Dust Analysis in FK-Ising Percolation and Convergence to $\mathrm{SLE}(16/3, 16/3-6)$

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N.B This version will remain an unpublished manuscript. It corresponds to an earlier version of the preprint [GW18] before we learned about [BC16, appendix A]. The purpose of the preprint [GW18] is to write down the complete proof of the convergence of FK-Ising percolation to $\mathrm{SLE}(16/3, 16/3-6)$ *via excursion-decomposition. The proof there is simpler than this version due to the observation made in [BC16] that the Lebsegue measure of times* $\{t, \theta_t = 0\}$ *is a.s. zero. This observation suggests that the dust analysis carried in Section 3 in the present draft (even though quite natural in its own) can be essentially avoided.*

Abstract

We give a simplified and complete proof of the convergence of the chordal exploration process in critical FK-Ising percolation to chordal $\mathrm{SLE}_\kappa(\kappa-6)$ with $\kappa = 16/3$. Our proof follows the classical excursion-construction of $\mathrm{SLE}_\kappa(\kappa-6)$ processes in the continuum and we are thus lead to analyse the behaviour of the driving function of the discrete system when Dobrushin boundary conditions collapse to a single point (this corresponds to analysing the contribution of microscopic excursions and this is what we call "dust analysis"). Our proof is very different from [KS15, KS16] as it only relies on the convergence to the chordal SLE_κ process in Dobrushin boundary conditions and does not require the introduction of a new observable. Still, it relies crucially on the powerful topological framework developed in [KS17] as well as its follow-up paper [CDCH$^+$14]. Also it relies in an essential way on the strong RSW Theorem proved in [CDCH16].

We end the paper with a detailed sketch of the convergence to radial $\mathrm{SLE}_\kappa(\kappa-6)$ when $\kappa = 16/3$ as well as the derivation of Onsager's one-arm exponent $1/8$.

Contents

1 Introduction

Random cluster model on a finite graph $\Omega = (V(\Omega), E(\Omega)) \subset \mathbb{Z}^2$ is a probability measure on bond configurations $\omega = (\omega_e : e \in E(\Omega)) \in \{0,1\}^{E(\Omega)}$:

$$\phi_{p,q,\Omega}[\omega] \propto p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)},$$

where $o(\omega)$ (resp. $c(\omega)$) denotes the number of open edges (resp. closed edges) in ω and $k(\omega)$ denotes the number of clusters in ω. This model was introduced by Fortuin and Kasteleyn in 1969 and this model is closely related to the Ising model and the Potts model. When $q \geq 1$, the model enjoys FKG inequality which makes it possible to consider the infinite volume measures of the model. For $q \geq 1$, there exists a critical value p_c for each q such that, for $p > p_c$, any infinite volume measure has an infinite cluster; whereas, for $p < p_c$, any infinite volume measure has no infinite cluster. This dichotomy does not tell what happens at criticality $p = p_c$ and the critical phase is of great interest. When $q \in [1,4]$, the critical phase is believed to be conformally invariant and the interface at criticality is conjectured to converge to SLE_κ where

$$\kappa = 4\pi / \arccos(-\sqrt{q}/2).$$

Conformal invariance is proved for $q = 2$ in the celebrated works [Smi10, CS12] while the convergence to SLE_κ is proved in [CDCH+14]. When $q = 2$, the random-cluster model is also called FK-Ising percolation. Precisely, the conclusion proved in [Smi10, CS12, CDCH+14] is the following: consider the critical FK-Ising percolation on a simply connected domain Ω with Dobrushin boundary conditions, the interface converges in law to SLE_κ. What about the convergence with other boundary conditions? The simplest boundary conditions after the Dobrushin one is the fully wired boundary conditions. The convergence of the interface with fully wired boundary conditions is the main topic of this article.

Theorem 1.1. *Let $(\Omega; a, b, c)$ be either a Jordan domain or the upper half plane $\mathbb{H}$ with three marked points a, b, c on its boundary. Let $(\Omega^\delta; a^\delta, b^\delta, c^\delta)$ be a sequence of discrete domains on $\delta\mathbb{Z}^2$ converging to $(\Omega; a, b, c)$ in the Carathéodory sense. Then, as $\delta \to 0$, the exploration path of the critical FK-Ising model in the domain $(\Omega^\delta; a^\delta, b^\delta)$ with Dobrushin wired/free boundary conditions and targeted at c^δ, converges weakly to the chordal $\mathrm{SLE}_\kappa(\kappa-6)$ from a to c with force point at b and with $\kappa = 16/3$. The case $a \equiv b$ corresponds to an exploration path in a fully wired (or fully free) domain.*

The same conclusion was also proved in [KS15], but our proof is very different from the one there. In [KS15], the authors constructed the so-called holomorphic observable for fully wired boundary conditions which is a generalization of the observable constructed in [CS12] for Dobrushin boundary conditions; and then extract information from the observable to characterize the scaling limit. Our approach is different and it only relies on the convergence to the chordal SLE process and the powerful topological tool developed in [KS17].

In order to explain our approach, let us first describe the connection between $\mathrm{SLE}_\kappa(\kappa-6)$ and SLE_κ. Fix $\kappa \in (4, 8)$, the process $\mathrm{SLE}_\kappa(\kappa-6)$ is the Loewner chain (see Section 2.2) with the driving function W which is the solution to the following SDE system:

$$dW_t = \sqrt{\kappa}dB_t + \frac{(\kappa-6)dt}{W_t - V_t}, \quad W_0 = 0; \quad dV_t = \frac{2dt}{V_t - W_t}, \quad V_0 = x \geq 0, \tag{1.1}$$

where B is a standard one-dimensional Brownian motion. The corresponding Loewner chain is called $\mathrm{SLE}_\kappa(\kappa-6)$ in $\mathbb{H}$ from 0 to ∞ with force point x. Set $\theta_t = V_t - W_t$, we find that $\theta_t/\sqrt{\kappa}$ is a Bessel process of dimension $3 - 8/\kappa$. Note that θ_t is the renormalized harmonic measure (see Section 3.2) of the right side of $\eta[0,t]$ union $[0, x]$.

As the process is scaling invariant, one can define the process in any simply connected domain via conformal image. The process has the following special property—target-independence: Suppose $(\Omega; a, b, c)$ is a simply connected domain with three distinct degenerated prime ends a, b, c on the boundary in counterclockwise order. Then an $\mathrm{SLE}_\kappa(\kappa-6)$ in Ω from a to c with force point b, then, up to the disconnection time—the first hitting time of the boundary arc ∂_{bc}, it has the same law as an SLE_κ in Ω from a to b, up to the disconnection time. This target-independent property allows us to decompose $\mathrm{SLE}_\kappa(\kappa-6)$ process into SLE_κ excursions as follows.

Fix some cut-off $\epsilon > 0$, define T_1^ϵ to be the first time that θ reaches ϵ and define S_1^ϵ to be the first time after T_1^ϵ that θ hits zero. Generally, define T_{k+1}^ϵ to be the first time after S_k^ϵ that θ reaches ϵ and define S_{k+1}^ϵ to be the first time after T_{k+1}^ϵ that θ hits zero. For $t > 0$, suppose g_t is the conformal map corresponding to the Loewner chain in the definition of η and denote by x_t the preimage of V_t under g_t. Then, by the above target-independence, we see that, for each $k \geq 1$, the conditional law of $(\eta(t), T_k^\epsilon \leq t \leq S_k^\epsilon)$ given $(\eta(t), t \leq T_k^\epsilon)$ is the same as SLE_κ in $\mathbb{H} \setminus \eta[0, T_k^\epsilon]$ from $\eta(T_k^\epsilon)$ to $x_{T_k^\epsilon}$ up to the disconnection time. In other words, the conditional law of $(g_{T_k^\epsilon}(\eta(t)), T_k^\epsilon \leq t \leq S_k^\epsilon)$ is the same as SLE_κ in $\mathbb{H}$ from 0 to ϵ up to the disconnection time. Roughly speaking, $\mathrm{SLE}_\kappa(\kappa-6)$ can be constructed by concatenating a sequence of i.i.d. SLE_κ excursions. In particular, we have the following decomposition of θ: $\{(\theta(t)/\sqrt{\kappa}, T_k^\epsilon \le t \le S_k^\epsilon)\}_k$ are i.i.d. Bessel excursions. Each of them is a Bessel process starting from ϵ and stopped at the first hitting time of zero.

In our approach of proving Theorem 1.1, we wish to follow the above excursion construction of $\mathrm{SLE}_\kappa(\kappa-6)$. Suppose the same setup as in Theorem 1.1 and suppose γ^n is the interface in Ω^{δ_n} from a^{δ_n} to c^{δ_n} where the lattice size $\delta_n \to 0$. Fix some conformal map $\phi^n : (\Omega^{\delta_n}; a^{\delta_n}, c^{\delta_n}) \to (\mathbb{H}; 0, \infty)$ (resp. $\phi : (\Omega; a, c) \to (\mathbb{H}; 0, \infty)$) such that $\phi^n(b^{\delta_n}) \to \phi(b)$. Denote by $\eta^n = \phi^n(\gamma^n)$. Denote by W^n its driving function and θ^n the renormalized harmonic measure of the right side of $\eta^n[0,t]$ union $[0, \phi^n(b^{\delta_n})]$. The goal of Theorem 1.1 is to show the convergence of η^n to $\eta \sim \mathrm{SLE}_\kappa(\kappa-6)$ in distribution. To this end, we first introduce stopping times $T_k^{n,\epsilon}$ and $S_k^{n,\epsilon}$ for η^n which are the analogs of T_k^ϵ and S_k^ϵ for η, see Section 5.1. These stopping times decompose θ^n into excursions $\{(\theta^n(t), T_k^{n,\epsilon} \le t \le S_k^{n,\epsilon})\}_k$ and dusts $\{\theta^n(t), S_k^{n,\epsilon} \le t \le T_{k+1}^{n,\epsilon}\}$. Our strategy is as follows.

1. First, we argue that $\{\eta^n\}_n$ is tight, see details in Section 2.3. For any convergent subsequence, which we still denote by $\{\eta^n\}_n$, we know that the limiting process η is a continuous curve with continuous driving function W. Moreover, $W^n \to W$ and $\eta^n \to \eta$ locally uniformly. The key ingredient in the first step is the topological framework developed in [KS17] and Russo-Symour-Welsh bounds proved in [CDCH16].

2. Second, we argue that $\theta^n \to \theta$ locally uniformly. This fact seems intuitive, but it is not as easy as one expects. We prove the convergence in Section 4.

3. Third, we argue that the stopping times converge: $T_k^{n,\epsilon} \to T_k^\epsilon, S_k^{n,\epsilon} \to S_k^\epsilon$. Although we have $\eta^n \to \eta, W^n \to W$ and $\theta^n \to \theta$ locally uniformly, the convergence of the stopping times still requires certain technical works. One difficulty one faces is that one cannot rely on a stopping time for the limiting curve without possibly ruining the domain Markov property for the discrete exploration paths. It will be proved in Section 5.2.

4. Fourthly, we use the convergence of the interface with Dobrushin boundary conditions [CS12] to conclude that, for each $k \ge 1$, the process $(\theta(t), T_k^\epsilon \le t \le S_k^\epsilon)$ is a Bessel excursion and it is independent of $(\theta(t), t \le T_k^\epsilon)$. There are several subtleties in this step. The first one is that, although $\theta^n \to \theta$, $T_k^{n,\epsilon} \to T_k^\epsilon$ and $S_k^{n,\epsilon} \to S_k^\epsilon$, we still need to control the processes on the intervals $[T_k^{n,\epsilon} \wedge T_k^\epsilon, T_k^{n,\epsilon} \vee T_k^\epsilon]$. The second one is that the Markov property of η^n or θ^n does not pass to the limit η or θ automatically. This is related to the convergence of the conditional distributions which can be quite delicate to conclude in general. See discussions in Section 5.3.

5. Fifthly, we control the dusts $\{\theta^n(t), S_k^{n,\epsilon} \le t \le T_{k+1}^{n,\epsilon}\}$. In this step, we restrict to the case when Ω is flat near c. The key ingredient in this step is a uniform estimate, Proposition 3.7, whose proof in Section 3 is based on the strong Russo-Symour-Welsh bounds from [CDCH16].

6. Combining the previous two steps, we conclude that θ is indeed a Bessel process. In the fourth step, we have shown that θ has the law of Bessel excursion on the intervals $[T_k^\epsilon, S_k^\epsilon]$; and in the fifth step, we control the process on the intervals $[S_k^\epsilon, T_{k+1}^\epsilon]$ in a uniform way. From these, we see that θ is an "approximate Bessel process". We show that the approximate Bessel processes converge to Bessel process in Section 2.1, and then conclude that θ is indeed a Bessel process. The conclusions derived in the fourth and fifth steps are both crucial in this step, see Section 5.4. The dependency structure is also rather delicate here: we do not need to show that the limiting dust

components $\{\theta(t), S_k^{n,\epsilon} \le t \le T_{k+1}^{n,\epsilon}\}$ are independent of what happened before, but we crucially need of course that the excursions $(\theta(t), T_k^\epsilon \le t \le S_k^\epsilon)$ are independent of $(\theta(t), t \le T_k^\epsilon)$.

7. Next, we argue that η is an $\mathrm{SLE}_\kappa(\kappa-6)$. In other words, we wish to argue that W_t solves the SDE system (1.1) from the fact that θ is a Bessel process. Recall that W is the driving function of η and θ is defined as the renormalized harmonic measure of the right side of η. It is not immediate how to get information on W out of θ. Naively, the first trivial attempt is through the convergence from the discrete to the continuum, as one has in the discrete
$$\theta^n(t) = W_t^n + \int_0^t \frac{2ds}{\theta^n(s)}.$$
Combining with the facts that $W^n \to W$ and $\theta^n \to \theta$, it is tempting to conclude
$$\theta(t) = W_t + \int_0^t \frac{2ds}{\theta_s}. \tag{1.2}$$
However, it is not hard to find examples where the integral term does not necessarily converge as $W^n \to W$ and $\theta^n \to \theta$. In fact, the relation (1.2) still holds, but we prove it using the fact that η is a continuous curve with continuous driving function W and that η satisfies the Russo-Symour-Welsh bounds. With (1.2) at hand, one can conclude that η is indeed an $\mathrm{SLE}_\kappa(\kappa-6)$, see Section 5.4.

8. As we restrict to the case when Ω is flat near c in the fifth step, we only prove Theorem 1.1 in this case in the previous step. Finally, we prove the conclusion in the general case in Section 5.5 using a standard coupling argument.

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2 Preliminaries

2.1 Approximate Bessel process

Let $(X_t, t \ge 0)$ be a Bessel process of dimension d. When $d > 1$, it is a semimartingale and a strong solution to the SDE:
$$X_t = X_0 + B_t + \frac{d-1}{2}\int_0^t \frac{ds}{X_s}, \tag{2.1}$$
where B is a standard one-dimensional Brownian motion, see [RY94, Chapter XI]. When $d \in (1,2)$, it almost surely assumes the value zero on a nonempty random set with zero Lebesgue measure. Standard excursion theory shows that if we decompose X according to zero points, then it gives a Poisson point process of Bessel excursions of the same dimension.

Fix $d \in (1,2)$, let $X = (X_t, t \ge 0)$ be a Bessel process of dimension d starting from zero. We will decompose the process according to zero points. For $\epsilon > 0$, define sequences of stopping times: set $S_0^\epsilon = 0$, for $k \ge 0$,
$$T_{k+1}^\epsilon = \inf\{t > S_k^\epsilon : X_t \ge \epsilon\}, \quad S_{k+1}^\epsilon = \inf\{t > T_{k+1}^\epsilon : X_t = 0\}.$$

We know that

$$\left(X_t, S_k^\epsilon \le t \le T_{k+1}^\epsilon\right), \quad k \ge 0$$

are i.i.d; and that

$$\left(X_t, T_k^\epsilon \le t \le S_k^\epsilon\right), \quad k \ge 0$$

are i.i.d and their common law is Bessel excursion of dimension d starting from ϵ and stopped when it hits zero. Fix $t_0 > 0$, define $N^\epsilon = \sup\{k : S_k^\epsilon \le t_0\}$. Roughly speaking, consider the process X on the time interval $[0, t_0]$,the quantity N^ϵ is the number of excursions before t_0 that have heights greater than ϵ.

Next, we will introduce some approximate Bessel processes. These processes are the same as X on the time intervals $\cup_k (T_k^\epsilon, S_k^\epsilon)$. We will show that, under certain conditions, these approximate Bessel processes will converge to the Bessel process as $\epsilon \to 0$.

