

FINITELY STRICTLY SINGULAR OPERATORS BETWEEN JAMES SPACES

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ABSTRACT. An operator $T: X \rightarrow Y$ between Banach spaces is said to be finitely strictly singular if for every $\varepsilon > 0$ there exists n such that every subspace $E \subseteq X$ with $\dim E \geq n$ contains a vector x such that $\|Tx\| < \varepsilon\|x\|$. We show that, for $1 \leq p < q < \infty$, the formal inclusion operator from J_p to J_q is finitely strictly singular. As a consequence, we obtain that the strictly singular operator with no invariant subspaces constructed by C. Read is actually finitely strictly singular. These results are deduced from the following fact: if $k \leq n$ then every k -dimensional subspace of \mathbb{R}^n contains a vector x with $\|x\|_{\ell_\infty} = 1$ such that $x_{m_i} = (-1)^i$ for some $m_1 < \dots < m_k$.

1. INTRODUCTION

Recall that an operator $T: X \rightarrow Y$ between Banach spaces is said to be ***strictly singular*** if for every $\varepsilon > 0$ and every infinite dimensional subspace $E \subseteq X$ there is a vector x in the unit sphere of E such that $\|Tx\| < \varepsilon$. Furthermore, T is said to be ***finitely strictly singular*** if for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for every subspace $E \subseteq X$ with $\dim E \geq n$ there exists a vector x in the unit sphere of E such that $\|Tx\| < \varepsilon$. Finitely strictly singular operators are also known in literature as ***superstrictly singular***. Note that

$$\text{compact} \quad \Rightarrow \quad \text{finitely strictly singular} \quad \Rightarrow \quad \text{strictly singular},$$

and that each of these three properties defines a closed subspace in $L(X, Y)$. Actually, each property defines an operator ideal. We refer the reader to [2, 7, 9, 10, 11, 14] for more information about strictly and finitely strictly singular operators. All the Banach spaces in this paper are assumed to be over real scalars.

We say that a subspace $E \subseteq X$ is invariant under an operator $T: X \rightarrow X$ if $\{0\} \neq E \neq X$ and $T(E) \subseteq E$. Every compact operator has invariant subspaces by [1]. On the other hand, Read constructed in [12] an example of a strictly singular operator without nontrivial closed invariant subspaces (this answered a question of Pełczyński). Read's operator acts on an infinite direct sum which involves James spaces. Recall

that James' p -space J_p is a sequence space consisting of all sequences $x = (x_n)_{n=1}^\infty$ in c_0 satisfying $\|x\|_{J_p} < \infty$ where

$$\|x\|_{J_p} = \left(\sup \left\{ \sum_{i=1}^{n-1} |x_{k_{i+1}} - x_{k_i}|^p : 1 \leq k_1 < \dots < k_n, n \in \mathbb{N} \right\} \right)^{\frac{1}{p}}$$

is the norm in J_p . For more information on James' spaces we refer the reader to [3, 6, 7, 8, 13].

It was an open question whether every finitely strictly singular operator has invariant subspaces. Some partial results in this direction were obtained in [2, 11]. We answer this question in the negative by showing that the operator in [12] is, in fact, finitely strictly singular. As an intermediate result, we prove that the formal inclusion operator from J_p to J_q with $1 \leq p < q < \infty$ is finitely strictly singular. The latter statement in a certain sense refines the result of Milman [9] that the formal inclusion operator from ℓ_p to ℓ_q with $1 \leq p < q < \infty$ is finitely strictly singular.

Milman's proof is based on the fact that every k -dimensional subspace E of \mathbb{R}^n contains a vector "with a flat", namely, a vector x with sup-norm one with (at least) k coordinates equal in modulus to 1. For such a vector, one has $\|x\|_{\ell_q} \ll \|x\|_{\ell_p}$. The proofs of our results are based on the following refinement of this observation. We will show that x can be chosen so that these k coordinates have alternating signs. For such a "highly oscillating" vector x one has $\|x\|_{J_q} \ll \|x\|_{J_p}$. More precisely, a finite or infinite sequence of real numbers in $[-1, 1]$ will be called a **zigzag** of order k if it has a subsequence of the form $(-1, 1, -1, 1, \dots)$ of length k . Our results will be based on the following theorem; two different proofs of it will be presented in Sections 2 and 3.

Theorem 1. *For every $k \leq n$, every k -dimensional subspace of \mathbb{R}^n contains a zigzag of order k .*

Corollary 2. *Let $k \in \mathbb{N}$, then every k -dimensional subspace of c_0 contains a zigzag of order k .*

Proof. Let F be a subspace of c_0 with $\dim F = k$. For every $n \in \mathbb{N}$, define $P_n: c_0 \rightarrow \mathbb{R}^n$ via $P_n: (x_i)_{i=1}^\infty \mapsto (x_i)_{i=1}^n$. Let n_1 be such that $\dim P_{n_1}(F) = k$. There exists n_2 such that every vector in F attains its norm on the first n_2 coordinates. Indeed, define $g: F \setminus \{0\} \rightarrow \mathbb{N}$ via $g(x) = \max\{i : |x_i| = \|x\|_\infty\}$. Then g is upper semi-continuous, hence bounded on the unit sphere of F , so that we put $n_2 = \max\{g(x) : x \in F, \|x\| = 1\}$.

