Overcompleteness of sequences of reproducing kernels in model spaces

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January 10, 2005

Abstract

We give necessary conditions and sufficient conditions for sequences of reproducing kernels $(k_\Theta(\cdot,\lambda_n))_{n\geq 1}$ to be overcomplete in a given model space K^p_Θ where Θ is an inner function in H^∞, $p \in (1,\infty)$, and where $(\lambda_n)_{n\geq 1}$ is an infinite sequence of pairwise distinct points of $\mathbb{D}$. Under certain conditions on Θ we obtain an exact characterization of overcompleteness. As a consequence we are able to describe the overcomplete exponential systems in $L^2(0,a)$.

AMS Subject Classification: 46E22, 46B15.

1 Introduction

Given a Banach space X and a sequence $(x_n)_{n\geq 1} \subset X$, the question of completeness of sequences $(x_n)_{n\geq 1}$ in X is classical and appears in many problems. In this paper, we deal with a stronger property than completeness.

Definition 1.1 *Let X be a Banach space. An infinite sequence $(x_n)_{n\geq 1}$ whose terms are pairwise distinct is* overcomplete *in X if every infinite subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ is complete in X, i.e.* $\text{span}\{x_{n_k} : k \geq 1\} = X$, *where* span *denotes the closed linear hull.*

One might expect that overcomplete sequences were rare, but in fact V. Klee [13] proved that every separable Banach space contains an overcomplete sequence. Such sequences (also known as hypercomplete or densely-closed sequences) have been much studied in the theory of the geometry of Banach spaces, originally because of their links with the existence of bases. See the book of Singer [17] for further details.

In this paper, we study the following problem due to N. Nikolski and considered previously in [9].

Problem 1.1 *Find necessary and sufficient conditions concerning the inner function Θ and the sequence $(\lambda_n)_{n\geq 1}$ of $\mathbb{D}$ in order to obtain overcompleteness of $(k_\Theta(\cdot,\lambda_n))_{n\geq 1}$ in the model space K^p_Θ.*

In fact overcompleteness of $(k_\Theta(\cdot,\lambda_n))_{n\geq 1}$ in K^p_Θ is equivalent to the following assertion: if $f \in K^q_\Theta$ satisfies $f(\lambda_{n_p}) = 0$ for $(\lambda_{n_p})_{p\geq 1}$ an infinite subsequence of $(\lambda_n)_{n\geq 1}$, then $f = 0$.

The characterization of overcompleteness is linked to the same problem for completeness, which is rather difficult, even in the special case of sequences of exponential type (see [3, 14] for partial

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results in this direction).

The plan of the paper is the following. The next section contains preliminary material on Hardy spaces and inner functions. In Section 3, we study reflexive Banach spaces X of holomorphic functions on a domain Ω admitting evaluations E_λ at points $\lambda \in \Omega$. We give necessary conditions and sufficient conditions for the overcompleteness of $(E_{\lambda_n})_{n\geq 1}$ in X. The main result of this section is the following :

if $X \cap H^\infty(\Omega)$ is dense in X, then the overcompleteness of $(E_{\lambda_n})_{n\geq 1}$ implies the strong relative compactness of $(E_{\lambda_n})_{n\geq 1}$.

In Section 4, we provide a characterization of the overcomplete sequences of exponentials, i.e.

$$(e^{i\mu_n t})_{n\geq 1} \text{ is overcomplete in } L^2(0,a) \Longleftrightarrow \sup_{n\geq 1} |\mu_n| < \infty.$$

The main result of Section 5 is a geometric necessary and sufficient condition for the overcompleteness of $k_\Theta(\cdot, \lambda_n)_{n\geq 1}$ in reflexive spaces K^p_Θ, holding for a wide class of inner functions Θ. We also study the links between overcompleteness of sequences of reproducing kernels and properties of minimality or uniform minimality of all their infinite subsequences. We conclude with some illustrative examples analysed using the theory of Toeplitz operators.

2 Preliminaries

For $1 \leq p \leq +\infty$, H^p will denote the standard Hardy space of the open unit disk $\mathbb{D}$ in $\mathbb{C}$, which we identify with the subspace of functions $f \in L^p(\mathbb{T})$ for which $\hat{f}(n) = 0$ for all $n < 0$ [5, 10]. Here $\mathbb{T}$ denotes the unit circle with normalized Lebesgue measure. Recall that a function $\Theta \in H^\infty$ is called *inner* if $|\Theta(\zeta)| = 1$ for almost $\zeta \in \mathbb{T}$. We associate with each inner function Θ the model space K^p_Θ defined by

$$K^p_\Theta := H^p \cap \Theta\overline{H^p_0} = \{f \in H^p : \langle f, \Theta g\rangle = 0,\ g \in H^q\},$$

where $\overline{H^p_0} = \{\bar{f} : f \in H^p : f(0) = 0\}$ and where p and q are conjugate exponents.

For $p \in (1,\infty)$, Beurling's theorem ([10], Chap. II) states that every nontrivial closed invariant subspace of H^p for $S^* : f \longmapsto \dfrac{f - f(0)}{z}$ is of the form K^p_Θ. The study of the subspaces K^p_Θ is relevant in various subjects such as rational approximation [8, 11, 16], Toeplitz operators [4, 6] and spectral theory for general linear operators [15]. The reproducing kernels in the subspaces K^q_Θ are the functions $k_\Theta(.,\lambda) \in K^p_\Theta$ such that $f(\lambda) = \langle f, k_\Theta(.,\lambda)\rangle$ for $\lambda \in \mathbb{D}$ and $f \in K^q_\Theta$. By [12] they are given by

$$k_\Theta(z,\lambda) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \overline{\lambda}z}.$$

Recall that if Θ is an inner function in H^∞, then Θ has a canonical decomposition of the form

$$\Theta(z) = e^{i\alpha} z^N \prod_{n\geq 1} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n}z} \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\right) \tag{1}$$

where $\alpha \in \mathbb{R}$, $a_n \neq 0$, $\sum_{n\geq 1}(1 - |a_n|) < \infty$ and where μ is a non negative singular measure.