For $\epsilon > 0$, suppose $\tilde{X}_t^\epsilon$ is a continuous process with the following properties. Set $\tilde{S}_0^\epsilon = 0$. Let $\tilde{T}_1^\epsilon$ be the first time that the process exceeds ϵ. After $\tilde{T}_1^\epsilon$, the process evolves according to (2.1) until it hits zero at time $\tilde{S}_1^\epsilon$. For $k \ge 0$, let $\tilde{T}_{k+1}^\epsilon$ be the first time after $\tilde{S}_k^\epsilon$ that the process exceeds ϵ. After $\tilde{T}_{k+1}^\epsilon$, the process evolves according to (2.1) until it hits zero at time $\tilde{S}_{k+1}^\epsilon$. For $t_0 > 0$, define similarly $\tilde{N}^\epsilon = \sup\{k : \tilde{S}_k^\epsilon \le t_0\}$.

Proposition 2.1. *Fix $d \in (1, 2)$. Suppose $\tilde{X}^\epsilon$ satisfies the following assumptions.*

(1) For each $k \ge 1$, the processes

$$(\tilde{X}_t^\epsilon, t \le \tilde{T}_k^\epsilon), \quad \textit{and} \quad (\tilde{X}_t^\epsilon, \tilde{T}_k^\epsilon \le t \le \tilde{S}_k^\epsilon),$$

are independent.

(2) There exist a sequence of identically distributed random variables $\{Y_k^\epsilon\}_{k \ge 0}$ such that, for all k, we have almost surely,

$$\tilde{T}_{k+1}^\epsilon - \tilde{S}_k^\epsilon \le Y_k^\epsilon.$$

(N.B. Note that we are not claiming the independence of the Y_k^ϵ here).

(3) We have $\epsilon^{-1}\mathbb{E}[Y_0^\epsilon] \to 0$ as $\epsilon \to 0$.

Then the process $\tilde{X}^\epsilon$ converges in law to the Bessel process of dimension d with the topology of uniform convergence on compact subsets of $[0, \infty)$.

Proof. First, we construct a coupling of the process $\tilde{X}^\epsilon$ with a Bessel process X of dimension d. Run $\tilde{X}_t^\epsilon$ for $t \in [0, \tilde{T}_1^\epsilon]$ and X_t for $t \in [0, S_1^\epsilon]$ independently; given $\tilde{X}^\epsilon$ up to $\tilde{T}_1^\epsilon$ and X_t up to S_1^ϵ, set

$$\tilde{X}_{\tilde{T}_1^\epsilon + t}^\epsilon = X_{T_1^\epsilon + t}, \quad 0 \le t \le S_1^\epsilon - T_1^\epsilon.$$

Generally, for $k \ge 1$, given $\tilde{X}_t^\epsilon$ up to $\tilde{T}_k^\epsilon$ and X_t up to S_k^ϵ, set

$$\tilde{X}_{\tilde{T}_k^\epsilon + t}^\epsilon = X_{T_k^\epsilon + t}, \quad 0 \le t \le S_k^\epsilon - T_k^\epsilon.$$

Define $M^\epsilon = \sup\{n : \sum_{k=1}^n (S_k^\epsilon - T_k^\epsilon) \le t_0\}$. Then it is clear that $N^\epsilon, \tilde{N}^\epsilon \le M^\epsilon$. To obtain the conclusion, it is sufficient to show, for any $\delta > 0$ and $t_0 > 0$,

$$\lim_{\epsilon \to 0} \mathbb{P}\left[\sup_{0 \le t \le t_0} |\tilde{X}_t^\epsilon - X_t| \ge \delta\right] = 0. \tag{2.2}$$

Note that, $\tilde{X}^\epsilon$ and X agree on $[0, t_0]$ up to translation of time by an amount at most

$$\sum_{k \le M^\epsilon} (T^\epsilon_{k+1} - S^\epsilon_k) + \sum_{k \le M^\epsilon} \left(\tilde{T}^\epsilon_{k+1} - \tilde{S}^\epsilon_k\right).$$

Thus, for any $r > 0$ small, we have

$$\begin{aligned}\mathbb{P}\left[\sup_{0 \le t \le t_0} |\tilde{X}^\epsilon_t - X_t| \ge \delta\right] \le \mathbb{P}\left[\exists s, t \in [t_0 + 1] \text{ such that } |s-t| < r, |X_s - X_t| \ge \delta - 2\epsilon\right] \\ + \mathbb{P}\left[\sum_{k \le M^\epsilon} (T^\epsilon_{k+1} - S^\epsilon_k) \ge r/2\right] + \mathbb{P}\left[\sum_{k \le M^\epsilon} \left(\tilde{T}^\epsilon_{k+1} - \tilde{S}^\epsilon_k\right) \ge r/2\right].\end{aligned}$$

Since X is continuous, we know that, for fixed $\delta > 0$,

$$\mathbb{P}\left[\exists s, t \in [t_0 + 1] \text{ such that } |s-t| < r, |X_s - X_t| \ge \delta - 2\epsilon\right] \to 0, \quad \text{as } r \to 0.$$

By [She09, Proposition 3.6], we have almost surely

$$\epsilon M^\epsilon \to 0, \quad \sum_{k \le M^\epsilon} (T^\epsilon_{k+1} - S^\epsilon_k) \to 0, \quad \text{as } \epsilon \to 0.$$

Therefore, for fixed $r > 0$,

$$\mathbb{P}\left[\sum_{k \le M^\epsilon} (T^\epsilon_{k+1} - S^\epsilon_k) \ge r/2\right] \to 0, \quad \text{and} \quad \mathbb{P}\left[M^\epsilon \ge 1/\epsilon\right] \to 0, \quad \text{as } \epsilon \to 0.$$

Combining these, to derive (2.2), it is sufficient to show, for any $r > 0$,

$$\mathbb{P}\left[\sum_{k \le 1/\epsilon} \left(\tilde{T}^\epsilon_{k+1} - \tilde{S}^\epsilon_k\right) \ge r/2\right] \to 0, \quad \text{as } \epsilon \to 0. \tag{2.3}$$

Since $\tilde{T}^\epsilon_{k+1} - \tilde{S}^\epsilon_k \le Y^\epsilon_k$ for all k. When $\epsilon^{-1}\mathbb{E}[Y^\epsilon_0] \to 0$, we have

$$\mathbb{P}\left[\sum_{k \le 1/\epsilon} \left(\tilde{T}^\epsilon_{k+1} - \tilde{S}^\epsilon_k\right) \ge r/2\right] \le \mathbb{P}\left[\sum_{k \le 1/\epsilon} Y^\epsilon_k \ge r/2\right] \le \frac{2\mathbb{E}[Y^\epsilon_0]}{\epsilon r} \to 0,$$

as desired. □

2.2 Chordal Loewner chain

Suppose that K is a is compact subset of $\overline{\mathbb{H}}$. We call K an $\mathbb{H}$**-hull** if $K = \overline{\mathbb{H} \cap K}$ and $H = \mathbb{H} \setminus K$ is simply connected. Riemann's mapping theorem asserts that there exists a unique conformal map Ψ from H onto $\mathbb{H}$ such that $\Psi(z) = z + O(1/z)$, as $z \to \infty$. In particular, there exists real $a = a(K)$ such that

$$\Psi(z) = z + 2a/z + o(1/z), \quad \text{as } z \to \infty.$$

The quantity $a(K)$ is a non-negative increasing function of the set K, and we call it the *half-plane capacity* of K and denote it by $\mathrm{hcap}(K)$.

We list some some estimates of the half-plane capacity which are useful in the later sections. For their proof, see for instance[KS17, Lemma A.13].

Lemma 2.2.

- *If* $K \subset B(x, \epsilon)$ *for some* $x \in \mathbb{R}$, *then* $\mathrm{hcap}(K) \le \epsilon^2/2$.
- *If* $K \cap (\mathbb{R} \times \{\epsilon i\}) \neq \emptyset$, *then* $\mathrm{hcap}(K) \ge \epsilon^2/4$.
- *If* $K \subset [-l, l] \times [0, \epsilon]$, *then* $\mathrm{hcap}(K) \le \frac{l\epsilon}{2\pi}(1 + o(1))$ *as* $\epsilon/l \to 0$.

The **chordal Loewner chain** with a continuous driving function $W : [0, \infty) \to \mathbb{R}$ is the solution for the following ODE: for $z \in \overline{\mathbb{H}}$,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z.$$

This solution is well-defined up to the swallowing time

$$T(z) := \inf \big\{ t : \inf_{s \in [0,t]} |g_s(z) - W_s| > 0 \big\}.$$

For $t \ge 0$, define $K_t := \{z \in \overline{\mathbb{H}} : T(z) \le t\}$, then $g_t(\cdot)$ is the unique conformal map from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ with the expansion $g_t(z) = z + 2t/z + o(1/z)$ as $z \to \infty$.

We record two lemmas 2.3 and 2.4 in as follows. These two properties hold deterministically and will be useful later in the paper.

Lemma 2.3. *[LSW04, Lemma 2.1]. There is a constant* $C > 0$ *such that the following holds. Let* $(W_t, t \ge 0)$ *be a continuous driving function and* $(K_t, t \ge 0)$ *be the corresponding Loewner chain. Set*

$$k(t) := \sqrt{t} + \max\{|W(s) - W(0)| : s \in [0, t]\}.$$

Then, for all $t \ge 0$, *we have*

$$C^{-1} k(t) \le \mathrm{diam}(K_t) \le C k(t).$$

Lemma 2.4. *[MS16b, Lemma 3.3]. Suppose that* η *is a continuous path in* $\overline{\mathbb{H}}$ *from 0 to* ∞ *that admits a continuous Loewner driving function* W. *Let* (g_t) *be the corresponding family of conformal maps. For each* t, *let* X_t *be the right most point of* $g_t(\eta[0, t]) \cap \mathbb{R}$. *If the Lebesgue measure of* $\eta \cap \mathbb{R}$ *is zero, then* X *solves the integral equation*

$$X_t = \int_0^t \frac{2ds}{X_s - W_s}, \quad X_0 = 0^+. \tag{2.4}$$

Chordal SLE_κ is the chordal Loewner chain with driving function $W = \sqrt{\kappa} B$ where B is a one-dimensional Brownian motion. For $\kappa > 0$, the SLE_κ process is almost surely a continuous transient curve in $\mathbb{H}$ from 0 to ∞. When $\kappa \in [0, 4]$, the curve is simple; when $\kappa \in (4, 8)$, the curve is self-touching; and when $\kappa \ge 8$, the curve is space-filling. (See [Law05] and references therein).

Chordal $\mathrm{SLE}_\kappa(\rho)$ with force point $V_0 = x \in \mathbb{R}$ is the chordal Loewner chain with driving function W solving the following SDEs:

$$dW_t = \sqrt{\kappa} dB_t + \frac{\rho dt}{W_t - V_t}, \quad W_0 = 0; \quad dV_t = \frac{2dt}{V_t - W_t}, \quad V_0 = x. \tag{2.5}$$

For $\kappa > 0$ and $\rho > -2$, define $\theta_t = V_t - W_t$. The process $\theta_t/\sqrt{\kappa}$ is a Bessel process of dimension $1 + 2(\rho + 2)/\kappa > 1$, hence $V_t - W_t$ is well-defined for all times. This implies the existence and uniqueness of the solution to (2.5). It is proved in [MS16a] that $\mathrm{SLE}_\kappa(\rho)$

with $\rho > -2$ is almost surely generated by continuous and transient curves. Suppose $(\Omega; a, b, c)$ is a simply connected domain Ω with three marked points (degenerate prime ends) a, b, c on the boundary in counterclockwise order. We define $\mathrm{SLE}_\kappa(\rho)$ in Ω from a to c with force point b as the image of $\mathrm{SLE}_\kappa(\rho)$ in $\mathbb{H}$ from 0 to ∞ with force point 1 under the conformal map $\phi : (\mathbb{H}; 0, 1, \infty) \to (\Omega; a, b, c)$. In this article, we are interested in $\mathrm{SLE}_\kappa(\kappa - 6)$ as it has the following target-independent property.

Lemma 2.5. *[SW05]. Suppose η is an $\mathrm{SLE}_\kappa(\kappa-6)$ in Ω from a to c with force point b and define S to be the first time that η hits the boundary arc ∂_{bc}, then $(\eta(t), 0 \le t \le S)$ has the same law as an SLE_κ in Ω from a to b up to the first time that it hits the boundary arc ∂_{bc}.*

Suppose η is an $\mathrm{SLE}_\kappa(\kappa-6)$ in $\mathbb{H}$ from 0 to ∞ with force point $x \ge 0$. Then the process $\theta_t = V_t - W_t$ is the renormalized harmonic measure (see Definition 3.6) of the right side of $\eta[0, t]$ union $[0, x]$. On the other hand, we find

$$d\theta_t = -\sqrt{\kappa} dB_t + \frac{(\kappa - 4)dt}{\theta_t}.$$

Thus the process $\theta_t/\sqrt{\kappa}$ is a Bessel process of dimension $3 - 8/\kappa$. Note that,

$$3 - 8/\kappa \in (1, 2), \quad \text{when} \quad \kappa \in (4, 8).$$

This is important when we will apply Proposition 2.1 in the proof of Theorem 1.1.

2.3 Convergence of curves: the chordal case

In this section, we recall the main result of [KS17]. Let X be the set of continuous oriented unparameterized curves, that is, continuous mappings from $[0, 1]$ to $\mathbb{C}$ modulo reparameterization. We equip X with the metric

$$d_X(\gamma_1, \gamma_2) = \inf_{\varphi_1, \varphi_2} \sup_{t \in [0,1]} |\gamma_1(\varphi_1(t)) - \gamma_2(\varphi_2(t))|, \tag{2.6}$$

where the infimum is over all increasing homeomorphisms $\varphi_1, \varphi_2 : [0, 1] \to [0, 1]$. The topology on (X, d_X) gives rise to a notion of weak convergence for random curves on X.

We call $(\Omega; a, b)$ a Dobrushin domain if Ω is a bounded simply connected domain Ω with two distinct degenerate prime ends a, b on the boundary. We denote by ∂_{ab} or (ab) the boundary arc of $\partial\Omega$ from a to b in counterclockwise order.

Let $X_{\mathrm{simple}}(\Omega; a, b)$ be the collection of continuous simple curves in Ω from a to b such that they only touch the boundary $\partial\Omega$ in $\{a, b\}$. In other words, $X_{\mathrm{simple}}(\Omega; a, b)$ is the collection of continuous simple curves γ such that

$$\gamma(0) = a, \quad \gamma(1) = b, \quad \gamma(0, 1) \subset \Omega.$$

Let $X_0(\Omega; a, b)$ be the closure of the space $X_{\mathrm{simple}}(\Omega; a, b)$ in the metric topology (X, d_X). We often consider some reference sets $X_0(\mathbb{U}; -1, +1)$ and $X_0(\mathbb{H}; 0, \infty)$ where the latter can be understood by extending the above definition to curves defined on the Riemann sphere.

Since choral SLE is invariant under scaling, we can define chordal SLE in $(\Omega; a, b)$ via conformal image: suppose ϕ is any conformal map from $\mathbb{H}$ onto Ω that sends $0, \infty$ to a, b, we define chordal SLE in Ω from a to b by the image of chordal SLE in $\mathbb{H}$ from 0 to ∞ by ϕ. Note that SLE_κ is in $X_{\mathrm{simple}}(\Omega; a, b)$ almost surely when $\kappa \le 4$ and it is in $X_0(\Omega; a, b)$ almost surely when $\kappa > 4$.

We call $(Q; x_1, x_2, x_3, x_4)$ a **quad** if Q is simply connected subset of $\mathbb{C}$ with four distinct boundary points x_1, x_2, x_3, x_4. The four points are in counterclockwise order. We denote by $d_Q((x_1x_2), (x_3x_4))$ the extremal distance between (x_1x_2) and (x_3x_4) in Q. We say that a curve γ crosses Q if there exists a subinterval $[s, t]$ such that $\gamma(s, t) \subset Q$ and $\gamma[s, t]$ intersects both (x_1x_2) and (x_3x_4).

For any curve $\gamma \in X_0(\Omega; a, b)$ and any time τ, define $\Omega(\tau)$ to be the connected component of $\Omega \setminus \gamma[0, \tau]$ with b on the boundary. Consider a quad $(Q; x_1, x_2, x_3, x_4)$ in $\Omega(\tau)$ such that (x_2x_3) and (x_4x_1) are contained in $\partial\Omega(\tau)$. We say that Q is **avoidable** if it does not disconnect $\gamma(\tau)$ from b in $\Omega(\tau)$.