Put $n = \max\{n_1, n_2\}$. Since $P_n(F)$ is a k -dimensional subspace of \mathbb{R}^n , by Theorem 1 there exists $x \in F$ such that $P_n x$ is a zigzag of order k . It follows from our definition of n that x is a zigzag of order k in F . \square

Suppose that $1 \leq p < q$. Since $\|x\|_{J_p}$ is defined as the supremum of ℓ_p -norms of certain sequences, $\|\cdot\|_{\ell_q} \leq \|\cdot\|_{\ell_p}$ implies $\|\cdot\|_{J_q} \leq \|\cdot\|_{J_p}$. It follows that $J_p \subseteq J_q$ and the formal inclusion operator $i_{p,q} : J_p \rightarrow J_q$ has norm 1. We show next that it is finitely strictly singular. The proof is analogous to that of Proposition 3.3 in [14]. The main difference, though, is that we use Corollary 2 instead of the simpler lemma from [9, 14].

Theorem 3. *If $1 \leq p < q < \infty$ then the formal inclusion operator $i_{p,q} : J_p \rightarrow J_q$ is finitely strictly singular.*

Proof. Given any $x \in J_p$, then $|x_i - x_j|^q \leq (2\|x\|_\infty)^{q-p}|x_i - x_j|^p$ for every $i, j \in \mathbb{N}$, so that $\|x\|_{J_q} \leq (2\|x\|_\infty)^{1-\frac{p}{q}}\|x\|_{J_p}^{\frac{p}{q}}$. Fix an arbitrary $\varepsilon > 0$. Let $k \in \mathbb{N}$ be such that $(k-1)^{\frac{1}{p}-\frac{1}{q}} > \frac{1}{\varepsilon}$. Suppose that E is a subspace of J_p with $\dim E = k$. By Corollary 2, there is a zigzag $z \in E$ of order k . By the definition of norm in J_p , we have $\|z\|_{J_p} \geq 2(k-1)^{\frac{1}{p}}$.

Put $y = \frac{z}{\|z\|_{J_p}}$. Then $y \in E$ with $\|y\|_{J_p} = 1$. Obviously, $\|y\|_\infty \leq \frac{1}{2}(k-1)^{-\frac{1}{p}}$, so that

$$\|i_{p,q}(y)\|_{J_q} = \|y\|_{J_q} \leq (k-1)^{\frac{1}{q}-\frac{1}{p}}\|y\|_{J_p}^{\frac{p}{q}} < \varepsilon.$$

Hence, $i_{p,q}$ is finitely strictly singular. \square

We will now use Theorem 3 to show that the strictly singular operator T constructed by Read in [12] is finitely strictly singular. Let us briefly outline those properties of T that will be relevant for our investigation. The underlying space X for this operator is defined as the ℓ_2 -direct sum of ℓ_2 and Y , $X = (\ell_2 \oplus Y)_{\ell_2}$, where Y itself is the ℓ_2 -direct sum of an infinite sequence of J_p -spaces $Y = \left(\bigoplus_{i=1}^{\infty} J_{p_i}\right)_{\ell_2}$, with (p_i) a certain strictly increasing sequence in $(2, +\infty)$. The operator T is a compact perturbation of $0 \oplus W_1$, where $W_1 : Y \rightarrow Y$ acts as a weighted right shift, that is,

$$W_1(x_1, x_2, x_3, \dots) = (0, \beta_1 x_1, \beta_2 x_2, \beta_3 x_3, \dots), \quad x_i \in J_{p_i}$$

with $\beta_i \rightarrow 0$. Note that one should rather write $\beta_i i_{p_i, p_{i+1}} x_i$ instead of $\beta_i x_i$. Clearly, it suffices to show that W_1 is finitely strictly singular.

For $n \in \mathbb{N}$, define $V_n : Y \rightarrow Y$ via

$$V_n(x_1, x_2, x_3, \dots) = (0, \beta_1 x_1, \dots, \beta_n x_n, 0, 0, \dots), \quad x_i \in J_{p_i}.$$

It follows from $\beta_i \rightarrow 0$ that $\|V_n - W_1\| \rightarrow 0$. Since finitely strictly singular operators from Y to Y form a closed subspace of $L(Y)$, it suffices to show that V_n is finitely strictly singular for every n . Given $n \in \mathbb{N}$, one can write

$$V_n = \sum_{i=1}^n \beta_i j_{i+1} i_{p_i, p_{i+1}} P_i,$$

where $P_i: Y \rightarrow J_{p_i}$ is the canonical projection and $j_i: J_{p_i} \rightarrow Y$ is the canonical inclusion. Thus, V_n is finitely strictly singular because finitely strictly singular operators form an operator ideal. This yields the following result.

Theorem 4. *Read's operator T is finitely strictly singular.*

In the remaining two sections, we present two different proofs of Theorem 1, one based on combinatorial properties of polytopes and the other based on the geometry of the set of all zigzags and algebraic topology.

2. PROOF OF THEOREM 1 VIA COMBINATORIAL PROPERTIES OF POLYTOPES

By a **polytope** in \mathbb{R}^k we mean a convex set which is the convex hull of a finite set. A set is a polytope iff it is bounded and can be constructed as the intersection of finitely many closed half-spaces. A **facet** of P is a face of (affine) dimension $k - 1$. We refer the reader to [5, 15] for more details on properties of polytopes.