Definition 2.1 *Let Θ be an inner function in H^∞. The spectrum of Θ is denoted by $\sigma(\Theta)$ and is defined to be the complement in $\overline{\mathbb{D}}$ of the set $\{\xi \in \overline{\mathbb{D}} : \frac{1}{\Theta}$ can be analytically continued in a (full) neighborhood of $\xi\}$.*

It follows from [15], p. 63, that $\sigma(\Theta) \cap \mathbb{T} = \{\xi \in \mathbb{T} : \liminf_{z\to\xi} |\Theta(z)| = 0\}$ and if Θ has the canonical decomposition (1), then $\sigma(\Theta) = \operatorname{clos}\{a_n : n \geq 1\} \cup \operatorname{supp}\mu$, where $\operatorname{supp}(\mu)$ denotes the support of μ and clos denotes the closure.

A useful fact concerning the spectrum of an inner function is contained in the following proposition.

Proposition 2.1 ([15], p. 65) *Let Θ be an inner function and $p \in (1, \infty)$. The set $\mathbb{T} \setminus \sigma(\Theta)$ coincides with the set of points ξ such that every function in the model space K^p_Θ admits an analytic continuation across ξ.*

We shall also require another set associated with Θ, defined as follows.

Definition 2.2 *Let Θ be an inner function with the canonical decomposition* (1). Then define the Ahern–Clark set E_Θ [1] by:

$$E_\Theta := \left\{ \zeta \in \mathbb{T} \,:\, \sum_{n\geq 1} \frac{1-|a_n|^2}{|\zeta - a_n|^2} + 2\int_{\mathbb{T}} \frac{d\mu(t)}{|t-\zeta|^2} < +\infty \right\}.$$

Note that $\mathbb{T} \setminus \sigma(\Theta) \subset E_\Theta$, but as we shall see later these sets can be distinct. Also recall that the set E_Θ is an open set relative to $\mathbb{T}$. When Θ is an inner function on $\mathbb{D}$ and ζ_0 is a point in $\mathbb{T}$, one says that Θ has an *angular derivative in the sense of Carathéodory* at ζ_0 if Θ has a non tangential limit at ζ_0 of modulus 1 and in addition the derivative Θ' of Θ has a non tangential limit at ζ_0. We have the following characterization of such points:

Proposition 2.2 *Let Θ be an inner function and $\zeta_0 \in \mathbb{T}$. Then the following assertions are equivalent:*
(i) Θ has an angular derivative in the sense of Carathéodory at ζ_0.
(ii) $\liminf\limits_{\substack{z\in\mathbb{D}\\ z\to\zeta_0}} \dfrac{1-|\Theta(z)|^2}{1-|z|^2} < \infty\,.$
(iii) $\zeta_0 \in E_\Theta$.

The equivalence between (i) and (ii) follows from Carathéodory's Theorem [16] and for the equivalence between (ii) and (iii) see [7].

Finally, we need the notion of minimal sequences.

Definition 2.3 *Let $(x_n)_{n\geq 1}$ be a sequence of a Banach space X. Then $(x_n)_{n\geq 1}$ is called minimal if for every $n \geq 1$, we have $x_n \notin \operatorname{span}\{x_k : k \neq n\}$. Moreover, $(x_n)_{n\geq 1}$ is called uniformly minimal if* $\inf_{n\geq 1} \operatorname{dist}(x_n/\|x_n\|, \operatorname{span}\{x_k : k \neq n\}) > 0$.

A standard application of the Hahn–Banach theorem gives the following characterization of minimality and uniform minimality ([15], p. 131).

Proposition 2.3 *Let $(x_n)_{n\geq 1}$ be a sequence of a Banach space X.*

1. *$(x_n)_{n\geq 1}$ is minimal if and only if there exists a sequence $(x_n^*)_{n\geq 1}$ in X^* satisfying $< x_n, x_k^* >= \delta_{n,k}$ where $\delta_{n,k}$ is the Kronecker symbol. Such a sequence is called a* biorthogonal sequence *of $(x_n)_{n\geq 1}$.*

2. *$(x_n)_{n\geq 1}$ is uniformly minimal if and only if there exists a biorthogonal sequence $(x_n^*)_{n\geq 1}$ of $(x_n)_{n\geq 1}$ such that* $\sup_{n\geq 1} \|x_n\|\|x_n^*\| < \infty$.

3 Overcomplete sequences in reflexive Banach spaces

First of all, we recall a useful lemma.

Lemma 3.1 ([2]) *Let $(y_n)_{n\geq 1}$ be a sequence in a Banach space X satisfying $\inf\limits_{n\geq 1} \|y_n\| > 0$ and such that $(y_n)_{n\geq 1}$ tends weakly to 0. Then $(y_n)_{n\geq 1}$ has a subsequence $(y_{n_p})_{p\geq 1}$ which is a basic sequence, i.e., a Schauder basis in its span.*

Now, we can give a general necessary condition for overcompleteness.

