite cases depending whether  $b$  is an extreme point of the unit ball of  $H^\infty(\mathbb{D})$  or not. Let us recall that if  $X$  is a linear space and  $S$  is a convex subset of  $X$ , then an element  $x \in S$  is called an extreme point of  $S$  if it is not a proper convex combination of any two

distinct points in  $S$ . Then, it is well known (see [4, page 125]) that a function  $f$  is an extreme point of the unit ball of  $H^\infty(\mathbb{D})$  if and only if

$$\int_{\mathbb{T}} \log(1 - |f(\zeta)|) d\zeta = -\infty.$$

The following result is a generalization of results of Helson [7] and Sarason [11]. The equivalence of (i), (ii) and (iii) were proved in [11, page 42] under the assumption that  $b$  is an extreme point. Our contribution is the last two parts. The mere assumption of continuity implies analyticity and this observation has interesting applications.

**Theorem 4.1.** *Let  $b$  be in the unit ball of  $H^\infty(\mathbb{D})$  and let  $I$  be an open arc of  $\mathbb{T}$ . Then the following are equivalent:*

- (i)  $b$  has an analytic continuation across  $I$  and  $|b| = 1$  on  $I$ ;
- (ii)  $I$  is contained in the resolvent set of  $X^*$ ;
- (iii) any function  $f$  in  $\mathcal{H}(b)$  has an analytic continuation across  $I$ ;
- (iv) any function  $f$  in  $\mathcal{H}(b)$  has a continuous extension to  $\mathbb{D} \cup I$ ;
- (v)  $b$  has a continuous extension to  $\mathbb{D} \cup I$  and  $|b| = 1$  on  $I$ .

**Proof:** (i)  $\implies$  (ii): since  $|b| = 1$  on an open interval, it is clear that  $b$  is an extreme point of the unit ball of  $H^\infty(\mathbb{D})$ . In that case, we know that the characteristic function of the operator  $X^*$  (in the theory of Sz-Nagy and Foias) is  $b$  (see [10]). But then this theory tells us that  $\sigma(X^*) = \sigma(b)$  (see [9, Theorem 2.3.4., page 102]). Therefore, if  $b$  has an analytic continuation across  $I$  and  $|b| = 1$  on  $I$ , then  $I$  is contained in the complement of  $\sigma(b)$  and thus  $I$  is contained in the resolvent set of  $X^*$ .

(ii)  $\implies$  (iii): for  $f \in \mathcal{H}(b)$ , we have

$$f(\omega) = \langle f, k_\omega^b \rangle_b = \langle f, (Id - \bar{\omega}X^*)^{-1}k_0^b \rangle_b.$$

Now if  $I$  is contained in the resolvent set of  $X^*$ , then the vector valued function  $\omega \mapsto (Id - \omega X^*)^{-1}k_0^b$ , thought of as an  $\mathcal{H}(b)$ -valued function, can be continued analytically across  $I$  and thus the condition (iii) follows.

(iii)  $\implies$  (iv): is clear.

(iv)  $\implies$  (v): let  $\omega_0 \in \mathbb{D}$  such that  $b(\omega_0) \neq 0$ . Since  $\frac{1 - \overline{b(\omega_0)}b(z)}{1 - \overline{\omega_0}z}$  belongs to  $\mathcal{H}(b)$ , it has a continuous extension to  $\mathbb{D} \cup I$ . Therefore  $b$  also has a continuous extension to  $\mathbb{D} \cup I$ . Now let  $\zeta_0$  be a point of  $I$ . An application of the principle of uniform boundedness shows that the functional on  $\mathcal{H}(b)$  of evaluation at  $\zeta_0$  is bounded. Let  $k_{\zeta_0}^b$  denote the corresponding kernel function. The family  $k_{\omega}^b$  tends weakly to  $k_{\zeta_0}^b$  as  $\omega$  tends to  $\zeta_0$  from  $\mathbb{D}$ . Thus, for any  $z \in \mathbb{D}$ , we also have

$$\begin{aligned} k_{\zeta_0}^b(z) &= \langle k_{\zeta_0}^b, k_z^b \rangle_b = \lim_{\omega \rightarrow \zeta_0} \langle k_{\omega}^b, k_z^b \rangle_b \\ &= \lim_{\omega \rightarrow \zeta_0} \frac{1 - \overline{b(\omega)}b(z)}{1 - \overline{\omega}z} = \frac{1 - \overline{b(\zeta_0)}b(z)}{1 - \overline{\zeta_0}z}. \end{aligned}$$

In particular, the function  $\frac{1 - \overline{b(\zeta_0)}b(z)}{z - \zeta_0}$  is in  $H^2(\mathbb{C}_+)$ , which is possible only if  $|b(\zeta_0)| = 1$ . Hence we get that  $|b| = 1$  on  $I$ .