A polytope P is centrally symmetric iff it can be represented as the absolutely convex hull of its vertices, that is, $P = \text{conv}\{\pm \bar{u}_1, \dots, \pm \bar{u}_n\}$ where $\pm \bar{u}_1, \dots, \pm \bar{u}_n$ are the vertices of P . Clearly, P is centrally symmetric iff it can be represented as the intersection of finitely many centrally symmetric ‘‘bands’’. More precisely, there are vectors $\bar{a}_1, \dots, \bar{a}_m \in \mathbb{R}^k$ such that $\bar{u} \in P$ iff $-1 \leq \langle \bar{u}, \bar{a}_i \rangle \leq 1$ for all $i = 1, \dots, m$, and the facets of P are described by $\{u \in P : \langle \bar{u}, \bar{a}_i \rangle = 1\}$ or $\{u \in P : \langle \bar{u}, -\bar{a}_i \rangle = 1\}$ as $i = 1, \dots, m$.

A **simplex** in \mathbb{R}^k is the convex hull of $k + 1$ points with non-empty interior. A polytope P in \mathbb{R}^k is **simplicial** if all its faces are simplexes (equivalently, if all the facets of P are simplexes). Every polytope can be perturbed into a simplicial polytope by an iterated ‘‘pulling’’ procedure, see e.g., [5, Section 5.2] for details. We will outline a slight modification of the procedure such that it preserves the property of being centrally symmetric. Suppose that P is a centrally symmetric polytope with vertices, say $\pm \bar{u}_1, \dots, \pm \bar{u}_n$. Pull \bar{u}_1 ‘‘away from’’ the origin, but not too far, so that it does not reach any affine hyperplane spanned by the facets of P not containing \bar{u}_1 ; denote the resulting point \bar{u}'_1 . Let $Q = \text{conv}\{\bar{u}'_1, -\bar{u}_1, \pm \bar{u}_2, \dots, \pm \bar{u}_n\}$. By [5, 5.2.2, 5.2.3] this

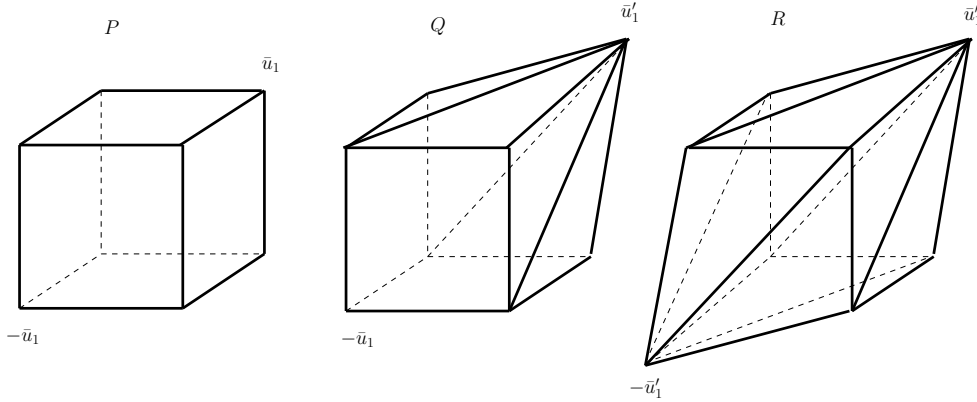


FIGURE 1. Pulling out the first pair of vertices.

procedure does not affect the facets of P not containing \bar{u}_1 , while all the facets of Q containing \bar{u}'_1 become pyramids having apex at \bar{u}'_1 . Note that no facet of P contains both \bar{u}_1 and $-\bar{u}_1$. Hence, if we put $R = \text{conv}\{\pm\bar{u}'_1, \pm\bar{u}_2, \dots, \pm\bar{u}_n\}$, then, by symmetry, all the facets of R containing $-\bar{u}'_1$ become pyramids with apex at $-\bar{u}'_1$, while the rest of the facets (in particular, the facets containing \bar{u}'_1) are not affected.

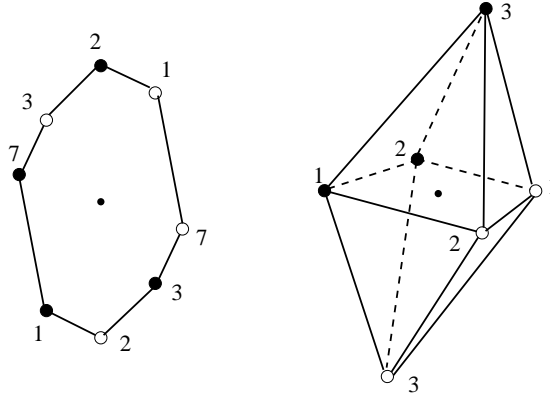
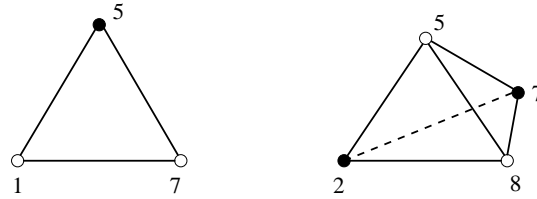
Now iterate this procedure with every other pair of opposite vertices. Let P' be the resulting polytope, $P' = \text{conv}\{\pm\bar{u}'_1, \dots, \pm\bar{u}'_n\}$. Clearly, P' is centrally symmetric and simplicial as in [5, 5.2.4]. It also follows from the construction that if F is a facet of P' then all the vertices of P corresponding to the vertices of F belong to the same facet of P .