Theorem 3.1 *Let X be a reflexive Banach space and $(x_n)_{n\geq 1} \subset X$ a bounded infinite sequence of pairwise distinct vectors. If $(x_n)_{n\geq 1}$ does not contain a uniformly minimal subsequence (so, in particular if $(x_n)_{n\geq 1}$ is overcomplete in X), then $(x_n)_{n\geq 1}$ is strongly relatively compact.*

Proof: Suppose that $(x_n)_{n\geq 1}$ is not strongly relatively compact. As $(x_n)_{n\geq 1}$ is bounded, we can find $y \in X$ and a subsequence $(x_{n_k})_{k\geq 1}$ tending weakly to y such that $\inf\limits_{k\geq 1} \|x_{n_k} - y\| > 0$.

First case: $y = 0$. Using Lemma 3.1, we obtain a subsequence of $(x_{n_k})_{k\geq 1}$ which forms a basis in its span. In particular this subsequence is uniformly minimal, which proves that $(x_n)_{n\geq 1}$ is not overcomplete in X.

Second case: $y \neq 0$. Using once more Lemma 3.1, we can find a subsequence $(x_{n_{k_p}} - y)_{p\geq 1}$ which is a basic sequence. It follows that $\bigcap\limits_{i\geq 1} \text{span}\,\{x_{n_{k_p}} - y : p \geq i\} = \{0\}$. Indeed, since $(x_{n_{k_p}} - y)_{p\geq 1}$ is a basic sequence, for any $z \in \text{span}\{x_{n_{k_p}} - y : p \geq 1\}$, there exists a unique scalar sequence $(a_{n_p})_{p\geq 1}$ such that $z = \sum_{p\geq 1} a_{n_p}(x_{n_{k_p}} - y)$. The minimality of $(x_{n_{k_p}} - y)_{p\geq 1}$ implies that $a_{n_p} = 0$ for $p \geq 1$ if, in addition, $z \in \bigcap_{i\geq 1} \text{span}\,\{x_{n_{k_p}} - y : p \geq i\} = \{0\}$.

Since $y \neq 0$, there exists $i_0 \in \mathbb{N}$ such that $y \notin \text{span}\,\{x_{n_{k_p}} - y : p \geq i_0\}$. Hence we get that $\mathfrak{X} = (y, x_{n_{k_p}} - y)_{p\geq i_0}$ is a basic sequence, and thus a uniformly minimal sequence. Let $(y^*, (x_{n_{k_p}} - y)^*)_{p\geq i_0}$ be the biorthogonal sequence of $\mathfrak{X}$ such that $\sup_{p\geq i_0} \|x_{n_{k_p}} - y\|\|(x_{n_{k_p}} - y)^*\| < \infty$. One can check that $((x_{n_{k_p}} - y)^*)_{p\geq i_0}$ is also a biorthogonal sequence for $(x_{n_{k_p}})_{p\geq i_0}$. Since $(x_{n_{k_p}})_{p\geq i_0}$ is bounded and $\inf_{p\geq i_0} \|x_{n_{k_p}} - y\| > 0$, it follows that $\sup_{p\geq i_0} \|x_{n_{k_p}}\|\|(x_{n_{k_p}} - y)^*\| < \infty$. Therefore, $(x_{n_{k_p}})_{p\geq i_0}$ is uniformly minimal. In particular, $(x_n)_{n\geq 1}$ is not overcomplete, which ends the proof.

□

In the rest of the section, we consider a reflexive complex Banach space X and Ω a domain in $\mathbb{C}$. Moreover suppose that the mapping $f \longmapsto f$ is well-defined and continuous from X into $\text{Hol}(\Omega)$ (the space of holomorphic function on Ω equipped with the topology of the uniform convergence on compact subsets). It is a well-known fact that the evaluations $E_\lambda : f \mapsto f(\lambda)$ for $\lambda \in \Omega$, are continuous. In this context, we can relax the hypothesis under which we can give a necessary condition for overcompleteness.

Theorem 3.2 *Suppose that $X \cap H^\infty(\Omega)$ is dense in X and let $(\lambda_n)_{n\geq 1}$ be an infinite sequence of pairwise distinct points in Ω. If $(E_{\lambda_n})_{n\geq 1}$ does not contain a uniformly minimal subsequence (so, in particular if $(E_{\lambda_n})_{n\geq 1}$ is overcomplete in X^*), then $(E_{\lambda_n})_{n\geq 1}$ is strongly relatively compact.*

Proof: By Theorem 3.1, it suffices to show that $\sup\limits_{n\geq 1} \|E_{\lambda_n}\| < +\infty$. Assume that $\sup\limits_{n\geq 1} \|E_{\lambda_n}\| = +\infty$ and let $(y_n)_{n\geq 1}$ be defined by $y_n = E_{\lambda_n}/\|E_{\lambda_n}\|$. For all $f \in H^\infty(\Omega) \cap X$, we have $|\langle f, y_n\rangle| = |f(\lambda_n)|/\|E_{\lambda_n}\| \leq \|f\|_\infty/\|E_{\lambda_n}\| \to 0$ as $n \to \infty$. Since $H^\infty(\Omega) \cap X$ is dense in X, we get that $(y_n)_{n\geq 1}$ tends weakly to 0 and using Lemma 3.1, we find a subsequence $(y_{n_p})_{p\geq 1}$ which is a basic sequence and in particular is uniformly minimal. Hence $(E_{\lambda_{n_p}})_{p\geq 1}$ cannot be overcomplete in X^*.

□

An obvious sufficient condition for overcompleteness is given by the following proposition, which follows immediately from the principle of isolated zeros.

Proposition 3.1 *Let $(\lambda_n)_{n\geq 1}$ be an infinite sequence of pairwise distinct points in Ω. If the closure of $(\lambda_n)_{n\geq 1}$ is a subset of Ω, then $(E_{\lambda_n})_{n\geq 1}$ is overcomplete in X^*.*

4 Overcomplete sequences in K^p_Θ, $1 < p < \infty$

Before investigating overcompleteness in the reflexive model spaces K^p_Θ, it is natural to consider the problem in H^p where the reproducing kernels are $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$, for $\lambda \in \mathbb{D}$.