(v)  $\implies$  (i): follows from standard facts based on the Schwarz's reflection principle. □

As we have seen in the proof of Theorem 4.1, one of the conditions (i) – (v) implies that  $b$  is an extreme point of the unit ball of  $H^\infty(\mathbb{D})$ . Thus, the continuity (or equivalently, the analytic continuation) of  $b$  or of the elements of  $\mathcal{H}(b)$  on the boundary completely depend on  $b$  being an extreme point or not. If  $b$  is not an extreme point of the unit ball of  $H^\infty(\mathbb{D})$  and if  $I$  is an open arc of  $\mathbb{T}$ , then there exists necessarily a function  $f \in \mathcal{H}(b)$  such that  $f$  has not a continuous extension to  $\mathbb{D} \cup I$ . On the opposite case, if  $b$  is an extreme point such that  $b$  has continuous extension to  $\mathbb{D} \cup I$  with  $|b| = 1$  on  $I$ , then all the functions  $f \in \mathcal{H}(b)$  are continuous on  $I$  (and even can be continued analytically across  $I$ ).

Theorem 4.1 shows that the de Branges–Rovnyak spaces  $\mathcal{H}(b)$  have a remarkable property, i.e. continuity on an open arc of  $\mathbb{T}$  of all functions of  $\mathcal{H}(b)$  is enough to imply the analyticity of these functions. This property enables us to show that the result of Dyakonov [5] concerning the Bernstein's inequality in the model spaces is sharp in the

sense that we couldn't extend it to all de Branges–Rovnyak spaces. The definition of de Branges–Rovnyak spaces of the upper half plane is similar to its counterpart for the unit disc. First, we make precise a little more the transfer of the unit disc to the upper half plane  $\mathbb{C}_+$ . We consider  $\gamma$  the conformal map from  $\mathbb{C}_+$  onto  $\mathbb{D}$  defined by

$$\gamma(z) = \frac{z-i}{z+i}, \quad z \in \mathbb{C}_+,$$

and we denote by  $U$  the (unitary) map from  $L^2(\mathbb{T})$  onto  $L^2(\mathbb{R})$  defined by

$$(4.1) \quad (Uf)(x) := \frac{1}{\sqrt{\pi}} \frac{1}{x+i} f\left(\frac{x-i}{x+i}\right), \quad x \in \mathbb{R}, f \in L^2(\mathbb{T}).$$

Then it is well known (see [8, pages 247–248]) that  $U$  maps  $H^2(\mathbb{D})$  onto  $H^2(\mathbb{C}_+)$ . Moreover, if  $\varphi \in L^\infty(\mathbb{T})$ , then

$$(4.2) \quad UT_\varphi = T_{\varphi \circ \gamma}U.$$

Now let  $b$  be in the unit ball of  $H^\infty(\mathbb{D})$  and let  $b_1 = b \circ \gamma$ . Then, using (4.2), basic arguments show that  $U$  maps unitarily  $\mathcal{H}(b)$  onto  $\mathcal{H}(b_1)$ . Using this unitary transform, we can obviously state the analogue of Theorem 3.2 and Theorem 4.1 in the upper half plane  $\mathbb{C}_+$ .

**Corollary 4.2.** *Let  $b_1$  be a point of the unit ball of  $H^\infty(\mathbb{C}_+)$ . Then the following are equivalent:*

- (i) *the operator  $f \longrightarrow f'$  is a bounded operator from  $\mathcal{H}(b_1)$  into  $H^2(\mathbb{C}_+)$ ;*
- (ii)  *$b_1$  is an inner function and  $b'_1 \in H^\infty(\mathbb{C}_+)$ .*

**Proof:** Using [5, Theorem 1], the only thing to prove is that if (i) holds, then  $b_1$  is inner. But, if for any function  $f$  in  $\mathcal{H}(b_1)$ , we have  $f' \in H^2(\mathbb{C}_+)$ , then in particular,  $f$  has a continuous extension to  $\mathbb{C}_+ \cup \mathbb{R}$ . Thus, using the analogue of Theorem 4.1 in the upper half plane, we see that  $b_1$  has a continuous extension to  $\mathbb{C}_+ \cup \mathbb{R}$  and  $|b_1| = 1$  on  $\mathbb{R}$ , which means  $b_1$  is an inner function.  $\square$

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