We will call a polytope P **marked** if the following assumptions are satisfied:

- (i) P is simplicial, centrally symmetric, and has a non-empty interior.
- (ii) Every vertex is assigned a natural number, called its **index**, such that two vertices have the same index iff they are opposite to each other.
- (iii) All the vertices of P are painted in two colors, say, black and white, so that opposite vertices have opposite colors.

See Figure 2 for examples of marked polytopes. A face of a marked polytope is said to be **happy** if, when one lists its vertices in the order of increasing indices, the colors of the vertices alternate. For example, the front top facet of the marked polytope in the right hand side of Figure 2 is happy. See Figure 3 for more examples of happy faces.

We will reduce Theorem 1 to the claim that every marked polytope has a happy facet, which we will prove afterwards. Suppose that $k \leq n$ and E is a subspace of \mathbb{R}^n with $\dim E = k$. Let $\{\bar{b}_1, \dots, \bar{b}_k\}$ be a basis of E . We need to find a linear combination

FIGURE 2. Examples of marked polytopes in \mathbb{R}^2 and \mathbb{R}^3 .FIGURE 3. Examples of happy simplexes in \mathbb{R}^2 and \mathbb{R}^3 .

of these vectors $\bar{x} := a_1\bar{b}_1 + \dots + a_k\bar{b}_k$ such that \bar{x} is a zigzag. Let B be the $n \times k$ matrix with columns $\bar{b}_1, \dots, \bar{b}_k$, and let $\bar{u}_1, \dots, \bar{u}_n$ be the rows of B . If $\bar{a} = (a_1, \dots, a_k)$, then $x_i = \langle \bar{u}_i, \bar{a} \rangle$ as $i = 1, \dots, n$. Thus, it suffices to find $\bar{a} \in \mathbb{R}^k$ such that the vector $(\langle \bar{u}_i, \bar{a} \rangle)_{i=1}^n$ is a zigzag of order k .

Let P be the centrally symmetric convex polytope spanned by $\bar{u}_1, \dots, \bar{u}_n$, i.e., $P = \text{conv}\{\pm\bar{u}_1, \dots, \pm\bar{u}_n\}$. Then some of the $\pm\bar{u}_i$'s will be the vertices of P , while the others might end up inside P . Suppose that $\pm\bar{u}_{m_1}, \dots, \pm\bar{u}_{m_r}$ are the vertices of P , so that $P = \text{conv}\{\pm\bar{u}_{m_1}, \dots, \pm\bar{u}_{m_r}\}$. Following the ‘‘pulling’’ procedure that was described before, construct a simplicial centrally symmetric polytope $P' = \text{conv}\{\pm\bar{u}'_{m_1}, \dots, \pm\bar{u}'_{m_r}\}$. Every vertex of P' is either \bar{u}'_{m_i} or $-\bar{u}'_{m_i}$ for some i . Paint the vertex white in the former case and black in the latter case; assign index i to this vertex. This way we make P' into a marked polytope.

We claim that happy facets of P' correspond to zigzags. Indeed, suppose that P' has a happy facet. Then this facet (or the facet opposite to it) is spanned by some $-\bar{u}'_{m_{i_1}}, \bar{u}'_{m_{i_2}}, -\bar{u}'_{m_{i_3}}, \bar{u}'_{m_{i_4}}$, etc, for some $1 \leq i_1 < \dots < i_k \leq r$. It follows that $-\bar{u}_{m_{i_1}}, \bar{u}_{m_{i_2}}, -\bar{u}_{m_{i_3}}, \bar{u}_{m_{i_4}}$, etc, are all contained in the same facet of P . Hence, they are contained in an affine hyperplane, say L , such that P ‘‘sits’’ between L and $-L$. Let \bar{a} be the vector defining L , that is, $L = \{\bar{u} : \langle \bar{u}, \bar{a} \rangle = 1\}$. Since P is between L and $-L$, we have

$-1 \leq \langle \bar{u}, \bar{a} \rangle \leq 1$ for every \bar{u} in P . In particular, $-1 \leq x_i = \langle \bar{u}_i, \bar{a} \rangle \leq 1$ for $i = 1, \dots, n$. On the other hand, it follows from $-\bar{u}_{m_{i_1}}, \bar{u}_{m_{i_2}}, -\bar{u}_{m_{i_3}}, \bar{u}_{m_{i_4}}, \dots \in L$ that $x_{m_{i_1}} = -1$, $x_{m_{i_2}} = 1$, $x_{m_{i_3}} = -1$, $x_{m_{i_4}} = 1$, etc. Hence, \bar{x} is a zigzag of order k .

Thus, to complete the proof, it suffices to show that *every marked polytope has a happy facet*. Throughout the rest of this section, P will be a marked polytope in \mathbb{R}^k ; \mathcal{F}_j stands for the set of all j -dimensional faces of P for $j = 0, \dots, k-1$. In particular, \mathcal{F}_{k-1} is the set of all facets of P , while \mathcal{F}_0 is the set of all vertices of P .