Theorem 4.1 *Let $p \in (1,\infty)$ and $(\lambda_n)_{n\geq 1}$ an infinite sequence of pairwise distinct points in $\mathbb{D}$. The sequence $(k_{\lambda_n})_{n\geq 1}$ is overcomplete in H^p if and only if* $\sup_{n\geq 1} |\lambda_n| < 1$.

Proof: In order to apply the results of Section 3, set $\Omega = \mathbb{D}$, $X = H^q$ where p and q are conjugate. In this context, for $\lambda \in \mathbb{D}$, E_λ can be identified with k_λ. By Proposition 3.1, the condition $\sup_{n\geq 1} |\lambda_n| < 1$ implies that $(k_{\lambda_n})_{n\geq 1}$ is overcomplete in H^p. Conversely, by Theorem 3.2 the overcompleteness of $(k_{\lambda_n})_{n\geq 1}$ implies in particular that $\sup_{n\geq 1} \|k_{\lambda_n}\|_p < \infty$. Now, it is known ([15], p. 188) that $\|k_{\lambda_n}\|_p \asymp \frac{1}{(1-|\lambda_n|^2)^{1/q}}$. Therefore, $\sup_{n\geq 1} \|k_{\lambda_n}\|_p < \infty$ if and only if $\sup_{n\geq 1} |\lambda_n| < 1$.

□

The study of sequences of reproducing kernels in the model spaces K^p_Θ is often considered under the geometrical condition $\sup |\Theta(\lambda_n)| < 1$ [12]. In this case we have the following result.

Theorem 4.2 *Let $p \in (1,\infty)$ and $(\lambda_n)_{n\geq 1}$ an infinite sequence of pairwise distinct points in $\mathbb{D}$. Suppose $\sup\limits_{n\geq 1} |\Theta(\lambda_n)| < 1$; then $(k_\Theta(\cdot,\lambda_n))_{n\geq 1}$ is overcomplete in K^p_Θ if and only if $\sup\limits_{n\geq 1} |\lambda_n| < 1$.*

Proof: Set $\Omega = \mathbb{D}$, $X = K^q_\Theta$ where p and q are conjugate. For $\lambda \in \mathbb{D}$, the evaluation E_λ on X can be identified with $k_\Theta(\cdot,\lambda)$. By Proposition 3.1, the second condition is sufficient for the overcompleteness. By Theorem 3.2, overcompleteness implies in particular that $\sup_{n\geq 1} \|k_\Theta(\cdot,\lambda_n)\|_p < \infty$. But we have

$$\|k_\Theta(\cdot,\lambda_n)\|_p^p \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1-\overline{\Theta(\lambda_n)}\Theta(e^{it})}{1-\overline{\lambda_n}e^{it}}\right|^p dt \geq (1-|\Theta(\lambda_n)|)^p \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-\overline{\lambda_n}e^{it}|^p} dt.$$

Since $\sup_{n\geq 1} |\Theta(\lambda_n)| < 1$, there exists a positive constant c such that $\|k_\Theta(\cdot,\lambda_n)\|_p^p \geq c\|k_{\lambda_n}\|_p^p$. It follows that $\sup_{n\geq 1} \|k_{\lambda_n}\|_p < \infty$, and hence $\sup_{n\geq 1} |\lambda_n| < 1$, as shown in the proof of Theorem 4.1.

□

The study of bases of exponentials in $L^2(0,a)$ provided the original motivation for the development of the functional model approach in [12]. In the remainder of this section we discuss in more detail overcompleteness of exponentials. Some preliminaries are needed to translate the problem into the language of model spaces.

If $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, then we define the conformal mapping $\phi : \mathbb{C}_+ \to \mathbb{D}$ by $\phi(z) = \frac{z-i}{z+i}$. The operator $(Uf)(z) = \frac{1}{\pi(z+i)} f(\phi(z))$ maps H^2 unitarily onto the Hardy space $H^2(\mathbb{C}_+)$. The corresponding transformation for functions in H^∞ is $f \mapsto f \circ \phi$; it maps inner functions in $\mathbb{D}$ into inner functions in $\mathbb{C}_+$. We have then $UK_\Theta = H^2(\mathbb{C}_+) \ominus (\Theta\circ\phi)H^2(\mathbb{C}_+)$, and $U(k^\Theta_\lambda)$ is the reproducing kernel for the point $\phi(\lambda)$.

The Blaschke factor corresponding to $\mu \in \mathbb{C}_+$ is $b^+_\mu(z) = \frac{z-\mu}{z-\bar\mu}$ and the Blaschke product with zeros $(\mu_n)_{n\geq 1}$ is $B^+(z) = \prod_{n\geq 1} c_{\mu_n} b^+_{\mu_n}(z)$, the coefficients c_{μ_n} being chosen as to make all terms positive at $z = i$.

Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Fourier transform. Then $\mathcal{F}U$ maps H^2 unitarily onto $L^2(0,\infty)$. If $\Theta_a(z) = e^{a\frac{z+1}{z-1}}$, then $\mathcal{F}U$ maps K_{Θ_a} unitarily onto $L^2(0,a)$; the reproducing kernel $k_{\Theta_a}(\cdot,\lambda)$ ($\lambda \in \mathbb{D}$) is mapped (up to a nonzero constant) into $e^{i\mu t}$, where $\mu = -\overline{\phi^{-1}(\lambda)}$. Note that $|\Theta_a(\lambda_n)| = e^{-a\operatorname{Im}\mu_n}$ and thus $\sup_{n\geq 1} |\Theta_a(\lambda_n)| < 1$ if and only if $\inf_{n\geq 1} \operatorname{Im}\mu_n > 0$.