Definition 2.6. *Suppose $\{(\Omega_n; a_n, b_n)\}_n$ is a sequence of Dobrushin domains. For each n, suppose $\mathbb{P}_n$ is a probability measure supported on $X_0(\Omega_n; a_n, b_n)$. We say that the collection $\Sigma(M) = \{\mathbb{P}_n\}_n$ satisfies **Condition C2** if there exists a constant $M > 0$ such that for any $\mathbb{P}_n \in \Sigma(M)$, any stopping time $0 \le \tau \le 1$, and any avoidable quad $(Q; x_1, x_2, x_3, x_4)$ of $\Omega_n(\tau)$ such that $d_Q((x_1x_2), (x_3x_4)) \ge M$, we have*

$$\mathbb{P}_n[\gamma[\tau, 1] \textit{crosses } Q \,|\, \gamma[0, \tau]] \le 1/2.$$

For a probability measure $\mathbb{P}$ on curves in Ω, let ϕ be a conformal map on Ω. We denote by $\phi\mathbb{P}$ the pushforward of $\mathbb{P}$ by ϕ. For the Dobrushin domain $(\Omega_n; a_n, b_n)$, let ψ_n be any conformal map from $(\Omega_n; a_n, b_n)$ onto $(\mathbb{U}; -1, +1)$. Given the family $\Sigma(M)$ as above, define the family

$$\Sigma_{\mathbb{U}}(M) = \{\psi_n\mathbb{P}_n : \mathbb{P}_n \in \Sigma(M)\}.$$

Theorem 2.7. *If the family $\Sigma(M)$ satisfies Condition C2, then the family $\Sigma_{\mathbb{U}}(M)$ is tight in the topology induced by (2.6). Suppose $\mathbb{P}_\infty$ is a limiting measure of the family $\Sigma_{\mathbb{U}}(M)$, then the following statements hold $\mathbb{P}_\infty$ almost surely.*

(1) There exists $\beta > 0$ such that γ has a Hölder continuous parameterization for the Hölder exponent β.

(2) The tip $\gamma(t)$ of the curve lies on the boundary of the connected component of $\mathbb{U}\setminus\gamma[0, t]$ with 1 on the boundary for all t.

(3) The curve γ is transient: i.e. $\lim_{t\to 1} \gamma(t) = 1$.

Suppose $\gamma_n \sim \mathbb{P}_n$ and let ϕ_n be any conformal map from $(\Omega_n; a_n, b_n)$ onto $(\mathbb{H}; 0, \infty)$. We parameterize $\eta_n := \phi_n(\gamma_n)$ by the half-plane capacity. Let W_n be the driving process of η_n. Then

(4) $\{W_n\}_n$ is tight in the metrizable space of continuous function on $[0, \infty)$ with the topology of local uniform convergence.

(5) $\{\eta_n\}_n$ is tight in the metrizable space of continuous function on $[0, \infty)$ with the topology of local uniform convergence.

Moreover, if the sequence converges in any of the topologies (4) and (5) above it also converges in the other topology and the limits agree in the sense that the limiting curve is driven by the limiting driving process.

Proof. [KS17, Theorem 1.5, Corollary 1.7, Proposition 2.6]. □

Lemma 2.8. *Assume the same as in Theorem 2.7 and suppose $\{\eta_n\}$ is any convergent subsequence and η is the limiting process. Then η satisfies all the requirements in Lemma 2.4 almost surely.*

Proof. Theorem 2.7 guarantees that η is generated by a continuous curve with a continuous Loewner driving function W. We only need to check that $\eta\cap\mathbb{R}$ has zero Lebesgue measure. For convenience, we couple all η_n and η in the same space so that $\eta_n \to \eta$ and $W_n \to W$ locally uniformly almost surely.

It is sufficient to prove there exists $\alpha > 0$ such that, for all $x \in \mathbb{R}$ and $\epsilon > 0$, we have

$$\mathbb{P}[\eta \cap B(x, \epsilon) \neq \emptyset] \leq 10(\epsilon/|x|)^{\alpha}. \tag{2.7}$$

With (2.7) at hand, we see that $\mathbb{P}[\eta \text{ hits } x] = 0$ for all $x \in \mathbb{R} \setminus \{0\}$, thus $\eta \cap \mathbb{R}$ has zero Lebesgue measure. We only need to prove (2.7) for $x \geq 4\epsilon$. Let $A_x(r, R)$ be the semi-annulus in $\mathbb{H}$ with center at x and inradius r and outradius R. It is proved in [KS17, Proposition 2.6] that Condition C2 implies the following property: there exists $\alpha' > 0$ such that, for any avoidable quad Q in $\mathbb{H}$,

$$\mathbb{P}[\eta_n \text{ crosses } Q] \leq 10 \exp(-\alpha' d_Q((x_1x_2), (x_2x_4))).$$

We apply this property to $Q = A_x(2\epsilon, x/2)$, then there exists $\alpha > 0$ such that

$$\mathbb{P}[\eta_n \text{ crosses } A_x(2\epsilon, x/2)] \leq 10(\epsilon/x)^{\alpha}.$$

For $T > 0$, denote by $\|\eta_n - \eta\|_{\infty,T} = \inf\{|\eta_n(t) - \eta(t)| : 0 \leq t \leq T\}$. Then we have

$$\begin{aligned}\mathbb{P}[\eta[0, T] \cap B(x, \epsilon) \neq \emptyset] &\leq \mathbb{P}[\eta_n \text{ crosses } A_x(2\epsilon, x/2)] + \mathbb{P}[\|\eta_n - \eta\|_{\infty,T} \geq \epsilon] \\ &\leq 10(\epsilon/x)^{\alpha} + \mathbb{P}[\|\eta_n - \eta\|_{\infty,T} \geq \epsilon].\end{aligned}$$

Let $n \to \infty$, we have

$$\mathbb{P}[\eta[0, T] \cap B(x, \epsilon) \neq \emptyset] \leq 10(\epsilon/x)^{\alpha}.$$

Thus

$$\begin{aligned}\mathbb{P}[\eta \cap B(x, \epsilon) \neq \emptyset] &\leq \mathbb{P}[\eta[0, T] \cap B(x, \epsilon) \neq \emptyset] + \mathbb{P}[\eta \cap B(x, \epsilon) \neq \emptyset, \eta[0, T] \cap B(x, \epsilon) = \emptyset] \\ &\leq 10(\epsilon/x)^{\alpha} + \mathbb{P}[\eta \cap B(x, \epsilon) \neq \emptyset, \eta[0, T] \cap B(x, \epsilon) = \emptyset].\end{aligned}$$

Let $T \to \infty$, as η is transient, the second term goes to zero and we obtain (2.7). This completes the proof. ☐

2.4 Application to exploration paths of FK-Ising percolation

Let us start by recalling some useful facts on FK-Ising percolation that we will need later in this paper. The reader may consult [DC13] for general background on FK-Ising percolation.

We will consider finite subgraphs $\Omega = (V(\Omega), E(\Omega)) \subset \mathbb{Z}^2$. For such a graph, we denote by $\partial\Omega$ the inner boundary of Ω:

$$\partial\Omega = \{x \in V(\Omega) : \exists y \notin V(\Omega) \text{ such that } \{x, y\} \in E(\mathbb{Z}^2)\}.$$

A configuration $\omega = (\omega_e : e \in E(\Omega))$ is an element of $\{0, 1\}^{E(\Omega)}$. If $\omega_e = 1$, the edge e is said to be open, otherwise e is said to be closed. The configuration ω can be seen as a subgraph of Ω with the same set of vertices $V(\Omega)$, and the set of edges given by open edges $\{e \in E(\Omega) : \omega_e = 1\}$.

Given a finite subgraph $\Omega \subset \mathbb{Z}^2$, boundary condition ξ is a partition $P_1 \sqcup \cdots \sqcup P_k$ of $\partial\Omega$. Two vertices are wired in ξ if they belong to the same P_i. The graph obtained from the configuration ω by identifying the wired vertices together in ξ is denoted by ω^{ξ}. Boundary conditions should be understood informally as encoding how sites are connected outside of Ω. Let $o(\omega)$ and $c(\omega)$ denote the number of open can dual edges of ω and $k(\omega^\xi)$ denote the number of maximal connected components of the graph ω^ξ.

The probability measure $\phi^\xi_{p,q,\Omega}$ of the *random cluster model* model on Ω with edge-weight $p \in [0,1]$, cluster-weight $q > 0$ and boundary condition ξ is defined by

$$\phi^\xi_{p,q,\Omega}[\omega] := \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega^\xi)}}{Z^\xi_{p,q,\Omega}},$$

where $Z^\xi_{p,q,\Omega}$ is the normalizing constant to make $\phi^\xi_{p,q,\Omega}$ a probability measure. For $q = 1$, this model is simply Bernoulli bond percolation.

If all the vertices in $\partial\Omega$ are pairwise wired (the partition is equal to $\partial\Omega$), it is called *wired boundary conditions*. The random cluster model with wired boundary conditions on Ω is denoted by $\phi^1_{p,q,\Omega}$. If there is no wiring between vertices in $\partial\Omega$ (the partition is composed of singletons only), it is called *free boundary conditions*. The random cluster model with free boundary conditions on Ω is denoted by $\phi^0_{p,q,\Omega}$.

We call critical *FK-Ising* model the random cluster model with

$$q = 2, \quad p = p_c(2).$$

For $q = 2$, we have a stronger version of RSW. Given a discrete quad $(Q; a, b, c, d)$, we denote by $d_Q((ab),(cd))$ the *discrete extremal distance* between (ab) and (cd) in Q, see [Che16, Section 6]. The discrete extremal distance is uniformly comparable to and converges to its continuous counterpart—the classical extremal distance.

Theorem 2.9. *[CDCH16, Theorem 1.1]. Fix $q = 2$. For each $L > 0$ there exists $c(L) > 0$ such that, for any quad $(Q; a, b, c, d)$ and any boundary conditions ξ, if $d_Q((ab),(cd)) \le L$, then*

$$\phi^\xi_{p_c(2),2,Q}[(ab) \leftrightarrow (cd)] \ge c(L).$$

The *medial lattice* $(\mathbb{Z}^2)^\diamond$ is the graph with the centers of edges of $\mathbb{Z}^2$ as vertex set, and edges connecting nearest vertices. This lattice is a rotated and rescaled version of $\mathbb{Z}^2$. The vertices and edges of $(\mathbb{Z}^2)^\diamond$ are called medial-vertices and medial-edges. We identify the faces of $(\mathbb{Z}^2)^\diamond$ with the vertices of $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$. A face of $(\mathbb{Z}^2)^\diamond$ is said to be black if it corresponds to a vertex of $\mathbb{Z}^2$ and white if it corresponds to a vertex of $(\mathbb{Z}^2)^*$. See more detail and figures in [DC13, Section 3].

Fix a Dobrushin domain $(\Omega; a, b)$ and consider a configuration ω together with its dual-configuration ω^*. The Dobrushin boundary condition is given by taking edges of ∂_{ba} to be open and the dual-edges of ∂^*_{ab} to be dual-open. Through every vertex of $\Omega^\diamond$, there passes either an open edge of Ω or a dual open edge of Ω^*. Draw self-avoiding loops on $\Omega^\diamond$ as follows: a loop arriving at a vertex of the medial lattice always makes a $\pm\pi/2$ turn so as not to cross the open or dual open edges through this vertex. The loop representation contains loops together with a self-avoiding path going from $a^\diamond$ to $b^\diamond$, see Fig. 2.1. This curve is called the *exploration path* in $\Omega^\diamond$ from $a^\diamond$ to $b^\diamond$. For $\delta > 0$, we consider the rescaled square lattice $\delta\mathbb{Z}^2$. The definitions of dual and medial Dobrushin domains extend to this context.

Theorem 2.10. *[CDCH+14, Theorem 2]. Suppose $(\Omega; a, b)$ is a bounded simply connected subset $\Omega \subset \mathbb{C}$ with two distinct boundary points (degenerate prime ends) a, b. Let $(\Omega^\delta; a^\delta, b^\delta)$ be a sequence of discrete Dobrushin domains on $\delta\mathbb{Z}^2$ converging to $(\Omega; a, b)$ in the Carathéodory sense: fix the conformal maps $\phi : (\Omega; a, b) \to (\mathbb{H}; 0, \infty)$ and $\phi^\delta : (\Omega^\delta; a^\delta, b^\delta) \to (\mathbb{H}; 0, \infty)$ so that $\phi^\delta \to \phi$ as $\delta \to 0$ uniformly on compact subsets of Ω.*

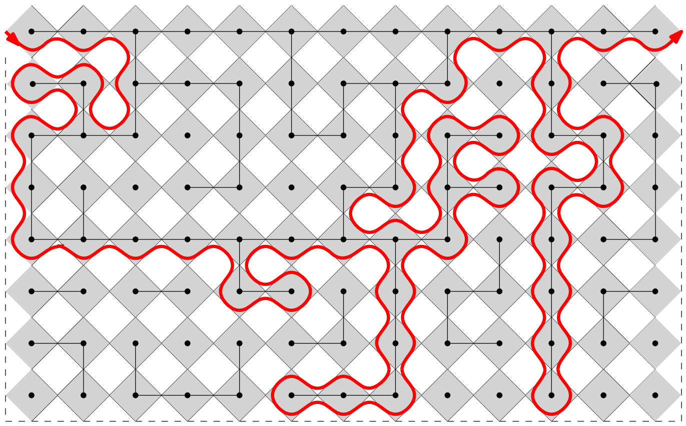

Figure 2.1

(1) Then the exploration path γ^δ of the critical FK-Ising model with Dobrushin boundary conditions in $(\Omega^\delta; a^\delta, b^\delta)$ converges in distribution for the topology induced by (2.6) to chordal $\mathrm{SLE}_{16/3}$ in Ω from a to b.

Suppose γ is an $\mathrm{SLE}_{16/3}$ in Ω from a to b. We parameterize $\phi^\delta(\gamma^\delta)$ (resp. $\eta = \phi(\gamma)$) by the half-plane capacity and let W^δ (resp. W) be the driving function. Let $\delta_n \to 0$, denote by $\gamma^n := \gamma^{\delta_n}$, $\eta^n := \phi^{\delta_n}(\gamma^{\delta_n})$ and $W^n := W^{\delta_n}$; and suppose $\{\eta^n\}$ is a convergent subsequence. We also have the followings

(2) W^n converges in distribution to W with the topology of local uniform convergence.

(3) η^n converges in distribution to η with the topology of local uniform convergence.

In the above, we have defined the exploration path with Dobrushin boundary conditions. Next, we will introduce the exploration path with wired boundary conditions. Consider a configuration in Ω with wired boundary conditions and draw its loop representation on $\Omega^\diamond$. Construct the exploration path from $a^\diamond$ to $b^\diamond$ as follows. Starting from $a^\diamond$, cut open the loop next to a and follow the loop clockwise until one of the following two cases happens: (1) the path reaches the target; (2) the path arrives at a point which is disconnected from the target. If case (1) happens, the path stops. If case (2) happens, cut open the loops next to the current position and follow the new loop clockwise until one of the two cases happens, and repeat the same strategy. Continue in this way until the path reaches $b^\diamond$. See Fig. 2.2.

2.5 On degenerate prime ends

In the course of our proof, we will need to apply the above convergence Theorem 2.10 at multiple occasions along the exploration procedure. In order to apply Theorem 2.10, we need to make sure that the tip of the exploration path ($a^\delta \to a$), as well as the marked point at the end of the dual arc ($b^\delta \to b$) are degenerate prime ends a.s. This will follow from the following general Lemma.

Lemma 2.11. *Let Ω be a bounded Jordan domain with some interior point x_0 and let $\gamma : [0,T] \to \bar{\Omega}$ be a continuous curve which avoids x_0. Then for any $t > 0$, the conformal map $f : \mathbb{U} \to \Omega(t)$ (where $\Omega(t)$ is the connected component of x_0 in $\Omega \setminus \gamma([0,t])$ can be continuously extended to $\bar{\mathbb{U}}$. In particular all points on $\partial\Omega(t)$ are degenerate prime ends. (N.B. $\Omega(t)$ may not be a Jordan domain anymore).*

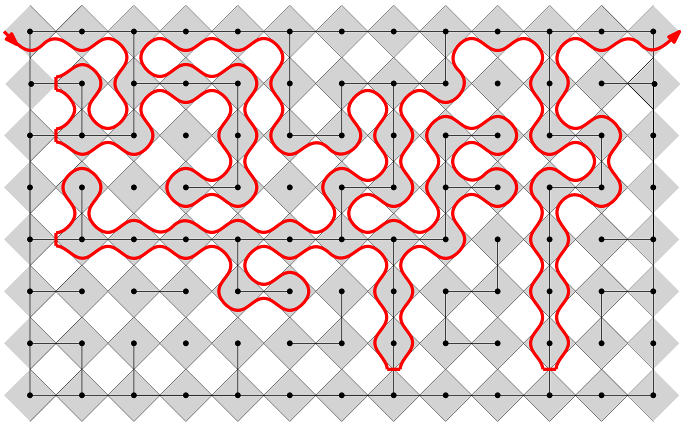

Figure 2.2

Remark 2.12. *Note that this statement is similar in flavour to the* visibility of the tip *statement in Theorem 2.7–Item(2). But it is independent of it : it does not follow from, nor imply the* visibility of the tip *property.*

This Lemma is not new: see for example Example 3.8 in [Law05] and its proof in [New92, pp 88–89]. We include a proof below for completeness.