By [5, 3.1.6], every $(k-2)$ -dimensional face E of P is contained in exactly two facets, say F and G ; in this case $E = F \cap G$. Suppose that $R \subseteq \mathcal{F}_{k-1}$. For $E \in \mathcal{F}_{k-2}$, we say that E is a boundary face of R if $E = F \cap G$ for some facets F and G such that $F \in R$ and $G \notin R$. The set of all boundary faces of R will be referred to as the **face boundary** of R and denoted $\tilde{\partial}R$. Clearly, $\tilde{\partial}R \subset \mathcal{F}_{k-2}$. If F is a single facet, we put $\tilde{\partial}F = \tilde{\partial}\{F\}$. Clearly, $\tilde{\partial}F$ is the set of all the facets of F .

For a face F of P we define its **color code** to be the list of the colors of its vertices in the order of increasing indices. For example, the color codes of the simplexes in Figure 3 are (wbw) and $(bwbw)$. Here b and w correspond to “black” and “white” respectively. A face in P will be said to be a **b -face** if its color code starts with b and a **w -face** otherwise.

Lemma 5. *Suppose that F is a facet of P . The following are equivalent:*

- (i) F is happy;
- (ii) $\tilde{\partial}F$ contains exactly one happy b -face;
- (iii) $\tilde{\partial}F$ has an odd number of happy b -faces;

Proof. Note that since F is a simplex, every face of F can be obtained by dropping one vertex of F and taking the convex hull of the remaining vertices. Hence, the color code of the face is obtained by dropping one symbol from the color code of F .

(i) \Rightarrow (ii) Suppose that F is happy, then its color code is either $(bwbw\dots)$ or $(wbwb\dots)$. In the former case, the only happy b -face of F is obtained by dropping the last vertex, while in the latter case the only happy b -face of F is obtained by dropping the first vertex.

(ii) \Rightarrow (iii) Trivial.

(iii) \Rightarrow (i) Suppose that $\tilde{\partial}F$ has an odd number of happy b -faces. Let E be a happy b -face in $\tilde{\partial}F$. Then the color code of E is the sequence $(bwbw\dots)$ of length $k-1$. The color code of F is obtained by inserting one extra symbol into this sequence. Note that inserting the extra symbol should not result in two consecutive b 's or w 's, as in this

case F would have exactly two happy b -faces (corresponding to removing each of the two consecutive symbols), which would contradict the assumption. Hence, the color code of F should be an alternating sequence, so that F is happy. \square

Lemma 6. *For every $R \subseteq \mathcal{F}_{k-1}$, the number of happy facets in R and the number of happy b -faces in $\tilde{\partial}R$ have the same parity.*

Proof. For $R \subseteq \mathcal{F}_{k-1}$, define the **parity** of R to be the parity of the number of happy b -faces in $\tilde{\partial}R$. Observe that if R and S are two disjoint subsets of \mathcal{F}_{k-1} , then the parity of $R \cup S$ is the sum of the parities of R and S (mod 2). It follows that the parity of R is the sum of the parities of all of the facets that make up R (mod 2). But this is exactly the parity of the number of happy facets in R by Lemma 5. \square

For every face F of P we write $-F$ for the opposite face. If R is a set of facets, we write $-R = \{-F : F \in R\}$. Also, we write $\bigcup R$ for the set theoretic union of all the facets in R .

Theorem 7. *Every marked polytope has a happy facet.*

Proof. We will prove a stronger statement: *every marked polytope in \mathbb{R}^k has an odd number of happy b -faces.* The proof is by induction on k . For $k = 1$, the statement is trivial. Let $k > 1$ and let P be a marked polytope in \mathbb{R}^k .

For every facet F , let \bar{n}_F be the normal vector of F , directed outwards of P . Fix a vector \bar{v} of length one such that \bar{v} is not parallel to any of the facets of P (equivalently, not orthogonal to \bar{n}_F for any facet F); it is easy to see that such a vector exists. By rotating P we may assume without loss of generality that $\bar{v} = (0, \dots, 0, 1)$. Let T be the projection from \mathbb{R}^k to \mathbb{R}^{k-1} such that $T: (x_1, \dots, x_{k-1}, x_k) \mapsto (x_1, \dots, x_{k-1})$. We can think of T as the orthogonal projection onto the ‘‘horizontal’’ hyperplane $\{\bar{x} \in \mathbb{R}^k : x_k = 0\}$ in \mathbb{R}^k . Let $Q = T(P)$. Since T is linear and surjective, Q is again a centrally symmetric convex polytope in \mathbb{R}^{k-1} with a non-empty interior.

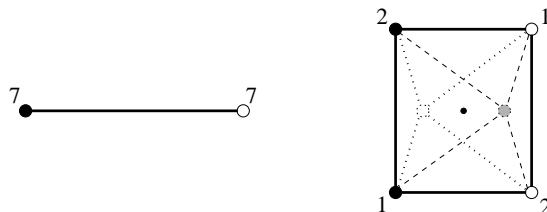


FIGURE 4. The images $T(P)$ of the polytopes in Figure 2.