Therefore, the previous results can then be adapted to the case of exponentials $e^{i\mu_n t}$, with $\inf_{n\geq 1} \operatorname{Im}\mu_n > 0$. Nevertheless we will see that the hypothesis $\inf_{n\geq 1} \operatorname{Im}\mu_n > 0$ can be removed.

Theorem 4.3 *Let $a > 0$ and $(\mu_n)_{n\geq 1}$ be an infinite sequence of pairwise distinct points in $\mathbb{C}$. Then $(e^{i\mu_n t})_{n\geq 1}$ is overcomplete in $L^2(0,a)$ if and only if* $\sup_{n\geq 1} |\mu_n| < \infty$.

Proof: Consider the sequence $(\mu^*_n)_{n\geq 1}$ defined as follows:

$$\mu^*_n = \begin{cases} \mu_n & \text{if } \operatorname{Im}\mu_n \geq 0, \\ \overline{\mu_n} & \text{if } \operatorname{Im}\mu_n < 0. \end{cases}$$

We will prove that

$$(e^{i\mu_n t})_{n\geq 1} \text{ is overcomplete in } L^2(0,a) \Longleftrightarrow (e^{i\mu_n^* t})_{n\geq 1} \text{ is overcomplete in } L^2(0,a). \qquad (2)$$

First we remark that for every infinite subset Λ of $\mathbb{N}^*$, considering the anti-linear bijection T defined by $Tf(t) = \overline{f(-t+a)}$ on $L^2(0,a)$, we have:

$$(e^{i\mu_n t})_{n\in\Lambda} \text{ is overcomplete in } L^2(0,a) \Longleftrightarrow (e^{i\overline{\mu_n} t})_{n\in\Lambda} \text{ is overcomplete in } L^2(0,a). \qquad (3)$$

If $\{n \geq 1 : \operatorname{Im}\mu_n < 0\}$ is finite or $\{n \geq 1 : \operatorname{Im}\mu_n \geq 0\}$ is finite, (2) follows from (3) and that fact that adding or deleting a finite set does not change the overcompleteness property. Otherwise, (2) follows from (3) and the fact that the union of two overcomplete sequences is overcomplete.

Let $\delta > 0$. Now, considering the unitary operator U on $L^2(0,a)$ defined by $Uf(t) = e^{i\delta t}f(t)$, we have:

$$(e^{i\mu_n^* t})_{n\in\Lambda} \text{ is overcomplete in } L^2(0,a) \Longleftrightarrow (e^{i(\mu_n^*+\delta)t})_{n\in\Lambda} \text{ is overcomplete in } L^2(0,a). \qquad (4)$$

Since $\inf_{n\geq 1}\operatorname{Im}(\mu_n^*+\delta) > 0$, by Theorem 4.2 and the translation of our problem into the language of model spaces, we get:

$$(e^{i(\mu_n^*+\delta)t})_{n\in\Lambda} \text{ is overcomplete in } L^2(0,a) \Longleftrightarrow \sup_{n\geq 1}|\mu_n^*+\delta| < \infty \Longleftrightarrow \sup_{n\geq 1}|\mu_n| < \infty.$$

Using (2) and (4), the proof of the theorem follows.

□

5 Overcompleteness in K^p_Θ in terms of $\sigma(\Theta)$ and E_Θ

The following result shows that we may assume, in the sequel, that Θ is an inner function which is not a finite Blaschke product and thus $\sigma(\Theta)\cap\mathbb{T} \neq \emptyset$.

Proposition 5.1 *Let $p \in (1,\infty)$, $(\lambda_n)_{n\geq 1}$ be an infinite sequence of pairwise distinct points in $\mathbb{D}$ and let Θ be a finite Blaschke product. Then $(k_\Theta(.,\lambda_n))_{n\geq 1}$ is overcomplete in K^p_Θ.*

Proof: Set $\Omega = \{z \in \mathbb{C} : |z| < R\}$ where $\frac{1}{R} = \max\{z \in \mathbb{D} : \Theta(z) = 0\} < 1$ and $X = K^q_\Theta$ where p and q are conjugate. For $\lambda \in \mathbb{D}$, the evaluation E_λ on X can be identified with $k_\Theta(\cdot,\lambda)$. Since $\operatorname{clos}(\{\lambda_n : n \geq 1\}) \subset \{z \in \mathbb{C} : |z| \leq 1\} \subset \Omega$, by Proposition 3.1, $(k_\Theta(.,\lambda_n))_{n\geq 1}$ is overcomplete in K^p_Θ.

□

Proposition 5.2 *Let $p \in [2,\infty)$, $(\lambda_n)_{n\geq 1}$ be an infinite sequence of pairwise distinct points in $\mathbb{D}$. We have the following sequence of implications:*

(SC) $\quad \inf_{n\geq 1}\operatorname{dist}(\lambda_n, \sigma(\Theta)\cap\mathbb{T}) > 0$

$\Downarrow$

(OVC) $\quad (k_\Theta(.,\lambda_n))_{n\geq 1}$ *is overcomplete in* K^p_Θ

$\Downarrow$

(NC_1) $\quad (k_\Theta(.,\lambda_n))_{n\geq 1}$ *is strongly relatively compact in* K^p_Θ

$\Downarrow$

(NC_2) $\quad \sup_{n\geq 1}\dfrac{1-|\Theta(\lambda_n)|^2}{1-|\lambda_n|^2} < \infty$

$\Downarrow$

(NC_3) $\quad \inf_{n\geq 1}\operatorname{dist}(\lambda_n, \mathbb{T}\setminus E_\Theta) > 0$

Moreover, for $p \in (1,2)$, $(SC) \Rightarrow (OVC) \Rightarrow (NC_1)$ remains true.