Proof of Lemma 2.11. Following [Law05, Proposition 3.7] (see also the continuity theorem p18 in [Pom92]), it is equivalent to the fact that $\mathbb{C} \setminus \Omega(t)$ is locally connected for any time $t > 0$. As we assumed that Ω is a Jordan domain, $\Omega(t)$ is the connected component of x_0 of a continuous curve $\eta : [0, 1] \to \mathbb{C}$. Following [Law05], a closed set $K \subset \mathbb{C}$ is *locally connected* if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $z, w \in K$ with $|z - w| < \delta$, there exists a connected set $K_1 \subset K$ with $z, w \in K_1$ and $\mathrm{diam}(K_1) \leq \epsilon$.

Suppose $\mathbb{C} \setminus \Omega(t)$ is not locally connected, i.e. one can find $\epsilon > 0$ and a sequence $\{z_n, w_n\}$ of points such that $|z_n - w_n| \to 0$ which do not satisfies the above property. As Ω is a bounded set, $\sup_n |z_n| \vee |w_n| < \infty$, we can then extract a convergent subsequence $\{\hat{z}_n, \hat{w}_n\}$ such that $\hat{z}_n, \hat{w}_n \to x^*$. If that point x^* is at positive distance from the curve η (in other words if x^* is in $\mathbb{C} \setminus \eta([0, 1])$), then it is immediate to reach a contradiction as $\mathbb{C} \setminus \eta([0, 1])$ is open. If on the other hand, x^* belongs to the range of η, then $x^* = \eta(t^*)$ for some t^* and one can find times u_n, (resp. v_n) such that $\hat{z}_n$ (resp. $\hat{w}_n$) is very close to $\eta(u_n)$ (resp. $\eta(v_n)$) in $\mathbb{C} \setminus \Omega(t)$ (for example by taking the closest points from $\hat{z}_n, \hat{w}_n$ to η). By extracting further, we can assume $u_n \to u$ and $v_n \to v$. Our hypothesis implies that there is no connected subset K_1 in $\mathbb{C} \setminus \Omega(t)$ of diameter less than $\epsilon/2$ connecting $\eta(u)$ to $\eta(v)$. As $\hat{z}_n, \hat{w}_n \to x^* = \eta(t^*)$, by the continuity of the curve η, we must have $\eta(u) = \eta(v) = x^*$. This gives us a contradiction by choosing $K_1 := \{x^*\}$. □

3 Uniform control on the dust

The goal of this section is to obtain a uniform control on the Loewner drift of the exploration process γ^δ when the approximate Bessel process is close to 0. Even though this paper mostly deals with the chordal case, we start with the radial case because it is simpler to state and also because it is needed in the detailed sketch for the radial convergence in Section 6. We then state and adapt the proof to the chordal case.

3.1 The radial case

Let us state the key estimate in the radial setting.

Proposition 3.1. *There exist constants $c, C > 0$ such that for all inner radius $r > 0$, all outer radius $R > r$, all $\epsilon \in (0, \frac{1}{4}]$, all radial lattice-domain Ω^δ which surrounds $B(0, 2r)$ and is surrounded by $B(0, R)$, all marked point $a^\delta \in \partial\Omega^\delta \subset \delta\mathbb{Z}^2$, then if the mesh δ is sufficiently small (the proof below gives $\delta \leq \exp(-\frac{100}{C_{R,r}}\frac{1}{\epsilon})$, but is highly sub-optimal, see Remark 3.4 below), the FK-radial exploration process starting from a^δ and targeting the origin will create, with probability at least $c > 0$, a dual arc of harmonic measure ϵ before accumulating a capacity larger than $C\,\epsilon^2$.*

The statement of this proposition is of course very intuitive. The difficulty lies in the fact that the geometry of Ω^δ can be arbitrary complicated. We will need to build appropriate quads inside Ω^δ which will have the following properties:

1. Their capacity is bounded by $C\,\epsilon^2$
2. They are built in such a way that the strong RSW—Theorem 2.9—can be used to identify arcs with large $\Omega(\epsilon)$-harmonic measure.

As it does not seem straighforward to come up with such quads, our approach is as follows:

1. We will first map Ω^δ to the disk using a conformal map $\phi : \Omega^\delta \to \mathbb{U}$
2. Then we will build the appropriate quads and domains on $\mathbb{U}$
3. Finally, and this will be the main step, we will argue that the image under ϕ^{-1} of these continuous quads can be accurately approximated by $\delta\mathbb{Z}^2$- quads which have the suitable monotony properties so that the capacity/harmonic measure are still satisfied for these approximated quads.

Let us start by stating an analog of Lemma 2.2 in the radial setting. Recall first that the **conformal radius** of a domain D seen from $z \in D$ is $|\varphi'(z)|^{-1}$ where φ is any conformal map from D onto $\mathbb{U}$ sending z to the origin. We denote this conformal radius by $\mathrm{CR}(z; D)$. For any compact subset $K \subset \overline{\mathbb{U}}$, let D be the connected component of $\mathbb{U} \setminus K$ that contains z. We define the **capacity** of K seen from z to be

$$\mathrm{cap}(z; K) = -\log \mathrm{CR}(z; D).$$

When z is the origin, we simply denote $\mathrm{CR}(0; D)$ and $\mathrm{cap}(0; K)$ by $\mathrm{CR}(D)$ and $\mathrm{cap}(K)$ respectively. We have similar estimates for the capacity as above.

Lemma 3.2 (Radial analog of Lemma 2.2)**.** *Suppose that K is a compact subset of $\overline{\mathbb{U}}$ such that $\mathbb{U} \setminus K$ is simply connected and contains the origin.*

(1) If $K \subset B(x, \epsilon)$ for some $x \in \partial\mathbb{U}$, then $\mathrm{cap}(K) \leq 4\epsilon^2$.

(2) If $K \cap B(0, 1 - \epsilon) \neq \emptyset$, then $\mathrm{cap}(K) \geq \epsilon^2/8$.

(3) If $K \subset \{z : 1 - \epsilon < |z| < 1, \arg(z) \in [-l, l]\}$ for some $l \in (0, \pi/2]$, then $\mathrm{cap}(K) \leq Cl\epsilon$ as long as $l \geq \epsilon$ where C is a universal constant.

Proof. First, one can check Item (1) by calculating $\text{cap}(B(1,\epsilon))$. Next, we prove Item (2) by contradiction. Suppose that there exists K such that $K \cap B(0,1-\epsilon) \neq \emptyset$ and that $\text{cap}(K) < \epsilon^2/8$. Then one can find a smooth simple curve $(\gamma(t), 0 \le t \le T)$ parameterized by the capacity such that $\gamma(0) \in \partial\mathbb{U}$, $|\gamma(T)| \le 1-\epsilon$ and $T < \epsilon^2/8$. Suppose that $(g_t(z), 0 \le t \le T)$ is the solution to the radial Loewner chain for γ with driving function $W_t = \exp(i\theta_t)$. For $z \in \mathbb{U} \setminus \gamma[0,T]$, we write $X_t^z = Re \log g_t(z)$. By the ODE for $g_t(z)$ (see [Law05]), we have

$$\frac{dX_t^z}{dt} = Re \frac{1 + g_t(z)e^{-i\theta_t}}{1 - g_t(z)e^{-i\theta_t}} \le \frac{1 + e^{X_t^z}}{1 - e^{X_t^z}}.$$

Thus, $\left(1 - e^{X_t^z}\right) dX_t^z \le 2dt$. Integrating both sides, we have

$$X_T^z - e^{X_T^z} - X_0^z + e^{X_0^z} \le 2T.$$

This is true for any $z \in \mathbb{U} \setminus \gamma[0,T]$. Let $z \to \gamma(T)$, we have

$$2T \ge -\epsilon - \log(1-\epsilon) \ge \epsilon^2/4.$$

This is a contradiction and completes the proof of Item (2).

Finally, we prove Item (3) by estimating the capacity of

$$\mathfrak{W}(\epsilon, l) := \{z : 1 - \epsilon < |z| < 1, \arg(z) \in [-l, l]\}.$$

Set $\phi(z) = i(1-z)/(1+z)$, then ϕ is the conformal map from $\mathbb{U}$ onto $\mathbb{H}$ that sends 0 to i and 1 to 0. Suppose $l \in (0, \pi/2]$ and $\epsilon \in (0, 1/2)$, one can check that

$$\phi(\mathfrak{W}(\epsilon, \mathfrak{l})) \subset \mathfrak{R}(\epsilon, l) := [-l, l] \times [0, 2\epsilon].$$

The domain $\mathbb{H} \setminus \mathfrak{R}(\epsilon, l)$ is a polygon with vertices $w_1 = -l, w_2 = -l + 2i\epsilon, w_3 = l + 2i\epsilon, w_4 = l$. Schwarz-Christoffel mappings give the conformal mapping from $\mathbb{H}$ onto to such polygons. By symmetry, we may write one such mapping

$$f(z) = \int^z \sqrt{\frac{(w-a)(w+a)}{(w-a-b)(w+a+b)}} dw,$$

where $w_1 = f(-a-b), w_2 = f(-a), w_3 = f(a), w_4 = f(b)$. By [KS17, Lemma A.10], we know that

$$a = \frac{l}{2}(1 + o(1)), \quad b = \frac{4\epsilon}{\pi}(1 + o(1)), \quad \text{as } \epsilon/l \to 0. \tag{3.1}$$

By symmetry, we know that $Re\, f(ui) = 0$ for $u > 0$. Let $y > 0$ be such that $f(yi) = i$. Then $g(z) := f^{-1}(z)/y$ is a conformal map from $\mathbb{H} \setminus \mathfrak{R}(\epsilon, l)$ onto $\mathbb{H}$ that fixes i. By (3.1), we have $y = 1 + O(l\epsilon)$. Therefore

$$\text{cap}(\mathfrak{R}(\epsilon, l)) = \log |g'(i)| = -\log |f'(yi)| - \log y \le O(l\epsilon),$$

as desired. □

We now proceed to the proof of Proposition 3.1. Let $\phi : \Omega^\delta \to \mathbb{U}$ be the conformal map which maps 0 to 0 and a^δ to 1. We first want to build an "outer" shape which has bounded capacity $O(\epsilon^2)$. Let us consider for this a domain $K = K_\epsilon$ as in Lemma 3.2 of side-length $l = 2\epsilon$. This domain has a capacity bounded by $C\epsilon^2$. We shall also consider four scaled copies of such domains: $K_1 = 20K$, $\tilde{K}_1 = 10K$, $K_2 = 4K$ and $\tilde{K}_2 = 2K$. See Fig. 3.1. By the same Lemma 3.2, these domains have capacity bounded by $O(\epsilon^2)$. Our first key Lemma (which is of deterministic nature) can be stated as follows:

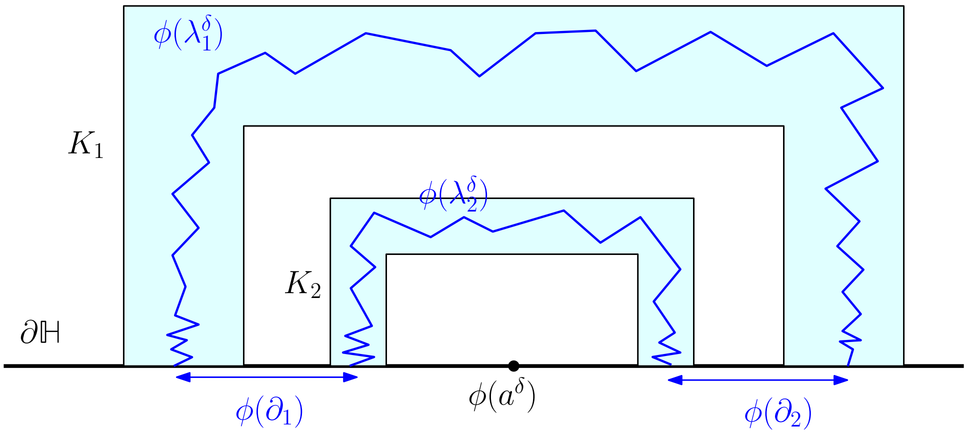

Figure 3.1: For simplicity, this figure as well as figures below are sketched in $\mathbb{H}$ instead of $\mathbb{U}$ as it should.

Lemma 3.3. *If δ is sufficiently small (i.e. $\delta \le \delta_0(\epsilon, r, R)$), there is a lattice path $\lambda_1 = \lambda_1^\delta$ in $\delta\mathbb{Z}^2 \cap \Omega^\delta$ which disconnects $\phi^{-1}(\partial K_1)$ from $\phi^{-1}(\partial \tilde{K}_1)$. Similarly there is a lattice path $\lambda_2 = \lambda_2^\delta$ which disconnects $\phi^{-1}(\partial K_2)$ from $\phi^{-1}(\partial \tilde{K}_2)$. See Fig. 3.1.*

Note here that there are no issues of subtle prime ends as Ω^δ is a $\delta\mathbb{Z}^2$ domain.

Proof. The proof relies on easy considerations of harmonic measure. Suppose one cannot find a path disconnecting say $\phi^{-1}(\partial K_1)$ from $\phi^{-1}(\partial \tilde{K}_1)$. This means that one can necessarily find a square Q_δ in $\delta\mathbb{Z}^2$ of side-length 3δ which intersects $\phi^{-1}(\partial K_1)$ as well as $\phi^{-1}(\partial \tilde{K}_1)$. As such, the conformal image $\phi(Q_\delta)$ intersects ∂K_1 and $\partial \tilde{K}_1$ and its diameter needs to be larger than $\mathrm{dist}(\partial K_1, \partial \tilde{K}_1) \ge \epsilon$. In particular the harmonic measure of $\phi(Q_\delta)$ seen from 0 in $\mathbb{U}$ (for the Brownian motion stopped when first exiting $\mathbb{U} \setminus f(Q_\delta)$) is larger than $\frac{1}{100}\epsilon$ (by easy considerations on Brownian motion). Now, by conformal invariance of harmonic measure, the harmonic measure of the square Q_δ seen from 0 (for the B.M. stopped when first exiting $\Omega^\delta \setminus Q_\delta$) needs to be larger than $\frac{1}{100}\epsilon$ as well. On the other hand, as Ω^δ is bounded ($\Omega^\delta \subset B(0,R)$), by monotony properties of harmonic measure, the above harmonic measure is smaller than the harmonic measure in $B(0,R)$ of Q_δ seen from 0. As the distance from Q_δ to the origin is larger than $\Omega(r)$ (which follows for example from Köbe's theorem), this later harmonic measure is smaller than $C_{R,r}(\log \frac{1}{\delta})^{-1}$. □

Remark 3.4. *Note that it is possible to obtain much better bounds on how small δ needs to be (with slightly more technical proofs though, this is why we sticked to that one). For example one way is to consider the extremal length in the annulus $A_1 := K_1 \setminus \tilde{K}_1$ from one of the arcs of A_1 intersecting $\partial\mathbb{U}$ to the other symetric arc. This extremal length is clearly bounded from above by some constant $M < \infty$. If a path as in Lemma 3.3 did not exist, then by designing an appropriate ρ-intensity on $\phi^{-1}(A_1)$ and using Beurling's estimate together with Koebe $\frac{1}{4}$-Theorem, one can show that the extremal length (which is conformally invariant) would need to be larger than $\Omega(1)\log(\frac{\epsilon^2}{20\delta})$ (ϵ^2 comes from Beurling here) which would yield a much better control on $\delta = \delta(\epsilon)$ in Lemma 3.3.*

End of the proof of Proposition 3.1. let Q be the lattice-quad in $\Omega^\delta \cap \delta\mathbb{Z}^2$ with two opposite arcs ∂_1 and ∂_2 along $\partial\Omega^\delta$ and its two other arcs are λ_1^δ and λ_2^δ. By construction, Q is a lattice-quad which has extremal length from ∂_1 to ∂_2 bounded from above by some constant $M < \infty$. It thus follows from the strong RSW Theorem 2.9 that uniformly in the boundary conditions around Q, there is a dual crossing from ∂_1 to ∂_2 in Q with probability larger than $c = c(M) > 0$. We now work in the domain $\mathbb{U}$ (using the map $\phi : \Omega^\delta \to \mathbb{U}$). The radial exploration path starts from $\phi(a^\delta)$. On the event that there is a dual crossing from $\phi(\partial_1)$ to $\phi(\partial_2)$, let τ be the first time the exploration path reaches the arc $\phi(\partial_1)$. (Recall the exploration path keeps dual edges on its right). By construction, the capacity of the interface stopped at $t = \tau$ is less than $C\epsilon^2$ (this follows from Lemma 3.2) and the harmonic measure of the dual arc of the exploration path at $t = \tau$ is larger (by monotony properties of harmonic measure) than the harmonic measure of the 4ϵ-long arc on the left of $\phi(a^\delta)$ and is thus larger than ϵ. This concludes our proof of Proposition 3.1. □

3.2 The chordal case

The additional slight technical difficulty in the chordal case is the fact that the harmonic measure is replaced by the notion of *renormalized harmonic measure* seen from the target b and $\partial\Omega$ may not be very regular around b (this issue obviously never arises with an interior target point in the radial case). Throughout this section, we shall assume that the boundary $\partial\Omega$ of our domain is smooth at least in a small neighbourhood of the target b (for Ω^δ, we shall even assume that $\partial\Omega^\delta$ is flat and oriented along e_x or e_y in a small neighbourhood of b^δ). This will allow us to define below an appropriate notion of **renormalized harmonic measure seen from** b. Later on in Section 5.5, we will explain how to derive the general result when $\partial\Omega$ is not necessarily smooth near b.