It follows from our choice of \bar{v} that the k -th coordinate of \bar{n}_F is non-zero for every facet F . Let R be the set of all the facets of P that “face upward”, that is,

$$R = \{ F \in \mathcal{F}_{k-1} : \text{the } k\text{-th coordinate of } \bar{n}_F \text{ is positive} \}.$$

Clearly, a facet F is in $-R$ iff the k -th coordinate of \bar{n}_F is negative. Hence, $-R \cap R = \emptyset$ and $-R \cup R = \mathcal{F}_{k-1}$. Observe that $\tilde{\partial}R = \tilde{\partial}(-R)$; hence $\tilde{\partial}R$ is centrally symmetric. Clearly, every vertical line (i.e., a line parallel to \bar{v}) that intersects the interior of P meets the boundary of P at exactly two points and meets the interior of Q at exactly one point. It follows that the restriction of T to $\bigcup R$ is a bijection between $\bigcup R$ and Q . The same is also true for $-R$. Therefore, the restriction of T to $\bigcup \tilde{\partial}R$ is a face-preserving bijection between $\bigcup \tilde{\partial}R$ and the boundary of Q . Under this bijection, the faces in $\tilde{\partial}R$ correspond to the facets of Q . Hence, this bijection induces a structure of a marked polytope on the boundary of Q , making Q into a marked polytope. It follows, by the induction hypothesis, that the boundary of Q has an odd number of happy b -facets. Hence, $\tilde{\partial}R$ has an odd number of happy b -faces. It follows from Lemma 6 that R has an odd number of happy facets.

Let m and ℓ be the numbers of all happy b -facets and w -facets in R , respectively. Then $m + \ell$ is odd. Observe that F is a happy b -facet iff $-F$ is a happy w -facet. It follows that $-R$ contains ℓ happy b -facets and m happy w -facets. Thus, the total number of happy b -facets of P is $m + \ell$, which we proved to be odd. \square

3. PROOF OF THEOREM 1 VIA ALGEBRAIC TOPOLOGY

Fix a natural number n and let B_∞^n and S_∞^{n-1} be, respectively, the unit ball and the unit sphere of ℓ_∞^n , i.e., $B_\infty^n = \{x \in \mathbb{R}^n : \max|x_i| \leq 1\}$ and $S_\infty^{n-1} = \{x \in \mathbb{R}^n : \max|x_i| = 1\}$. For $k \geq 1$ we define

$$\begin{aligned} \Gamma_k &= \{x \in B_\infty^n : x \text{ has at least } k \text{ alternating coordinates } \pm 1\}, \\ A_k^+ &= \{x \in B_\infty^n : x \text{ has at least } k \text{ alternating coordinates } \pm 1, \text{ starting with } 1\}, \\ A_k^- &= -A_k^+. \end{aligned}$$

Note that A_k^- is exactly the set of all zigzags of order k in \mathbb{R}^n . Put also $A_0^+ = A_0^- = \Gamma_0 = B_\infty^n$. For $k \geq 1$, $\Gamma_k, A_k^\pm \subset S_\infty^{n-1}$ and we have

$$\begin{aligned} A_k^+ \cup A_k^- &= \Gamma_k, \\ A_k^+ \cap A_k^- &= \Gamma_{k+1}. \end{aligned}$$

Note that the first relation above is true also for $k = 0$.

We start with a simple lemma.

Lemma 8. *Suppose p is a real polynomial of degree m , and there are $m+2$ real numbers $t_1 < t_2 < \dots < t_{m+2}$, such that $p(t_i) \geq 0$ for i odd and $p(t_i) \leq 0$ for i even. Then $p \equiv 0$.*

Proof. We do induction with respect to m . If $m = 0$, the result is obvious. If the lemma has been proved up to $m - 1$, and p is a polynomial of degree m , then p has at least one real root s . We write $p(t) = (t - s)q(t)$, and q (or $-q$) has a similar property, with respect to at least $m - 1$ values t_i —so we can apply induction. \square

Lemma 9. *There exists a sequence of subspaces $\pi_k \subset \mathbb{R}^n$, $\pi_k \supset \pi_{k+1}$, $\dim \pi_k = n - k$, such that, if P_k is the orthogonal projection onto π_k , then $P_k|_{A_k^+}$ is injective.*

Proof. For $1 \leq j \leq n$ we define the vectors $\zeta^j \in \mathbb{R}^n$ by the formula $\zeta_i^j = i^{j-1}$. One checks easily that the ζ^j 's are linearly independent. Define $\pi_0 = \mathbb{R}^n$, and, for $k \geq 1$, $\pi_k = (\text{span}\{\zeta^1, \dots, \zeta^k\})^\perp$.

Suppose that $x, y \in A_k^+$, and $P_k x = P_k y$. There exist scalars $\alpha_1, \dots, \alpha_k$, such that $x - y = \sum_{j=1}^k \alpha_j \zeta^j$. We have indices $1 \leq r_1 < \dots < r_k \leq n$ and $1 \leq s_1 < \dots < s_k \leq n$, such that $x_{r_l} = y_{s_l} = (-1)^{l-1}$. It follows that $x_{r_l} - y_{r_l} \geq 0$ for l odd and ≤ 0 for l even, while $x_{s_l} - y_{s_l} \leq 0$ for l odd and ≥ 0 for l even.