Proof: Let $p \in (1, \infty)$. Set $\Omega = \mathbb{C} \setminus (\sigma(\Theta) \cup \{\frac{1}{\bar{z}} : \Theta(z) = 0\})$ and $X = K_\Theta^q$ where p and q are conjugate. Using Proposition 2.1, X embeds continuously into $\mathrm{Hol}(\Omega)$. Then $(SC) \Longrightarrow (OVC)$ and $(OVC) \Longrightarrow (NC_1)$ applying respectively Proposition 3.1 and Theorem 3.2.

Now take $p \in [2, \infty)$. If (NC_1) is satisfied, then, obviously, $\sup_{n\geq 1} \|k_\Theta(\cdot, \lambda_n)\|_p < \infty$. Since $p \geq 2$ we have:

$$\sup_{n\geq 1} \frac{1 - |\Theta(\lambda_n)|^2}{1 - |\lambda_n|^2} \sup_{n\geq 1} \|k_\Theta(., \lambda_n)\|_2^2 \leq \sup_{n\geq 1} \|k_\Theta(., \lambda_n)\|_p^2 < \infty,$$

which implies that (NC_2) is satisfied. To prove that $(NC_2) \Longrightarrow (NC_3)$, take ζ_0 be a limit point of $(\lambda_n)_{n\geq 1}$ in $\mathbb{T}$. Then since $\liminf_{\substack{z\in\mathbb{D} \\ z\to\zeta_0}} \frac{1 - |\Theta(z)|^2}{1 - |z|^2} \leq \sup_{n\geq 1} \frac{1 - |\Theta(\lambda_n)|^2}{1 - |\lambda_n|^2} < \infty$, it follows from Proposition 2.2 that $\zeta_0 \in E_\Theta$. Since $\mathbb{T} \setminus E_\Theta$ is closed, there exists $\delta > 0$ such that for every n, $\mathrm{dist}(\lambda_n, \mathbb{T} \setminus E_\Theta) \geq \delta$.

□

In the case where $E_\Theta = \mathbb{T} \setminus \sigma(\Theta)$, Proposition 5.2 provides a characterization of overcomplete sequence of reproducing kernels in K_Θ^p for $p \geq 2$. The next theorem provides an explicit class of inner functions Θ for which $E_\Theta = \mathbb{T} \setminus \sigma(\Theta)$. First, recall that a sequence $(\alpha_n)_{n\geq 1} \subset \mathbb{D}$ is a *Stolz sequence* if there exists a finite subset e of $\mathbb{T}$ and a positive constant $c >$ such that for all $n \geq 1$, $\mathrm{dist}(\alpha_n, e) \leq c\,\mathrm{dist}(\alpha_n, \mathbb{T})$. If $(\alpha_n)_{n\geq 1}$ is a Stolz sequence and ζ is a limit point of $(\alpha_n)_{n\geq 1}$ then there exists a subsequence $(\alpha_{n_p})_{p\geq 1}$ and a Stolz angle

$$\Delta_\zeta := \{z \in \mathbb{D} : |\arg(1 - \overline{\zeta} z)| < \alpha,\ |z - \zeta| < \rho\} \qquad (0 < \alpha < \frac{\pi}{2},\ \rho < 2\cos\alpha),$$

such that $(\alpha_{n_p})_{p\geq 1} \subset \Delta_\zeta$ and $\lim_{p\to+\infty} \alpha_{n_p} = \zeta$. In other words, this means that $(\alpha_{n_p})_{p\geq 1}$ converges nontangentially to ζ.

Theorem 5.1 *Let $p \in [2, \infty)$ and $(\lambda_n)_{n\geq 1}$ be an infinite sequence of pairwise distinct points of $\mathbb{D}$. Let Θ be an inner function with the canonical decomposition* (1). If $(a_n)_{n\geq 1}$ is a Stolz sequence and if μ has a finite support, then

$$(k_\Theta(\cdot, \lambda_n))_{n\geq 1} \text{ is overcomplete in } K_\Theta^p \Leftrightarrow (SC) \Leftrightarrow (NC_1) \Leftrightarrow (NC_2) \Leftrightarrow (NC_3).$$

Proof: By Proposition 5.2, it is sufficient to prove that $\mathbb{T} \setminus E_\Theta = \mathbb{T} \cap \sigma(\Theta)$, or, equivalently, that $\mathbb{T} \setminus \sigma(\Theta) = E_\Theta$. The inclusion $\mathbb{T} \setminus \sigma(\Theta) \subset E_\Theta$ is true for any inner function Θ and follows from the definitions of $\sigma(\Theta)$ and E_Θ. Note also that $E_\Theta = E_B \cap E_{S_\mu}$ and $\sigma(\Theta) = \sigma(B) \cup \sigma(S_\mu)$. Therefore it suffices to prove that $E_B \subset \mathbb{T} \setminus \sigma(B)$ and $E_{S_\mu} \subset \mathbb{T} \setminus \sigma(S_\mu)$. Write $\mu = \sum_{\lambda\in\mathrm{supp}(\mu)} c_\lambda \delta_\lambda$ where $\mathrm{supp}(\mu)$ is the support of μ, $c_\lambda > 0$ and δ_λ is the Dirac measure at λ. If $\zeta_0 \in E_{S_\mu}$, then $\int_\mathbb{T} \frac{d\mu(t)}{|t - \zeta_0|^2} < \infty$, that is, $\sum_{\lambda\in\mathrm{supp}(\mu)} \frac{c_\lambda}{|\lambda - \zeta_0|^2} < \infty$. Since the support of μ is finite, we conclude that $\inf_{\lambda\in\mathrm{supp}(\mu)} |\lambda - \zeta_0| \inf_{\lambda\in\sigma(S_\mu)} |\lambda - \zeta_0| > 0$, and thus $\zeta_0 \in \mathbb{T} \setminus \sigma(S_\mu)$.