Definition 3.5. *Let $\Omega \subsetneq \mathbb{C}$ be a bounded simply connected domain which has a smooth boundary in some neighbourhood of $b \in \partial\Omega$. For any $A \subset \partial\Omega$, we define the* ***renormalized harmonic measure of*** *A to be*

$$\mathrm{RHM}_{\Omega,b}(A) := \lim_{u\to 0} \frac{1}{u} \mathbb{P}^{\,b+u\vec{n}}\big[B_\tau \in A\big]\,,$$

where $\vec{n}$ is the normal derivative of $\partial\Omega$ at b pointing inside Ω and τ is the first exit time that the Brownian motion B started at $b + u\vec{n}$ exits Ω.

If $a \neq b$ is a degenerate prime end on $\partial\Omega$, and if $\phi : (\Omega; a, b) \to (\mathbb{H}; 0, \infty)$ is the conformal map from Ω to $\mathbb{H}$ sending $a \to 0$, $b \to \infty$, and satisfying $\phi(b + u\vec{n}) \sim_{u\to 0} \frac{1}{u} i$, then the renormalized harmonic measure of A seen from b is the same as the classical RHM of $\phi(A)$ in the upper half plane $\mathbb{H}$ whose definition is given below.

Definition 3.6. *For any Borel set $A \subset \mathbb{R}$, we define the* ***renormalized harmonic measure*** *of A seen from infinity to be*

$$\mathrm{RHM}(A) = \mathrm{RHM}_{\mathbb{H}}(A) := \lim_{y\to\infty} \pi y\, \mathbb{P}^{\,i\cdot y}\big[B_\tau \in A\big]\,,$$

where τ is the first time the Brownian motion started at iy touches $\partial\mathbb{H}$. The multiplicative factor π is there so that $\mathrm{RHM}([0, L]) = L$. By conformal invariance of Brownian motion, we define in the same fashion the ***renormalized harmonic measure*** *for general hulls $H := \mathbb{H} \setminus K$ where K is any compact set of the plane as follows: for any subset $A \subset \partial H$, we define*

$$\mathrm{RHM}_H(A) := \lim_{y\to\infty} \pi y\, \mathbb{P}^{\,i\cdot y}\big[B_{\tau^H} \in A\big]\,.$$

We are now ready to state our key estimate in the chordal case.

Proposition 3.7. *There exist constants $c, C > 0$ such that for all inner radius $r > 0$, all outer radius $R > r$, all $\epsilon \in (0, \frac{1}{4}]$, all bounded lattice domain Ω^δ surrounded by $B(0,R)$, all marked point $a^\delta \in \partial\Omega^\delta \subset \delta\mathbb{Z}^2$, all marked point $b^\delta \in \partial\Omega^\delta$ such that $|b^\delta - a^\delta| > r$ and $\Omega^\delta \cap B(b^\delta, r)$ is identical to $B(b^\delta, r) \cap \mathbb{H}$ or $B(b^\delta, r) \cap (i\mathbb{H})$, then if the mesh δ is sufficiently small (the proof gives $\delta \leq \exp(-\frac{100}{C_{R,r}}\frac{1}{\epsilon})$), the FK-chordal exploration process η^δ from a^δ to b^δ will create, with probability at least $c > 0$, a dual arc of renormalized harmonic measure ϵ before accumulating a half-plane capacity larger than $C\,\epsilon^2$.*

The same statement holds if Ω^δ is $\mathbb{H} \cap \delta\mathbb{Z}^2 \setminus K^\delta$ where K^δ is a δ-lattice compact connected set which intersects $\mathbb{R}$.

Proof. The proof follows the same lines as in the radial case : let $\phi : \Omega^\delta \to \mathbb{H}$ be the conformal map such that $\phi(b^\delta + u\vec{n}) \sim_{u\to 0} \frac{1}{u} \cdot i$ and $\phi(a^\delta) = 0$. We build a quad Q via ϕ^{-1} in the same way (see Fig. 3.1). To show that this quad has the right properties, one needs to

1. adapt Lemma 3.3 to the chordal case.
2. rely on Lemma 2.2 to get the desired bound of $C\,\epsilon^2$ on the accumulated half-plane capacity.

To prove the analog of Lemma 3.3, one follows the same approach except one needs to replace the arguments based on the *harmonic measure* seen from 0 by arguments based instead on the **renormalised harmonic measure (RHM) seen from b^δ.** Indeed, suppose, one can find a δ-square Q_δ which intersects ∂K_1 and $\partial\tilde{K}_1$, this means its diameter is larger than $\mathrm{dist}(\partial K_1, \partial\tilde{K}_1) \geq \epsilon$. We need here to generalize slightly the notion of RHM as $\phi(Q_\delta)$ is not included in $\partial\mathbb{H}$, let us then define

$$\mathrm{RHM}(\phi(Q_\delta)) := \lim_{y\to\infty} \pi y\, \mathbb{P}^{\,i\cdot y}\big[B_\tau \in \phi(Q_\delta)\big]\,,$$

where τ is the first time the Brownian motion started at iy touches $\partial\mathbb{H} \cup \phi(Q_\delta)$. Since $\mathrm{diam}(\phi(Q_\delta)) \geq \epsilon$, it easily implies that its RHM is larger than $c_1\epsilon$. Now, as this quantity is conformally invariant, this means

$$\mathrm{RHM}_{\Omega^\delta, b^\delta}(Q_\delta) := \lim_{u\to 0} \frac{1}{u} \mathbb{P}^{\,b^\delta + u\vec{n}}\big[B_\tau \in Q_\delta\big] \geq c_1 \epsilon\,,$$

where τ is the first time the Brownian motion started at $b^\delta + u\vec{n}$ touches $\partial\Omega^\delta \cup Q_\delta$.

Then we consider the upper bound on $\mathrm{RHM}_{\Omega^\delta, b^\delta}(Q_\delta)$, to this end, we shall use our assumption that $\Omega^\delta \cap B(b^\delta, r)$ is identical to $B(b^\delta, r) \cap \mathbb{H}$ or $B(b^\delta, r) \cap (i\mathbb{H})$. Indeed, the fact the geometry is very simple around b^δ allows us to give an explicit expression of the RHM of Q_δ using (non-renormalized) harmonic measure, i.e. without limiting procedure $u \to 0$. More precisely, for any radius smaller than r, say $r_0 := \frac{r}{2}$, there is an explicit probability measure λ_{r_0} on $\partial B(b^\delta, r_0) \cap \Omega^\delta$, and a scaling parameter $k_{r_0} \asymp \frac{1}{r_0}$ such that

$$\mathrm{RHM}(Q_\delta) = k_{r_0} \int_{\partial B(b^\delta, r_0) \cap \Omega^\delta} \mathbb{P}^{\,x}\big[B_\tau \in Q_\delta\big] \lambda_{r_0}(dx)\,.$$

Now, as in the chordal case, it is easy to check that there is a constant $C_{r,R} < \infty$ s.t. uniformly in δ and $x \in \partial B(b^\delta, r_0) \cap \Omega^\delta$, $\mathbb{P}^{\,x}\big[B_\tau \in Q_\delta\big] \leq C_{r,R}(\log\frac{1}{\delta})^{-1}$. This gives us a contradiction and the rest of the proof is concluded in the exact same fashion. $\square$

4 Convergence of renormalized harmonic measure

Throughout this section, we will assume we are in the same setup as in Theorem 2.10. We thus have a sequence of domains $(\Omega^n; a^n, b^n)$ $(= (\Omega^{\delta_n}; a^{\delta_n}, b^{\delta_n}))$ which converge in Carathéodory sense to $(\Omega; a, b)$ and we are given conformal maps $\phi^n : \Omega^n \to \mathbb{H}$ and $\phi : \Omega \to \mathbb{H}$ satisfying the hypothesis in Theorem 2.10. Recall the main convergence result from that Theorem is its item (3) on the random curves in $\mathbb{H}$, $\eta^n := \phi^n(\gamma^n)$ and $\eta := \phi(\gamma)$ each parametrised by the half-plane capacity in $\mathbb{H}$. Furthermore recall that the Loewner driving function W of η is the limit in law of W^n, the driving function of η^n.

We now state the main result of this section.

Proposition 4.1. *Assume we are in the same setup as in Theorem 2.10. We also assume (using Skorokhod's representation theorem) that the random curves η^n and η (each parametrized by half-plane capacity) are coupled on the same probability space so that both W^n and η^n a.s. converge locally uniformly to W and η. Let $t \mapsto \theta^n(t)$ (resp. $t \mapsto \theta(t)$) denote the renormalized harmonic measure of the right boundary of $\eta^n([0,t])$ (resp. $\eta([0,t])$). Then θ^n a.s. converges to θ locally uniformly. I.e. for any $T > 0$, almost surely*

$$\|\theta^n - \theta\|_{\infty,T} = \sup_{t\in[0,T]} |\theta^n(t) - \theta(t)| \to 0, \quad \text{as } n \to \infty.$$

Proof. Let us fix some time $T > 0$. Recall we are in the setup of Theorems 2.7 and 2.10. By combining hypothesis of Theorem 2.7 with the estimate Lemma 2.3, one easily obtains that

$$M := \sup_n \mathrm{diam}(\eta^n[0,T]) \vee \mathrm{diam}(\eta[0,T]) < \infty \qquad \text{a.s.} \tag{4.1}$$

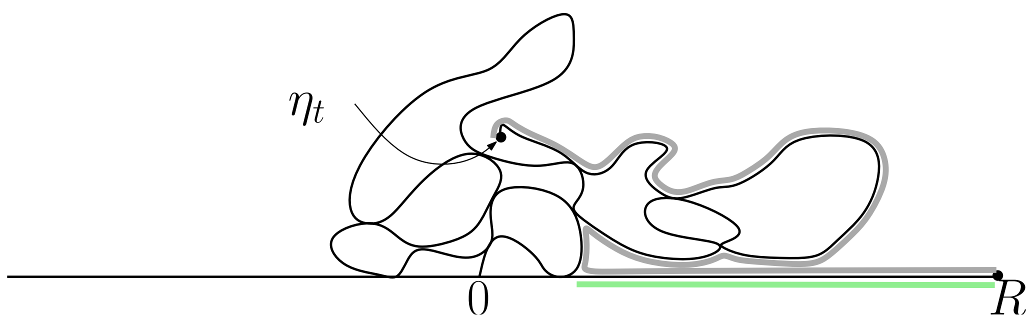

Figure 4.1: We express the harmonic measure on the right of the curve $\eta([0,t])$ as the difference of the harmonic measure of the grey and green arcs.

Our proof will be based on writing the renormalized harmonic function $\theta^n(t)$ as a difference of two quantities: indeed one has for any $t \in [0,T]$, and for any $R > M$,

$$\theta^n(t) = \mathrm{RHM}_{\mathbb{H}\setminus\eta^n[0,t]}(\eta^n(t), R) - \mathrm{RHM}_{\mathbb{H}\setminus\eta^n[0,t]}([0,R]). \tag{4.2}$$

Here $\mathrm{RHM}_{\mathbb{H}\setminus\eta^n[0,t]}(\eta^n(t), R)$ means the renormalized harmonic measure of the right boundary of $\eta^n[0,t]$ union $[0,R]$. See Fig. 4.1. Let g_t^n be the conformal map from the unbounded connected component of $\mathbb{H} \setminus \eta^n[0,t]$ onto $\mathbb{H}$ normalized at ∞. Now by conformal invariance of RHM, the first term is

$$\begin{aligned}\mathrm{RHM}_{\mathbb{H}\setminus\eta^n[0,t]}(\eta^n(t),R) &= g_t^n(R) - W^n(t)\\ &= R + O(\frac{(\mathrm{diam}\eta^n[0,t])^2}{R}) - W^n(t)\\ &= R + O(\frac{1}{R}) - W^n(t) \text{ by } (4.1)\\ &\to R + O(\frac{1}{R}) - W(t) \text{ uniformly on } [0,T]\,.\end{aligned}$$

Note that we also have

$$\mathrm{RHM}_{\mathbb{H}\setminus\eta[0,t]}(\eta(t),R) = R + O(\frac{1}{R}) - W(t)\,.$$

By letting $R \to \infty$ in the above two displayed equations and using (4.2), this concludes our proof modulo the remaining lemma below. □

Lemma 4.2. *For any $R > M$ (recall (4.1)),*

$$\mathrm{RHM}_{\mathbb{H}\setminus\eta^n[0,t]}([0,R]) \to \mathrm{RHM}_{\mathbb{H}\setminus\eta[0,t]}([0,R])$$

uniformly in $t \in [0,T]$ and the speed of convergence is independent of $R > M$.

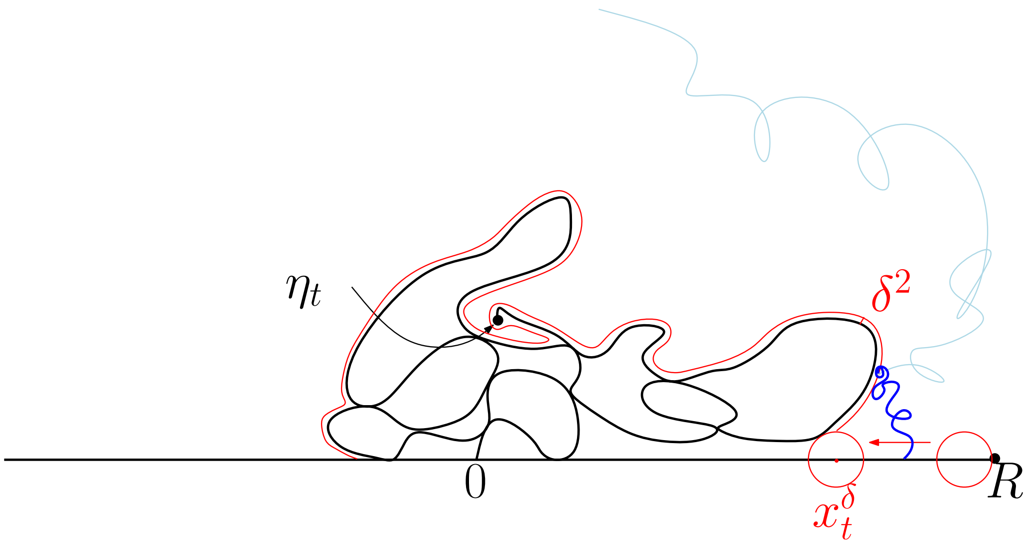

Figure 4.2: The point x_t^δ can be defined as the center of the δ-ball started at $+\infty$ and shifted towards the origin until it intersects for the first time the curve $\eta([0,t])$.