Let the polynomial p of degree $k - 1$ be given by $p(t) = \sum_{j=1}^k \alpha_j t^{j-1}$. If $r_l = s_l$ for all l , we obtain

$$\sum_j \alpha_j \zeta_{r_l}^j = \sum_j \alpha_j r_l^{j-1} = 0$$

for all $l = 1, \dots, k$. Thus p has k distinct zeros; it must be identically 0, whence $x = y$.

Suppose now that we have $r_l \neq s_l$ for at least one index l . We claim then that among the union of the indices r_l and s_l we can find $\iota_1 < \iota_2 < \dots < \iota_{k+1}$, such that $x_{\iota_l} - y_{\iota_l}$ have alternating signs. This can be achieved by induction with respect to k . For $k = 1$ we must have $r_1 \neq s_1$, so we may take $\iota_1 = \min\{r_1, s_1\}$, $\iota_2 = \max\{r_1, s_1\}$. For $k > 1$, there are two cases. If $r_1 = s_1$, we take $\iota_1 = r_1 = s_1$ and apply the induction hypothesis to obtain the rest. If $r_1 \neq s_1$, we take ι_1 as the lesser of the two and ι_2 as the other one, and then we continue “accordingly” to ι_2 (that is, taking as ι 's the rest of r 's if $\iota_2 = r_1$ and the rest of s 's if $\iota_2 = s_1$).

Now, the way ι_l have been chosen implies that $p(t)$ defined above satisfies the hypotheses of Lemma 8: it has degree $k - 1$ and the values it takes in $\iota_1, \dots, \iota_{k+1}$ have alternating signs. It must then be identically 0, which implies $x = y$. \square

Since $A_k^- = -A_k^+$, it follows that $P_k|_{A_k^-}$ is also injective.

Lemma 10. *If π_k, P_k are obtained in Lemma 9, then*

$$\Delta_k := P_k(\Gamma_k)$$

is a balanced, convex subset of π_k , with 0 as an interior point (in π_k). Moreover, $\Delta_k = P_k(A_k^-) = P_k(A_k^+)$ and $\partial\Delta_k = P_k(\Gamma_{k+1})$ (the boundary in the relative topology of π_k).

Proof. We will use induction with respect to k . The statement is immediately checked for $k = 0$ (note that $P_0 = I_{\mathbb{R}^n}$ and $\partial\Delta_0 = S_\infty^{n-1} = \Gamma_1$).

Assume the statement true for k ; we will prove its validity for $k+1$. By the induction hypothesis, we have

$$\Delta_{k+1} = P_{k+1}P_k(\Gamma_{k+1}) = P_{k+1}\partial\Delta_k = P_{k+1}\Delta_k$$

and is therefore a balanced, convex subset of π_{k+1} , with 0 as an interior point.

Take then $y \in \overset{\circ}{\Delta}_{k+1}$. Suppose $P_{k+1}^{-1}(y) \cap \partial\Delta_k$ contains a single point. Then $P_{k+1}^{-1}(y) \cap \Delta_k$ also contains a single point, and therefore $P_{k+1}^{-1}(y) \cap \pi_k$ is a support line for the convex set Δ_k . This line is contained in a support hyperplane (in π_k); but then the whole of Δ_k projects onto π_{k+1} on one side of this hyperplane, and thus y belongs to the boundary of this projection. Therefore y cannot be in $\overset{\circ}{\Delta}_{k+1}$.

The contradiction obtained shows that $P_{k+1}^{-1}(y) \cap \partial\Delta_k$ contains at least two points. But

$$\partial\Delta_k = P_k(\Gamma_{k+1}) = P_k(A_{k+1}^+) \cup P_k(A_{k+1}^-)$$

whence

$$P_{k+1}(\partial\Delta_k) = P_{k+1}(A_{k+1}^+) \cup P_{k+1}(A_{k+1}^-).$$

Since P_{k+1} restricted to each of the two terms in the right hand side is injective by Lemma 9, there exists a unique $z_+ \in A_{k+1}^+$ such that $y = P_{k+1}z_+$ and a unique $z_- \in A_{k+1}^-$ such that $y = P_{k+1}z_-$.

Take $x \in P_{k+1}^{-1}(y) \cap \partial\Delta_k$. Then either $x \in P_k(A_{k+1}^+)$ or $x \in P_k(A_{k+1}^-)$. If $x \in P_k(A_{k+1}^+)$ then $x = P_k z$ for some $z \in A_{k+1}^+$, so that $y = P_{k+1}x = P_{k+1}z$, which yields $z = z_+$; hence $x = P_k z_+$. Similarly, if $x \in P_k(A_{k+1}^-)$ then $x = P_k z_-$. It follows that $P_{k+1}^{-1}(y) \cap \partial\Delta_k \subseteq \{P_k z_+, P_k z_-\}$. Since $P_{k+1}^{-1}(y) \cap \partial\Delta_k$ contains at least two points, we conclude that $P_{k+1}^{-1}(y) \cap \partial\Delta_k = \{P_k z_+, P_k z_-\}$ and $P_k z_+ \neq P_k z_-$. It follows from $y = P_{k+1}z_\pm$ that $\overset{\circ}{\Delta}_{k+1} \subset P_{k+1}(A_{k+1}^\pm)$. But, Δ_{k+1} being a closed convex set with a nonempty interior, it is the closure of its interior $\overset{\circ}{\Delta}_{k+1}$; since the two sets on the right are closed, we have actually $\Delta_{k+1} = P_{k+1}(A_{k+1}^\pm)$.