It remains to check that $E_B \subset \mathbb{T} \setminus \sigma(B)$. Take $\zeta_0 \in E_B \cap \sigma(B)$. Since, $\zeta_0 \in E_B$, using Proposition 2.2, we know that B has a nontangential limit at ζ_0 with $|B(\zeta_0)| = 1$. Moreover, since $\zeta_0 \in \sigma(B) \cap \mathbb{T}$, there exists a sequence $(\alpha_n)_{n\geq 1}$ which tends to ζ_0 and satisfying $B(\alpha_n) = 0$ for $n \geq 1$. Since $(\alpha_n)_{n\geq 1}$ is a Stolz sequence, it follows that $B(\zeta_0) = 0$, which is absurd.

□

Note that $k_\Theta(\cdot, \lambda_n)$ strongly converges in K_Θ^2 if $\lambda_n \to \zeta \in E_\Theta$ nontangentially [1, 16]. Now, assuming that the sequence $(\lambda_n)_{n\geq 1}$ is a Stolz sequence, the conditions (NC_1), (NC_2) and (NC_3) are obviously equivalent with $p = 2$.

We now give a characterization of overcomplete sequences of reproducing kernels $(k_\Theta(\cdot, \lambda_n))_{n\geq 1}$ for some particular Blaschke products Θ whose sets of zeros are not necessarily Stolz sequences. If Θ is inner and $\alpha \in \mathbb{D}$, then we define $\Theta_\alpha = \frac{\Theta - \alpha}{1 - \overline{\alpha}\Theta}$. Then Θ_α is also an inner function and according to theorem of Frostman, for almost all $\alpha \in \mathbb{D}$, it is actually a Blaschke product.

Proposition 5.3 *Let $p \in [2, \infty)$ and $(\lambda_n)_{n\geq 1}$ is an infinite sequence of pairwise distinct points of $\mathbb{D}$. Let Θ be a Blaschke product and suppose that there exists $\alpha \in \mathbb{D}$ and a singular inner function S with finite support such that $\Theta = S_\alpha$. Then*

$$(k_\Theta(\cdot, \lambda_n))_{n\geq 1} \text{ is overcomplete in } K^p_\Theta \Leftrightarrow (SC) \Leftrightarrow (NC_1) \Leftrightarrow (NC_2) \Leftrightarrow (NC_3).$$

Proof: It is not difficult to check that the formula $U(f) = \sqrt{1-|\alpha|^2}\frac{f}{1-\overline{\alpha}\Theta}$ defines a unitary operator $U : K^p_S \to K^p_\Theta$ which maps (up to a nonzero constant) $k_S(\cdot, \lambda_n)$ into $k_\Theta(\cdot, \lambda_n)$. Therefore $(k_\Theta(\cdot, \lambda_n))_{n\geq 1}$ is overcomplete in K^p_Θ if and only if $(k_S(\cdot, \lambda_n))_{n\geq 1}$ is overcomplete in K^p_S. Moreover it follows from the very definition of the spectrum and Proposition 2.2 that $E_S = E_\Theta$ and $\sigma(S) \cap \mathbb{T} = \sigma(\Theta) \cap \mathbb{T}$. Applying Theorem 5.1, we conclude the proof of the proposition. □

Let $S(z) = e^{\frac{z-1}{z+1}}$, a singular inner function whose support is $\{-1\}$. For almost every $\alpha \in \mathbb{D}$, S_α is a Blaschke product. An easy calculation shows that the set of zeros of S_α, say $(a_n)_{n\geq 1}$, satisfies the equation

$$\left| a_n - \frac{\ln|\alpha|}{1 - \ln|\alpha|} \right| = \frac{1}{1 - \ln|\alpha|},$$

which means that the sequence $(a_n)_{n\geq 1}$ is on a circle tangent to $\mathbb{T}$ and thus $(a_n)_{n\geq 1}$ is not a Stolz sequence. Theorem 5.1 does not apply; however, Proposition 5.3 gives a criterion for overcompleteness in K_{S_α}.

In the introduction we have already mentioned the links between overcompleteness and minimality and uniform minimality. The next theorem gives the precise statements.

Theorem 5.2 *Let $p \in (1, \infty)$ and $(\lambda_n)_{n\geq 1}$ an infinite sequence of pairwise distinct points in $\mathbb{D}$.*

1. *The sequence $(k_\Theta(\cdot, \lambda_n))_{n\geq 1}$ is overcomplete in K^p_Θ if and only if it has no infinite subsequence which is minimal.*
2. *The sequence $(k_\Theta(\cdot, \lambda_n))_{n\geq 1}$ is strongly relatively compact in K^p_Θ if and only if it is bounded and has no infinite subsequence which is uniformly minimal.*