Proof. Let $\delta > 0$ be some small real number. Define (See Fig. 4.2)

$$x_t^\delta := \sup\{x \in \mathbb{R}_+ : B(x_t^\delta, \delta) \cap \eta[0,t] \neq \emptyset\}\,,$$

where $B(x_t^\delta, \delta)$ is the Euclidean ball centred around $x_t^\delta \in \partial\mathbb{H}$ of radius δ. Let us consider the domains

$$\begin{cases} H_t & := \mathbb{H} \setminus \mathrm{Hull}(\eta[0,t]) \\ H_t^\delta & := \mathbb{H} \setminus \left(\mathrm{Hull}(\eta[0,t])^{(\delta^2)} \cup B(x_t^\delta, \delta)\right)\,, \end{cases}$$

where $\mathrm{Hull}(\eta[0,t])^{(\delta^2)}$ is the δ^2-neighborhood in $\mathbb{H}$ of the hull generated by $\eta[0,t]$. See Fig. 4.2. Clearly, one has $H_t \subset H_t^\delta$ but we shall not use directly this fact. What we shall use instead is the fact that when $n = n(\delta)$ is sufficiently large then

$$H_t^n \subset H_t^\delta$$

where $H_t^n := \mathbb{H} \setminus \eta^n[0,t]$. Indeed this follows readily from the fact that $\eta^n \to \eta$ locally uniformly. This in turn implies immediately that for $n \geq n(\delta)$,

$$\mathrm{RHM}_{H_t^\delta}([0,R]) \leq \mathrm{RHM}_{\mathbb{H}\setminus\eta^n[0,t]}([0,R]) \tag{4.3}$$

Let us now show that there exists some continuous function $f : [0,1] \to [0,1]$ with $f(0) = 0$ s.t.

$$\mathrm{RHM}_{H_t}([0,R]) \leq \mathrm{RHM}_{H_t^\delta}([0,R]) + f(\delta) \tag{4.4}$$

To show (4.4), we consider Brownian motion starting at $\frac{1}{u}i$ and stopped the first time it hits ∂H_t^δ. Let τ^δ denote that stopping time. As $H_t \subset H_t^\delta$, one has $\tau^\delta \leq \tau$. Our goal is then to compare $\frac{1}{u}\mathbb{P}^{\frac{1}{u}i}\big[B_\tau \in [0,R]\big]$ with $\frac{1}{u}\mathbb{P}^{\frac{1}{u}i}\big[B_{\tau^\delta} \in [0,R]\big]$. The difference is given by

$$\frac{1}{u}\mathbb{P}^{\frac{1}{u}i}\big[B_{\tau^\delta} \in \partial\mathrm{Hull}(\eta[0,t])^{(\delta^2)} \setminus B(x_t^\delta,\delta) \text{ and } B_\tau \in [0,R]\big] + \frac{1}{u}\mathbb{P}^{\frac{1}{u}i}\big[B_{\tau^\delta} \in \partial B(x_t^\delta,\delta)\big]$$

The second term is less than the RHM of $B(x_t^\delta,\delta)$ in the full $\mathbb{H}$ and is thus bounded from above by $O(\delta)$ as $\delta \to 0$. For the first term, notice that

- B_{τ^δ} is at distance δ^2 from $\mathrm{Hull}(\eta[0,t])$
- Because of the definition of x_t^δ and $B(x_t^\delta,\delta)$, the Brownian motion needs to travel at distance at least δ between times τ^δ and τ in order to reach $[0,R]$.

Using Beurling's estimate, this happens with probability at most $O(\sqrt{\delta^2/\delta}) = O(\delta^{1/2})$. This gives us our desired bound with a continuous function f satisfying $f(\delta) = \Omega(\delta^{1/3})$.

We have thus shown that

$$\begin{aligned}\mathrm{RHM}_{H_t}([0,R]) &\leq \mathrm{RHM}_{H_t^\delta}([0,R]) + f(\delta)\\ &\leq \mathrm{RHM}_{\mathbb{H}\setminus\eta^n[0,t]}([0,R]) + f(\delta) \text{ for } n \geq n(\delta)\end{aligned}$$

Using the exact same proof in the reverse direction (by now defining a point $x_{n,t}^\delta$ which could possibly be very far from x_t^δ), we obtain that for $n \geq n(\delta)$,

$$\mathrm{RHM}_{\mathbb{H}\setminus\eta^n[0,t]}([0,R]) \leq \mathrm{RHM}_{H_t}([0,R]) + f(\delta)$$

which thus concludes the proof. □

5 Proof of Theorem 1.1

5.1 General setup

In this section, we explain the general setup as in Theorem 1.1. Suppose Ω is a bounded simply connected subset of $\mathbb{C}$ with three distinct boundary points (degenerate prime ends) a, b, c in counterclockwise order. Let $(\Omega^\delta; a^\delta, b^\delta, c^\delta)$ be a sequence of domains on $\delta\mathbb{Z}^2$ converging to $(\Omega; a, b, c)$ in the Carathéodory sense: there exist conformal map $\phi : (\Omega; a, c) \to$

$(\mathbb{H}; 0, \infty)$ and $\phi^\delta : (\Omega^\delta; a^\delta, c^\delta) \to (\mathbb{H}; 0, \infty)$ so that $\phi^\delta \to \phi$ as $\delta \to 0$ uniformly on compact subset of Ω and $\phi^\delta(b^\delta) \to \phi(b)$.

Consider the FK-Ising model on Ω^δ with Dobrushin boundary conditions: edges along ∂_{ba} are wired and the dual-edges of ∂^*_{ab} are dual-wired. Suppose γ^δ is the exploration path from a^δ to c^δ, as explained in Section 2.4. Suppose $\delta_n \to 0$, and denote by $\mathcal{A}^n = \mathcal{A}^{\delta_n}$ for $\mathcal{A} = \Omega, a, b, c, \gamma, \phi, S$ and denote $\eta^n = \phi^{\delta_n}(\gamma^{\delta_n})$. We parameterize η^n by the half-plane capacity and denote by W^n the driving function and by g^n_t the corresponding conformal maps in the definition of Loewner chain. Let $\theta^n(t)$ be the renormalized harmonic measure of the right side of $\eta^n[0,t]$ union $[0, \phi^n(b^n)]$ in $\mathbb{H} \setminus \eta^n[0,t]$ seen from ∞.

Definition 5.1 (Definition of stopping times $\{T^{n,\epsilon}_k, S^{n,\epsilon}_k\}_{k\geq 1}$)**.** *Fix $\epsilon \geq 10\sqrt{\delta_n}$. Define $T^{n,\epsilon}_1$ to be the first time that θ^n is greater than ϵ (if $\theta^n(0) = \phi^n(b^n) \geq \epsilon$, then $T^{n,\epsilon}_1 = 0$). Define $S^{n,\epsilon}_1$ to be the first time after $T^{n,\epsilon}_1$ that γ^n hits the boundary arc ∂_{bc}. Generally, for $k \geq 1$, let $T^{n,\epsilon}_{k+1}$ be the first time after $S^{n,\epsilon}_k$ that θ^n exceeds ϵ and define $S^{n,\epsilon}_{k+1}$ to be the first time after $T^{n,\epsilon}_{k+1}$ that γ^n hits the boundary arc ∂_{bc}.*

In this way, we decompose the process η^n as follows: from time $T^{n,\epsilon}_k$ to $S^{n,\epsilon}_k$, the exploration process is similar to the situation when the boundary conditions is Dobrushin; from time $S^{n,\epsilon}_k$ to $T^{n,\epsilon}_{k+1}$, we know little about the process, and we call this part as *dust*.

As the sequence $\{\gamma^n\}$ satisfies Condition C2 in Definition 2.6 (due to Theorem 2.9), from Theorem 2.7, both sequences $\{\gamma^n\}$ and $\{\eta^n\}$ are tight. We can extract subsequence, which we still denote by $\{\gamma^n\}$ and $\{\eta^n\}$, such that W^n converges in distribution to W and η^n converges in distribution to η locally uniformly, and that η satisfies the properties in Theorem 2.7. We couple them on the same probability space so that they converge almost surely.

For the limiting process η, recall from Theorem 2.7 that W is its driving function and let g_t be the corresponding conformal maps. Define $\theta(t)$ to be the renormalized harmonic measure of the right boundary of $\eta[0,t]$ union $[0, \phi(b)]$. Fix $T > 0$ large and define

$$\|\mathcal{A}^n - \mathcal{A}\|_{\infty,T} := \sup\{|\mathcal{A}^n(t) - \mathcal{A}(t)| : t \in [0,T]\},$$

for $\mathcal{A} = \eta, W, \theta$. By Proposition 4.1, we know that $\|\theta^n - \theta\|_{\infty,T} \to 0$ almost surely.

For the limiting process η, we define the stopping times similarly. Let T^ϵ_1 be the first time that θ is greater than ϵ. Define S^ϵ_1 to be the first time after T^ϵ_1 that θ hits zero. Generally, for $k \geq 1$, define T^ϵ_{k+1} to be the first time after S^ϵ_k that θ exceeds ϵ and define S^ϵ_{k+1} to be the first time after T^ϵ_{k+1} that θ hits zero.

The goal of Theorem 1.1 is to identify the law of η and our strategy is the following:

- First we argue that the following two processes are close:

$$\left(\theta^n(t), T^{n,\epsilon}_k \leq t \leq S^{n,\epsilon}_k\right) \quad \text{and} \quad \left(\theta(t), T^\epsilon_k \leq t \leq S^\epsilon_k\right).$$

- Then, using Theorem 2.10, we shall argue that $\left(\theta^n(t), T^{n,\epsilon}_k \leq t \leq S^{n,\epsilon}_k\right)$ converges in distribution to Bessel excursion, and thus $(\theta(t), T^\epsilon_k \leq t \leq S^\epsilon_k)$ is a Bessel excursion.

- For the dusts—$(\theta^n(t), S^{n,\epsilon}_k \leq t \leq T^{n,\epsilon}_{k+1})_k$—we control them in a uniform way thanks to Proposition 3.7 and argue that they will disappear as $\epsilon \to 0$. Then by Proposition 2.1, we conclude that θ is a Bessel process.

- Finally, we use Lemma 2.4 to extract W from θ and conclude that η is an $\mathrm{SLE}_\kappa(\kappa-6)$.

However, it is quite delicate to make this strategy work. The first issue is that, although the processes (η^n, W^n, θ^n) are close to (η, W, θ), we do not know a priori whether the stopping times $(T_k^{n,\epsilon}, S_k^{n,\epsilon})$ are close to the stopping times $(T_k^\epsilon, S_k^\epsilon)$. This will be proved in Proposition 5.2 and this turns out to be more technical than one might expect.

For the second item, the issue is that one needs to argue the processes are not moving much for θ^n and θ on

$$[T_k^{n,\epsilon} \wedge T_k^\epsilon, T_k^{n,\epsilon} \vee T_k^\epsilon].$$

This issue will be solved by equicontinuity, see Section 5.3.

For the third item, one issue is that Proposition 3.7 requires the domain to be flat near b (See Proposition 3.7 for a precise assumption). This is why we will restrict to such domains in Subsections 5.2 to 5.4 and will get back to the general setting of Theorem 1.1 only in Subsection 5.5.

Another issue concerns the convergence of conditional distribution, or the passage of Markov property to the limit. In the discrete, the exploration process η^n has domain Markov property and we know η^n converges to η. But the domain Markov property does not pass to η for free, as it was pointed out in [Sch00, Proposition 4.2 and Section 5] in the setting of loop-erased random walk. For simplicity, we first discuss the following two pieces

$$X^n := (\eta^n(t), 0 \le t \le T_1^{n,\epsilon}) \quad \text{and} \quad Y^n := (\eta^n(t), T_1^{n,\epsilon} \le t \le S_1^{n,\epsilon}).$$

Define the conformal map $G^n(\cdot) := g^n_{T_1^{n,\epsilon}}(\cdot) - W^n(T_1^{n,\epsilon})$. Note that G^n is a measurable function of X^n. In the limiting process η, we define

$$X := (\eta(t), 0 \le t \le T_1^\epsilon) \quad \text{and} \quad Y := (\eta(t), T_1^\epsilon \le t \le S_1^\epsilon).$$

Define the conformal map $G(\cdot) := g_{T_1^\epsilon}(\cdot) - W(T_1^\epsilon)$ and note again that G is a measurable function of X. At this point, we have $\eta^n \to \eta$ and $S_1^{n,\epsilon} \to S_1^\epsilon$, and hence we have the convergence of the concatenation of (X^n, Y^n) to the concatenation of (X, Y) in the metric (2.6), and we want to argue that the conditional law of Y_n given X_n converges to the conditional law of Y given X. However, this is generally false without further condition on (X^n, Y^n), see for example the discussion in [Gog94].

In our setting, we do have further properties below on the pair (X^n, Y^n) which allow us to conclude.

- As (η^n, W^n, θ^n) converges to (η, W, θ) and $T_1^{n,\epsilon} \to T_1^\epsilon$ almost surely, we see $G^n \to G$ in Carathéodory sense. (This follows for example from Caratheodory kernel theorem). As $\eta^n \to \eta$, together with equicontinuity and the fact that $T_1^{n,\epsilon} \to T_1^\epsilon$ a.s. we obtain that X^n converges to X. Consider the $G^n(\eta^n|_{t \ge T_1^{n,\epsilon}})$. The collection $\{G^n(\eta^n|_{t \ge T_1^{n,\epsilon}})\}_n$ satisfies Condition C2 due to Theorem 2.9, hence it is tight in the topologies in Theorem 2.7. Combining with the equicontinuity and $S_1^{n,\epsilon} \to S_1^\epsilon$, we will show in Lemma 5.8 that $G(Y)$ is the only possible subsequential limit, thus $G^n(Y^n)$ converges to $G(Y)$.

- Since Y^n is an exploration path in $\mathbb{H} \setminus X^n$ with Dobrushin boundary conditions (stopped at the disconnection time), using Theorem 2.10 and Lemma 2.5, we will obtain in Section 5.3 that $G^n(Y^n)$ converges to an $\mathrm{SLE}_\kappa(\kappa - 6)$ in $\mathbb{H}$ from 0 to ∞ with force point at ϵ (stopped at the disconnection time). This will be the purpose of Proposition 5.7 and the key point there will be that $G(Y)$ is independent of X.

Going back to the random process θ, we will see in Section 5.3 that by combining these three observations, one obtains: $(\theta(t), 0 \le t \le T_1^\epsilon)$ and $(\theta(t), T_1^\epsilon \le t \le S_1^\epsilon)$ are independent and $(\theta(t), T_1^\epsilon \le t \le S_1^\epsilon)$ has the law of Bessel excursion.

For general $k \geq 1$, the above argument applies to the following two pieces

$$(\eta^n(t), 0 \leq t \leq T_k^{n,\epsilon}) \quad \text{and} \quad (\eta^n(t), T_k^{n,\epsilon} \leq t \leq S_k^{n,\epsilon}).$$

As hinted above, Section 5.4 will combine the above analysis together with Section 3, as well as Proposition 2.1 in order to conclude that θ is a Bessel process. In the case of domains which are flat near b we will conclude the proof of Theorem 1.1 by relying on Lemma 2.4. Finally we will extend to result to general Jordan domains in Section 5.5 using a RSW coupling argument. (This latter argument is often implicit in the literature but is carefully written down here).

From now on (all the way to Subsection 5.5), we shall assume that $(\Omega; a, b)$ is r-flat around the target b, i.e. that there exists $r > 0$ such that $\Omega \cap B(b, r)$ is identical to $B(b, r) \cap \mathbb{H}$ or $B(b, r) \cap (i\mathbb{H})$ (this will allow us to rely on Proposition 3.7).

5.2 Convergence of discrete stopping times to their continuous analogs

In this section we shall prove the following key control on $\{T_k^{n,\epsilon}, T_k^\epsilon, S_k^{n,\epsilon}, S_k^\epsilon\}$

Proposition 5.2. *Assume we are in the above setup where, in particular, $\eta^n \to \eta$ locally uniformly and $W^n \to W$ locally uniformly. Then we have for any $k \geq 1$ and as $n \to \infty$,*

$$\begin{cases} T_k^{n,\epsilon} \to T_k^\epsilon \text{ in probability} \\ S_k^{n,\epsilon} \to S_k^\epsilon \text{ in probability} \end{cases}$$

We shall in fact need the following slightly preciser version. For any fixed $T > 0$, there exists a sequence $\{\alpha_n\}$ converging to zero such that the following holds:

$$\mathbb{P}\big[\exists k \geq 1, \text{ s.t. } T_k^\epsilon \leq T - 1 \text{ and } |T_k^{n,\epsilon} - T_k^\epsilon| > \alpha_n\big] \leq \alpha_n$$
$$\mathbb{P}\big[\exists k \geq 1, \text{ s.t. } S_k^\epsilon \leq T - 1 \text{ and } |S_k^{n,\epsilon} - S_k^\epsilon| > \alpha_n\big] \leq \alpha_n$$

Remark 5.3. *This is in the same flavour as [Wer07, Lemma 3.1] which would correspond to $S_k^{n,\epsilon} \to S_k^\epsilon$ a.s. (not to $T_k^{n,\epsilon} \to T_k^\epsilon$), the proof of which was left as a homework exercise. However, we find this exercise not that easy for the following reasons:*

(1) First, if we stop the joint exploration paths $(\eta^n, \tilde{\eta})$ at time S, indeed η^n is close to disconnecting and it is tempting to conclude by some careful use of RSW. But one important issue is that S is a stopping time for $\tilde{\eta}$ but not for η^n. Because of that, we are not allowed to use the discrete domain Markov property and a rather delicate analysis cannot be avoided it seems.