We want to show now that $\partial\Delta_{k+1} = P_{k+1}(\Gamma_{k+2})$. Suppose first that $y \in P_{k+1}(\Gamma_{k+2}) = P_{k+1}(A_{k+1}^+ \cap A_{k+1}^-)$; that is, $y = P_{k+1}z$ with $z \in A_{k+1}^+ \cap A_{k+1}^-$. Clearly, $y \in \Delta_{k+1}$. If $y \in \overset{\circ}{\Delta}_{k+1}$, then, defining z_+ and z_- as before, the injectivity of P_{k+1} on A_{k+1}^\pm implies $z = z_- = z_+$. This contradicts $P_k z_+ \neq P_k z_-$; consequently, $y \in \partial\Delta_{k+1}$.

Conversely, take $y \in \partial\Delta_{k+1} = \partial(P_{k+1}(\Delta_k))$. Again, take $z_+ \in A_{k+1}^+$, $z_- \in A_{k+1}^-$, such that $P_{k+1}z_+ = P_{k+1}z_- = y$. We have then $P_k z_+ \in \partial\Delta_k$ (if $P_k z_+ \in \overset{\circ}{\Delta}_k$, then $P_{k+1}z_+ = P_{k+1}P_k z_+$ must be in the interior of $P_{k+1}\Delta_k$, which is $\overset{\circ}{\Delta}_{k+1}$). Similarly, $P_k z_- \in \partial\Delta_k$.

If $P_k z_+ \neq P_k z_-$, then P_{k+1} applied to the whole segment $[P_k z_+, P_k z_-]$ is equal to y . Therefore the segment belongs to $\partial\Delta_k$. Since $\partial\Delta_k = P_k(A_{k+1}^+ \cup A_{k+1}^-)$, there exist two values x_1, x_2 either both in A_{k+1}^+ or both in A_{k+1}^- , such that $P_k x_1, P_k x_2 \in [P_k z_+, P_k z_-]$, and thus $P_{k+1}x_1 = P_{k+1}x_2 = y$. This contradicts the injectivity of P_{k+1} on A_{k+1}^\pm .

Therefore $P_k z_+ = P_k z_-$. But z_+ and z_- both belong to A_k^+ , on which P_k is injective. It follows that $z_+ = z_- \in A_{k+1}^+ \cap A_{k+1}^- = \Gamma_{k+2}$, and $P_{k+1}z_+ = y$. This ends the proof. \square

The main consequence of Lemma 10, in combination with Lemma 9, is the fact that the linear map P_{k-1} maps homeomorphically Γ_k into $\partial\Delta_{k-1}$, which is the boundary of a convex, balanced set, containing 0 in its interior.

Proof of Theorem 1. As noted above, P_{k-1} maps homeomorphically Γ_k onto the boundary of a convex, balanced set, containing 0 in its interior. Composing it with the map $x \mapsto \frac{x}{\|x\|}$, we obtain a homeomorphic map ϕ from Γ_k to S^{n-k} , which satisfies the relation $\phi(-x) = -\phi(x)$.

Suppose that E is a k -dimensional subspace of \mathbb{R}^n with no zigzags. Then $E \cap \Gamma_k = \emptyset$, so that the projection of Γ_k onto E^\perp does not contain 0. Composing this projection with the map $x \mapsto \frac{x}{\|x\|}$, we obtain a continuous map from $\psi : \Gamma_k \rightarrow S^{n-k-1}$, that satisfies $\psi(-x) = -\psi(x)$. Then the map $\Phi := \psi \circ \phi^{-1} : S^{n-k} \rightarrow S^{n-k-1}$ is continuous and satisfies $\Phi(-x) = -\Phi(x)$. This is however impossible: it is known that such a map does not exist (see, for instance, [4]). \square

Remark 11. In Theorem 1, the alternating sequence $(-1, 1, -1, 1, \dots)$ cannot generally be replaced with another ‘‘pattern’’, i.e., another sequence of length k of ± 1 ’s. Indeed, suppose that the pattern has two consecutive 1’s, say, in positions r and $r+1$. Let E be the subspace of \mathbb{R}^n defined by the relations $x_r + x_{r+1} + x_{r+2} = 0$ and $x_i = 0$ whenever $r+3 \leq i \leq n-k+r+1$. Then $\dim E = k$ and it is easy to see that no vector in $E \cap B_\infty^n$ has the required pattern.

On the other hand, it follows easily from Theorem 1 that for every subspace $E \subseteq \mathbb{R}^n$ with $\dim E = 2k - 1$, one can find a vector in $E \cap B_\infty^n$ with any given pattern of length k . Generally, $2k - 1$ is a sharp estimate, as the following example shows. Consider the pattern $(1, 1, \dots, 1)$ of length k . Consider the subspace $E \subset \mathbb{R}^n$ consisting of all the vectors whose first $2k - 1$ coordinates add up to zero, and the remaining coordinates are zero. Then $\dim E = 2k - 2$ and $E \cap B_\infty^n$ contains no vectors conforming to the pattern.

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