Proof: 1. By definition, an overcomplete sequence in a Banach space does not contain any infinite minimal subsequence. Conversely, if $(k_\Theta(\cdot, \lambda_n))_{n\geq 1}$ is not overcomplete, there exists an infinite subsequence $(k_\Theta(\cdot, \lambda_{n_p}))_{p\geq 1}$ which is not complete in K^p_Θ. By the Hahn–Banach theorem, there exists $g \in K^q_\Theta \setminus \{0\}$ such that $g(\lambda_{n_p}) = 0, p \geq 1$. Now, if m_p is the multiplicity of the zero at λ_{n_p} of g, the function Ψ_{n_p} defined by $\Psi_{n_p} = \frac{g}{(b_{\lambda_{n_p}})^{m_p}}$, with $b_{\lambda_{n_p}}(z) = \frac{z-\lambda_{n_p}}{1-\overline{\lambda_{n_p}}z}$, belongs to K^q_Θ ([15], p. 211). By construction $(\frac{\Psi_{n_p}}{\Psi_{n_p}(\lambda_{n_p})})_{p\geq 1}$ is a biorthogonal sequence of $(k_\Theta(\cdot, \lambda_{n_p}))_{p\geq 1}$. Therefore, the infinite subsequence $(k_\Theta(\cdot, \lambda_{n_p}))_{p\geq 1}$ is minimal.
2. By Theorem 3.1, if $(x_n)_{n\geq 1}$ is a bounded sequence in a reflexive Banach space which does not contain any uniformly minimal sequence is necessarily strongly relatively compact. Conversely, first note that

$$\|k_\Theta(\cdot, \lambda_n)\| \geq \left| \left\langle \frac{P_\Theta 1}{\|P_\Theta 1\|_q}, k_\Theta(\cdot, \lambda_n) \right\rangle \right| \frac{|1 - \Theta(0)\overline{\Theta(\lambda_n)}|}{\|P_\Theta 1\|_q} \geq \frac{1 - |\Theta(0)|}{\|P_\Theta 1\|_q}.$$

Therefore, there exists $c > 0$ such that $\inf_{n\geq 1} \|k_\Theta(\cdot, \lambda_n)\| \geq c$. It follows that

$$\text{dist}\left(\frac{k_\Theta(\cdot, \lambda_n)}{\|k_\Theta(\cdot, \lambda_n)\|}, \text{span}\{k_\Theta(\cdot, \lambda_k) : k \neq n\} \right) \leq \inf_{k\neq n} \frac{\|k_\Theta(\cdot, \lambda_n) - k_\Theta(\cdot, \lambda_k)\|}{c}.$$

Thus, if $(k_\Theta(\cdot, \lambda_n))_{n\geq 1}$ is strongly relatively compact, it is clear that $(k_\Theta(\cdot, \lambda_n))_{n\geq 1}$ is bounded and cannot have a uniformly minimal infinite subsequence.

□

By means of examples we obtain further information on the links between some of the conditions considered.

Proposition 5.4 *The condition (NC_3) is strictly weaker than (NC_2); furthermore, the condition (NC_1) is strictly weaker than (SC).*

Proof: We first construct an example where (SC) is not valid but (NC_1) is satisfied. Let

$$a_n = \frac{\frac{1}{n} + i(\frac{1}{2^n} - 1)}{\frac{1}{n} + i(\frac{1}{2^n} + 1)}$$

for $n \geq 1$. Since $1-|a_n|^2 \asymp \frac{1}{2^n}$, $(a_n)_{n\geq 1}$ is a Blaschke sequence. Let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence which converges to -1 and which satisfies the Stolz condition. Denote by B the Blaschke product associated with $(\lambda_n)_{n\geq 1}$. Since $\sigma(B) \cap \mathbb{T} = \{-1\}$ and $\lim_{n\to\infty} a_n = -1$, applying Theorem 5.1, it follows that $(k_B(\cdot, a_n))_{n\geq 1}$ is not overcomplete in K_B^2. Therefore there exists a subsequence $(a_{n_p})_{p\geq 1}$ of $(a_n)_{n\geq 1}$ such that $(k_B(\cdot, a_{n_p}))_{p\geq 1}$ is not complete in K_B^2. By Lemma 97 of [15], this is equivalent to the condition that $\ker T_{\overline{B}\Theta_1} \neq \{0\}$ where Θ_1 is the Blaschke product associated with $(a_{n_p})_{p\geq 1}$. By Coburn's lemma [15, Lemma 43, p. 318], it follows that $\{0\} = \ker T^*_{\overline{B}\Theta_1} = \ker T_{\overline{\Theta}_1 B}$. Applying once more Lemma 97 of [15], we deduce that the sequence $(k_{\Theta_1}(\cdot, \lambda_n))_{n\geq 1}$ is complete in $K^2_{\Theta_1}$. Obviously, we have $\sigma(\Theta_1) = \{-1\}$. Nevertheless we have $E_{\Theta_1} = \mathbb{T}$. Indeed, since $\mathbb{T} \setminus \sigma(\Theta_1) \subset E_{\Theta_1}$, we get $\mathbb{T} \setminus \{-1\} \subset E_{\Theta_1}$. By Definition 2.2, $-1 \in E_{\Theta_1}$ if and only if

$$\sum_{p\geq 1} \frac{1-|a_{n_p}|^2}{|1+a_{n_p}|^2} < \infty \,.$$

But this convergence follows from the estimate $1 - |a_{n_p}|^2 \asymp \frac{1}{2^{n_p}}$ and the existence of a constant $c > 0$ such that $|1 + a_{n_p}|^2 \geq \frac{c}{n_p^2}$. Therefore, we get $E_{\Theta_1} = \mathbb{T}$. Now, since $(\lambda_n)_{n\geq 1}$ is a Stolz sequence, $(k_{\Theta_1}(\cdot, \lambda_n))_{n\geq 1}$ converges in norm in $K^2_{\Theta_1}$, and then satisfies the condition (NC_1) but (SC) is not valid.

Moreover, if one takes Θ_1 defined as previously and $\lambda_n = a_n$, then $\Theta_1(\lambda_n) = 0$, which implies that (NC_2) does not hold, whereas (NC_3) is valid since $E_{\Theta_1} = \mathbb{T}$. □

Acknowledgments. The authors are grateful to the European Research Training Network in Analysis and Operators for financial support. They also wish to thank the referee for some valuable comments.

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