(2) Second, stopping the curve when it is close to disconnecting 0 from $\partial\mathbb{U}$ needs to be done with some care. For example, being close in $\|\cdot\|_\infty$ to disconnection does not prevent from having a dual harmonic arc with large harmonic measure seen from 0. This is why in our proof below, we rely on stopping times built from harmonic measure seen from 0 instead of Euclidean distance from disconnection.

(3) The stopping times S_k^ϵ are geometric disconnection times and may possibly be analysed through a different route using the absence of 6-arms events at large scales. On the other hand the stopping times T_k^ϵ, based on the R.H.M, are of a different nature and it is less clear how to carefully adapt a proof based on the 6 arms events in this case.

Let us start as a warm-up with the following Lemma

Lemma 5.4. *Assume for simplicity that we are in the case where $\theta^n(0) = \theta(0) = 0$ (no free arc at the beginning of the exploration). For any fixed $T \geq 2$, We have for any $u > 0$,*

$$\mathbb{P}\big[T_1^\epsilon \leq T-1 \text{ and } |T_1^{n,\epsilon} - T_1^\epsilon| > u\big] \to 0\,, \text{ as } n \to \infty.$$

Proof. For any $r < \epsilon/2$, define the stopping times $T_1^{n,\epsilon-2r}$ and $T_1^{n,\epsilon+2r}$ exactly as $T_1^{n,\epsilon}$. By definition and monotony, one clearly has

$$T_1^{n,\epsilon-2r} \leq T_1^{n,\epsilon} \leq T_1^{n,\epsilon+2r} \tag{5.1}$$

Now, let us show that there exists a function $f(r)$ which goes to zero as $r \to 0$, and which is such that uniformly in $n \geq n(r)$, one has

$$\mathbb{P}\big[T_1^{n,\epsilon+2r} - T_1^{n,\epsilon-2r} \geq f(r)\big] \leq f(r) \tag{5.2}$$

Remark 5.5. *Recall the main issue in the current proof is that any interaction between η and η^n may ruin the domain Markov property for η^n. The above estimate does not involve the limiting curve η in the joint coupling and it is therefore much safer to prove such an estimate using standard arguments.*

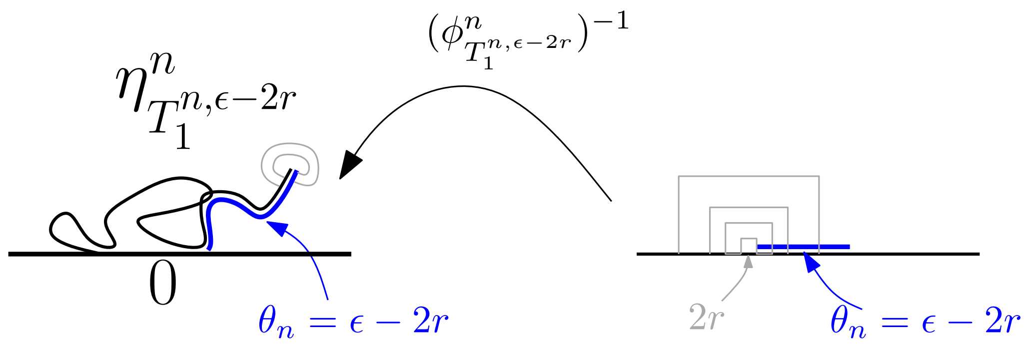

Figure 5.1

To Prove the estimate (5.2), we proceed exactly as in Section 3. Namely we use Strong-RSW in appropriate quads in the discrete domain $\mathbb{H} \setminus \eta^n[0, T_1^{n,\epsilon-2r}]$ which are defined as conformal images via $(\phi^n_{T_1^{n,\epsilon-2r}})^{-1}$ of well-chosen rectangular quads in $\mathbb{H}$. As the arguments are very similar to the ones in Section 3, we leave the details to the reader and refer to Fig. 5.1 for the construction of these quads.

Now, using that $\eta^n \to \eta$ locally uniformly and $W^n \to W$ locally uniformly, recall we have by Proposition 4.1 that $\|\theta^n - \theta\|_{\infty,T} \to 0$ as $n \to \infty$. In particular, if we define the event

$$E^{n,r} := \{\|\theta^n - \theta\|_{\infty,T} \leq r\}\,,$$

then we have for any $r > 0$, $\mathbb{P}\big[E^{n,r}\big] \to 1$ as $n \to \infty$. The main observation which remains in order to prove Lemma 5.4 is that on the event

$$\{\|\theta^n - \theta\|_{\infty,T} \leq r\}$$

we must have the inequality

$$T_1^{n,\epsilon-2r} \leq T_1^\epsilon \leq T_1^{n,\epsilon+2r}$$

at least if $T_1^{n,\epsilon+2r}$ is not too big (it needs to be less than T which it does with high probability on the event $T_1^\epsilon \le T-1$ thanks to the estimate (5.2)). This together with (5.1) implies readily that on the event $\{\|\theta^n - \theta\|_{\infty,T} \le r\} \cap \{T_1^{n,\epsilon+2r} - T_1^{n,\epsilon-2r} < f(r)\}$, one has $|T_1^{n,\epsilon} - T_1^\epsilon| < f(r)$ if $T_1^\epsilon \le T-1$. As $\liminf \mathbb{P}\big[E^{n,r} \cap \{T_1^{n,\epsilon+2r} - T_1^{n,\epsilon-2r} < f(r)\}\big] \ge 1-f(r)$, this concludes the proof of Lemma 5.4 by choosing r arbitrarily small. □

In order to prove Proposition 5.2, we would like to iterate the same idea to the later stopping times $S_k^{n,\epsilon}, T_k^{n,\epsilon}$ etc.

Proof of Proposition 5.2. Let us start by explaining in details how to handle the convergence of the next stopping time, i.e. $S_1^{n,\epsilon} \overset{\text{Prob.}}{\to} S_1^\epsilon$. Namely we wish to prove that for any $u > 0$,

$$\mathbb{P}\big[S_1^\epsilon \le T-1 \text{ and } |S_1^{n,\epsilon} - S_1^\epsilon| > u\big] \to 0\,, \text{ as } n \to \infty. \tag{5.3}$$

To prove this, we face two (slight) technical difficulties:

1. The first one is that $S_1^{n,\epsilon}$ will be close to S_1^ϵ only if the earlier stopping times $T_1^{n,\epsilon}$ and T_1^ϵ will be close as well. This must appear in the proof somewhere.

2. The second issue is that there is no monotonicity such as the one we used above (namely, $T_1^{n,\epsilon-2r} \le T_1^\epsilon \le T_1^{n,\epsilon+2r}$). We will still have an analog of the left inequality, but the R.H.S will be replaced by the inequality $S_1^\epsilon \le \liminf_{n\to\infty} S_1^{n,\epsilon}$ which can be seen as a deterministic statement given the fact that $\theta^n \to \theta$ uniformly on $[0,T]$.

Let us introduce the following stopping times which will have useful monotony properties:

$$\begin{cases} \tilde S_1^{n,\epsilon,2r} := \inf\{t > T_1^{n,\epsilon-2r}, \text{s.t. } \theta^n(t) \le 2r\} \\ \hat S_1^{n,\epsilon,2r} := \inf\{t > T_1^{n,\epsilon-2r}, \text{s.t. } \gamma^n \text{ hits the boundary arc } \partial_{bc}\} \end{cases}$$

Note first that it always the case that

$$\tilde S_1^{n,\epsilon,2r} \le S_1^{n,\epsilon} \tag{5.4}$$

Also, note that on the event $\{S_1^\epsilon \le T-1\} \cap \{\|\theta^n - \theta\|_{\infty,T} \le r\}$, we have that

$$\tilde S_1^{n,\epsilon,2r} \le S_1^\epsilon \tag{5.5}$$

Furthermore, exactly as for the estimate (5.2), one can prove in the same fashion that there exists a function $f(r)$ which goes to zero as $r \to 0$, and which is such that uniformly in $n \ge n(r)$, one has

$$\mathbb{P}\big[\hat S_1^{n,\epsilon,2r} - \tilde S_1^{n,\epsilon,2r} \ge f(r)\big] \le f(r) \tag{5.6}$$

Finally, using the estimate (5.2) as well as the equicontinuity of the set of functions $\{\theta^n\}_{n\ge1}$ restricted to the interval $[0,T]$ (this equicontinuity follows form the uniform convergence of θ^n towards the continuous θ), we deduce that for n large enough,

$$\mathbb{P}\big[\hat S_1^{n,\epsilon,2r} = S_1^{n,\epsilon}\big] \ge 1 - 2\,f(r)\,. \tag{5.7}$$

Indeed, one term $f(r)$ comes from the possibility that $T_1^{n,\epsilon+2r} \gg T_1^{n,\epsilon-2r}$ which could prevent the above equality to hold and is dominated thanks to (5.2), the second term $f(r)$ comes from the unlikely event that γ^n would hit the arc ∂_{bc} strictly between $T_1^{n,\epsilon-2r}$ and

$T_1^{n,\epsilon}$. This possibility is easily controlled using the equicontinuity of $\{\theta^n\}_{n\geq 1}$. Combining the above four estimates (5.4), (5.5), (5.6), (5.7), we obtain that for any $u > 0$,

$$\mathbb{P}\big[S_1^\epsilon - S_1^{n,\epsilon} < -u\big] \to 0 \text{ as } n \to \infty.$$

For the other direction, we rely on a completely different argument (already suggested in [Wer07, Lemma 3.1]) which is based on the Lemma stated below. Indeed it readily implies that

$$\mathbb{P}\big[S_1^\epsilon - S_1^{n,\epsilon} > +u\big] \to 0 \text{ as } n \to \infty.$$

which concludes our proof at least for $S_1^{n,\epsilon} \overset{\text{Prob.}}{\to} S_1^\epsilon$. □

Lemma 5.6. *On the event $S_1^\epsilon \leq T-1$,*

$$S_1^\epsilon \leq \liminf S_1^{n,\epsilon}$$

Proof. Let us argue by contradiction. Suppose this is not the case, then there exists $\alpha > 0$ s.t. for infinitely many $n_k \in \mathbb{N}$,

$$S_1^{n_k,\epsilon} \leq S_1^\epsilon - \alpha\,.$$

Using Beurling's estimate one has

$$\begin{cases} |\theta^{n_k}(T_1^{n_k,\epsilon}) - \epsilon| \leq 10\sqrt{\delta_{n_k}} \\ \theta^{n_k}(S_1^{n_k,\epsilon}) \leq 10\sqrt{\delta_{n_k}} \end{cases}$$

By possibly further extracting so that $T_1^{n_k,\epsilon}$ and $S_1^{n_k,\epsilon}$ both converge and using the fact that θ^{n_k} converges uniformly to θ on $[0,T]$, we thus reach a contradiction, as S_1^ϵ should then be smaller than $\lim S_1^{n_k,\epsilon}$. □

Proof of Proposition 5.2 continued. For the general case, $k \geq 2$, we can proceed inductively on $k \geq 1$. The induction hypothesis being that indeed, $T_j^{n,\epsilon} \overset{\text{Prob.}}{\to} T_j^\epsilon$ and $S_j^{n,\epsilon} \overset{\text{Prob.}}{\to} S_j^\epsilon$ for all $j \leq k-1$. Then, to propagate the induction hypothesis, we proceed as follows: say we have proved all stopping times converge in probability all the way to $T_k^{n,\epsilon} \overset{\text{Prob.}}{\to} T_k^\epsilon$ and we wish to control the next one, i.e. $S_k^{n,\epsilon} \overset{\text{Prob.}}{\to} S_k^\epsilon$. For the lower bound, we set up the following stopping times:

$$\begin{cases} \tilde{T}_2^{n,\epsilon,2r} := \inf\{t > \tilde{S}_1^{n,\epsilon,2r}, \theta^n(t) = \epsilon - 2r\} \\ \hat{T}_2^{n,\epsilon,2r} := \inf\{t > \hat{S}_1^{n,\epsilon,2r}, \theta^n(t) = \epsilon + 2r\} \\ \tilde{S}_2^{n,\epsilon,2r} := \inf\{t > \hat{T}_2^{n,\epsilon,2r}, \theta^n(t) = 2r\} \\ \hat{S}_2^{n,\epsilon,2r} := \inf\{t > \tilde{T}_2^{n,\epsilon,2r}, \text{s.t. } \gamma^n \text{ hits the boundary arc } \partial_{bc}\} \\ \tilde{T}_3^{n,\epsilon,2r} := \inf\{t > \tilde{S}_2^{n,\epsilon,2r}, \theta^n(t) = \epsilon - 2r\} \\ \ldots \end{cases}$$

The advantage of these definitions is that stopping times $\tilde{T}_k$ and $\hat{S}_k$ (resp. $\tilde{S}_k$ and $\hat{S}_k$) are close with high probability, and the following monotonies always hold:

$$\begin{aligned} \tilde{T}_k^{n,\epsilon,2r} &:= \inf\{t > \tilde{S}_{k-1}^{n,\epsilon,2r}, \theta^n(t) = \epsilon - 2r\} \leq T_k^\epsilon \\ \tilde{S}_k^{n,\epsilon,2r} &:= \inf\{t > \tilde{T}_k^{n,\epsilon,2r}, \theta^n(t) = \epsilon - 2r\} \leq S_k^\epsilon. \end{aligned}$$

The same proof as the one above implies that for all $k \geq 2$ and $u > 0$,

$$\mathbb{P}\big[T_k^\epsilon - T_k^{n,\epsilon} < -u\big] \vee \mathbb{P}\big[S_k^\epsilon - S_k^{n,\epsilon} < -u\big] \to 0 \text{ as } n \to \infty.$$

Now, for the upper bound, exactly as in Lemma 5.6, one has for all $k \geq 2$,

$$T_k^\epsilon \leq \liminf T_k^{n,\epsilon}, \quad S_k^\epsilon \leq \liminf S_k^{n,\epsilon}$$

which concludes the proof that one can iterate from $j \leq k-1$ to k in the same way as for $S_1^{n,\epsilon} \overset{\text{Prob.}}{\to} S_1^\epsilon$ above.

To conclude our proof of Proposition 5.2, one still need to handle a potentially large number of stopping times. Indeed the main statement in Proposition 5.2 provides a control on ALL stopping times $T_k^{n,\epsilon}$ or $S_k^{n,\epsilon}$ which arise below T. To conclude, we thus rely once again on the equicontinuity of $\{\theta^n\}$ on $[0,T]$ (which again follows from $\theta^n \to \theta$ uniformly on $[0,T]$). In particular, there is a random $\delta = \delta(\omega, \epsilon) > 0$ a.s., s.t. for all $n \geq 1$ and any $0 \leq s < t \leq T$, with $|s-t| < \delta$

$$|\theta^n(s) - \theta^n(t)| < \epsilon/2\,.$$

This implies readily that one cannot have more than $T\delta^{-1}$ stopping times before time T. Now by combining the fact that $\mathbb{P}\big[\delta(\omega,\epsilon) > \alpha\big] \to 1$ as $\alpha \searrow 0$ and a straightforward union bound argument, we conclude the proof of Proposition 5.2 with a choice of $\{\alpha_n\}_n$ converging sufficiently slowly to zero. ☐

5.3 Convergence in law to one Bessel excursion

In this section, we will show the following proposition.

Proposition 5.7. *The law of $(\theta(t)/\sqrt{\kappa}, T_1^\epsilon \leq t \leq S_1^\epsilon)$ is the same as a Bessel process of dimension $3 - 8/\kappa$ starting from ϵ and stopped when it reaches zero where $\kappa = 16/3$. Moreover, it is independent of $(\theta(t), t \leq T_1^\epsilon)$.*

We will give a detailed proof of Proposition 5.7 in this section, and most of the arguments can be applied verbatim for the future excursions.

From Proposition 5.2, we have

$$\mathbb{P}\left[S_1^\epsilon \leq T-1, |T_1^{n,\epsilon} - T_1^\epsilon| > \alpha_n, |S_1^{n,\epsilon} - S_1^\epsilon| > \alpha_n\right] \leq \alpha_n.$$

We may choose a subsequence $n_j \to \infty$ such that $\sum_j \alpha_{n_j} < \infty$. Then we have

$$\sum_j \mathbb{P}\left[S_1^\epsilon \leq T-1, |T_1^{n_j,\epsilon} - T_1^\epsilon| > \alpha_{n_j}, |S_1^{n_j,\epsilon} - S_1^\epsilon| > \alpha_{n_j}\right] < \infty.$$

By Borel-Cantelli Lemma, we have

$$T_1^{n_j,\epsilon} \to T_1^\epsilon, \quad S_1^{n_j,\epsilon} \to S_1^\epsilon, \quad \text{a.s. on } \{S_1^\epsilon \leq T-1\}.$$

Lemma 5.8. *Recall the definition of X^n, Y^n, G^n and X, Y, G as in Section 5.1. On the event $\{S_1^\epsilon \leq T-1\}$, the process $G^{n_j}(Y^{n_j})$ converges to $G(Y)$ almost surely.*

Proof. Consider the sequence $\{G^{n_j}(\eta^{n_j}|_{t \geq T_1^{n_j,\epsilon}})\}_j$, it satisfies Condition C2 by Theorem 2.9. Then it is tight as in Theorem 2.7. Suppose $G^{n_{k_j}}(\eta^{n_{k_j}}|_{t \geq T_1^{n_{k_j},\epsilon}})$ is a converging subsequence and the limit is $\tilde{\eta}$ with driving function $\tilde{W}$. We have the following observation.

- Applying Theorem 2.7 to $G^{n_{k_j}}(\eta^{n_{k_j}}|_{t\geq T_1^{n_{k_j},\epsilon}})$, we know that $W^{n_{k_j}}$, restricted to $[T_1^{n_{k_j},\epsilon}, T]$, converges to $\tilde{W}$ locally uniformly.
- Applying Theorem 2.7 to η^n, we know that W^n converges to W locally uniformly. In particular, W^n converges to W uniformly on $[0, T]$. The uniform convergence implies the equicontinuity of the sequence $\{W^n\}_n$ on the interval $[0, T]$.
- By the choice of n_j, we have $T_1^{n_j,\epsilon} \to T_1, S_1^{n_j,\epsilon} \to S_1^\epsilon$ almost surely on $\{S_1^\epsilon \leq T-1\}$.

Combining the above three facts, we conclude that $\tilde{W}$ coincides with W on the interval $[T_1^\epsilon, S_1^\epsilon]$. By Theorem 2.7 again, $\tilde{\eta}$ is the Loewner chain generated by $(\tilde{W}(t), T_1^\epsilon \leq t \leq S_1^\epsilon)$ and $G(Y)$ is the Loewner chain generated by $(W(t), T_1^\epsilon \leq t \leq S_1^\epsilon)$. Thus $\tilde{\eta}$ coincides with $G(Y)$. As $G(Y)$ is the unique subsequential limit of $\{G^{n_j}(Y^{n_j})\}_j$, we conclude that $G^{n_j}(Y^{n_j})$ converges to $G(Y)$. □

Proof of Proposition 5.7. Recall the definition of X^n, Y^n, G^n and X, Y, G as in Section 5.1.

First, we explain the convergence of G^{n_j}.

- Recall that G^{n_j} is the conformal map from $(\mathbb{H} \setminus \eta^{n_j}[0, T_1^{n_j,\epsilon}]; \eta^{n_j}(T_1^{n_j,\epsilon}), \infty)$ onto $(\mathbb{H}; 0, \infty)$ and G is the conformal map from $(\mathbb{H} \setminus \eta[0, T_1^\epsilon]; \eta(T_1^\epsilon), \infty)$ (normalized at ∞). As $\eta^n \to \eta, W^n \to W$ and $T_1^{n_j,\epsilon} \to T_1^\epsilon$, The map $G^{n_j} \to G$ locally uniformly almost surely on $\{S_1^\epsilon \leq T-1\}$ by Carathéodory kernel theorem.
- By Proposition 4.1, we have $\theta^n \to \theta$ locally uniformly. In particular, the sequence $\{\theta^n\}_n$ is equicontinuous on $[0, T]$. As $T_1^{n_j,\epsilon} \to T_1^\epsilon$, we conclude $\theta^{n_j}(T_1^{n_j,\epsilon}) \to \theta(T_1^\epsilon) = \epsilon$ almost surely on $\{S_1^\epsilon \leq T-1\}$.

Combining the above two items, we conclude that $G^{n_j} \to G$ in Carathéodory sense and the image of the dual arc under G^{n_j} converges to the interval $[0, \epsilon]$ almost surely on $\{S_1^\epsilon \leq T-1\}$.

Since Y^n is the exploration path in $\mathbb{H} \setminus X^n$ with Dobrushin boundary conditions. Combining the above convergence of G^{n_j} and Theorem 2.10, the sequence $G^{n_j}(Y^{n_j})$ converges in distribution to SLE_κ in $\mathbb{H}$ from 0 to ϵ. By the choice of n_j, we also have the convergence of the disconnection time: $S_1^{n_j,\epsilon} \to S_1^\epsilon$ almost surely on $\{S_1^\epsilon \leq T-1\}$. Therefore, $G^{n_j}(Y^{n_j})$ up to the disconnection time converges in distribution to SLE_κ in $\mathbb{H}$ from 0 to ϵ up to the disconnection time. By Lemma 2.5, we conclude that $G^{n_j}(Y^{n_j})$ (up to the disconnection time) converges in distribution to $\mathrm{SLE}_\kappa(\kappa-6)$ (up to the disconnection time) in $\mathbb{H}$ from 0 to ∞ with force point ϵ.

By Lemma 5.8, we have $G^{n_j}(Y^{n_j}) \to G(Y)$ almost surely. Combining with the above analysis, we know that $G(Y)$ has the law of $\mathrm{SLE}_\kappa(\kappa-6)$ (up to the disconnection time) in $\mathbb{H}$ from 0 to ∞ with force point ϵ. As θ is the corresponding renormalized harmonic measure, it has the same law as Bessel process starting from ϵ stopped at the first time that it reaches zero conditioned on $\{S_1^\epsilon \leq T-1\}$. As this is true for all $T \geq 2$, and $\mathbb{P}[S_1^\epsilon \leq T-1] \to 1$ as $T \to \infty$, we can remove the conditioning.

It thus remains to argue that $G(Y)$ is indeed independent of X. Let us show that for any bounded continuous functionals f and h on the space of continuous curves with the topology of local uniform convergence, one has $\mathbb{E}\big[f(G(Y))h(X)\big] = \mathbb{E}\big[f(G(Y))\big]\mathbb{E}\big[h(X)\big]$. As we have shown above that $X^{n_j} \to X$ a.s. and (Lemma 5.8) that $G^{n_j}(Y^{n_j}) \to G(Y)$ a.s., we readily have by dominated convergence theorem that

$$\begin{aligned}\mathbb{E}\big[f(G(Y))h(X)\big] &= \lim_{j\to\infty} \mathbb{E}\big[f(G^{n_j}(Y^{n_j}))h(X^{n_j})\big] \\ &= \lim_{j\to\infty} \mathbb{E}\big[\mathbb{E}\big[f(G^{n_j}(Y^{n_j})) \bigm| X^{n_j}\big]h(X^{n_j})\big]\end{aligned}$$

Now, observe that the above analysis in fact shows more than what we stated. Namely, on the event K that $G^{n_j} \to G$ and $\theta^{n_j}(T_1^{n_j,\epsilon}) \to \theta(T_1^\epsilon) = \epsilon$ (which happens with probability one as shown above), we have as argued above by Theorem 2.10 and Lemma 2.5 that the law of $G^{n_j}(Y^{n_j})$ (up to the disconnection time) given X^{n_j} converges in distribution to $\mathrm{SLE}_\kappa(\kappa-6)$ (up to the disconnection time) in $\mathbb{H}$ from 0 to ∞ with force point ϵ. This is nothing but saying that for any functional f as above, one has $\mathbb{E}\big[f(G^{n_j}(Y^{n_j})) \bigm| X^{n^j}\big] \to \mathbb{E}\big[f(\mathrm{SLE}_\kappa(\kappa-6))\big]$ almost surely on the event K. As f is bounded and $\mathbb{P}\big[K\big] = 1$, again by dominated convergence theorem, one has

$$\begin{aligned}
\mathbb{E}\big[f(G(Y))h(X)\big] &= \lim_{j\to\infty} \mathbb{E}\big[\mathbb{E}\big[f(G^{n_j}(Y^{n_j})) \bigm| X^{n_j}\big]h(X^{n_j})\big] \\
&= \mathbb{E}\big[\mathbb{E}\big[f(\mathrm{SLE}_\kappa(\kappa-6))\big]h(X)\big] \\
&= \mathbb{E}\big[f(\mathrm{SLE}_\kappa(\kappa-6))\big]\mathbb{E}\big[h(X)\big] = \mathbb{E}\big[f(G(Y))\big]\mathbb{E}\big[h(X)\big]
\end{aligned}$$

which thus concludes the proof. □

For general $k \geq 1$, by Proposition 5.2, we have

$$\mathbb{P}[S_k^\epsilon \leq T-1, \exists \ell \leq k \text{ s.t. } |T_\ell^{n,\epsilon} - T_\ell^\epsilon| > \alpha_n \text{ or } |S_\ell^{n,\epsilon} - S_\ell^\epsilon| > \alpha_n] \leq \alpha_n.$$

As $\sum_j \alpha_{n_j} < \infty$, we have

$$\sum_j \mathbb{P}[S_k^\epsilon \leq T-1, \exists \ell \leq k \text{ s.t. } |T_\ell^{n_j,\epsilon} - T_\ell^\epsilon| > \alpha_{n_j} \text{ or } |S_\ell^{n_j,\epsilon} - S_\ell^\epsilon| > \alpha_{n_j}] < \infty.$$

By Borel-Cantelli lemma, we have

$$T_\ell^{n_j,\epsilon} \to T_\ell^\epsilon, \quad S_\ell^{n_j,\epsilon} \to S_\ell^{n_j,\epsilon}, \quad \text{for all } \ell \leq k, \quad \text{a.s. on } \{S_k^\epsilon \leq T-1\}.$$

The proof of Proposition 5.7 also works for $(\theta(t), T_k^\epsilon \leq t \leq S_k^\epsilon)$.

Corollary 5.9. *For any $k \geq 1$, the law of $(\theta(t)/\sqrt{\kappa}, T_k^\epsilon \leq t \leq S_k^\epsilon)$ is the same as a Bessel process of dimension $3 - 8/\kappa$ starting from ϵ and stopped when it reaches zero where $\kappa = 16/3$. Moreover, it is independent of $(\theta(t), t \leq T_k^\epsilon)$.*

5.4 Convergence in law to a Bessel process

In this section, we will prove in Proposition 5.10 that θ considered as a whole process is indeed a Bessel and we will complete the proof of Theorem 1.1 (still in the case where Ω has a flat boundary near b, hypothesis which shall be removed in Subsection 5.5).

Proposition 5.10. *The law of $(\theta(t)/\sqrt{\kappa}, t \geq 0)$ is the same as a Bessel process of dimension $3 - 8/\kappa$ where $\kappa = 16/3$.*

Proof. In order to apply Proposition 2.1 to the process $\theta/\sqrt{\kappa}$, we need to check the three requirements in Proposition 2.1. The first item holds due to Corollary 5.9. It remains to check the other two items. To this end, we will need results in Section 3, this part is what we call dust analysis.

Suppose $C, c > 0$ are the universal constants as in Proposition 3.7. By Proposition 3.7, we have, for $k \geq 0$ and $\ell \geq 0$,

$$\mathbb{P}\left[T_{k+1}^{n,\epsilon} \leq S_k^{n,\epsilon} + (\ell+1)C\epsilon^2 \,|\, T_{k+1}^{n,\epsilon} > S_k^{n,\epsilon} + \ell C\epsilon^2\right] \geq c. \tag{5.8}$$

It is important that the constants c, C are uniform. Let $\mathcal{Z}$ be a random variable taking values in $\{1, 2, 3, \ldots\}$ such that

$$\mathbb{P}[\mathcal{Z} > \ell] = (1-c)^\ell, \quad \forall \ell \in \{0, 1, 2, \ldots\}.$$

From (5.8), we know that $T_{k+1}^{n,\epsilon} - S_k^{n,\epsilon}$ is stochastically dominated by $\mathcal{Z}C\epsilon^2$. By Proposition 5.2, for any $T > 0$ and $a > 0$

$$\begin{aligned}\mathbb{P}\left[T_{k+1}^{\epsilon} \le T-1, T_{k+1}^{\epsilon} - S_k^{\epsilon} \ge a\right] &\le \mathbb{P}\left[T_{k+1}^{\epsilon} \le T-1, T_{k+1}^{n,\epsilon} - S_k^{n,\epsilon} \ge a - 2\alpha_n\right] + 2\alpha_n \\ &\le \mathbb{P}\left[\mathcal{Z}C\epsilon^2 \ge a - 2\alpha_n\right] + 2\alpha_n.\end{aligned}$$

Let $n \to \infty$ and then $T \to \infty$, we have for any $a > 0$,

$$\mathbb{P}\left[T_{k+1}^{\epsilon} - S_k^{\epsilon} \ge a\right] \le \mathbb{P}\left[\mathcal{Z}C\epsilon^2 \ge a\right].$$

Thus $T_{k+1}^{\epsilon} - S_k^{\epsilon}$ is also stochastically dominated by $\mathcal{Z}C\epsilon^2$. Note that $\mathbb{E}[\mathcal{Z}C\epsilon^2] = \epsilon^2 C(1-c)/c$. This guarantees the other requirement in Proposition 2.1 and completes the proof. □

Proof of Theorem 1.1. Recall the notations at the beginning of Section 5.1. By Theorem 2.9, the collection of interfaces $\{\eta^\delta\}$ satisfies Condition C2. By Theorem 2.7, the sequence is tight. Suppose $\delta_n \to 0$ and $\{\eta^{\delta_n}\}_n$ is a convergent subsequence and the limit is denoted by η. Theorem 2.7 also gives that η is a continuous curve with continuous driving function W. We denote by θ_t the renormalized harmonic measure of the right side of $\eta[0,t]$ union $[0, \phi(b)]$ in $\mathbb{H} \setminus \eta[0,t]$ seen from ∞. By Lemma 2.8, we can apply Lemma 2.4 to η, thus

$$\theta_t + W_t = \int_0^t \frac{2ds}{\theta_s}, \quad \forall t \ge 0.$$

By Proposition 5.10, we know that $\theta(t)/\sqrt{\kappa}$ is a Bessel process of dimension $3 - 8/\kappa$. Thus $(W_t, \theta(t) + W_t : t \ge 0)$ solves (2.5), i.e. by setting $V_t = \theta_t + W_t$, we have

$$dW_t = \sqrt{\kappa} dB_t + \frac{(\kappa-6)dt}{W_t - V_t}, \quad dV_t = \frac{2dt}{V_t - W_t}.$$

Thus η is an $\mathrm{SLE}_\kappa(\kappa-6)$. As $\mathrm{SLE}_\kappa(\kappa-6)$ is the only subsequential limit, we have the convergence of the sequence. □

5.5 Coupling argument and the case of general Jordan domains

In the last subsections (since Subsection 5.2), we assumed our domains satisfied the assumption in Proposition 3.7, i.e. that there exists $r > 0$ such that $\Omega \cap B(b,r)$ is identical to $B(b,r) \cap \mathbb{H}$ or $B(b,r) \cap (i\mathbb{H})$. In order to extend the result to the class of Jordan domain as stated in Theorem 1.1, we use a classical coupling argument (see for example the strong mixing property in [DC13]) as follows: Let $(\Omega; a, b)$ be a Jordan domain with two distinct boundary points a, b. We wish to approximate Ω by a domain $\Omega^{(r)}$ which satisfies the above *flat condition* near b. Let $\gamma : [0,1] \to \mathbb{C}$ be the continuous simple curve drawing $\partial\Omega$ and let us assume that $\gamma(0) = a$ and $\gamma(1/2) = b$. Define the following times $s_1(r) := \inf\{t < 1/2, |\gamma(t) - b| = 2r\}$ and $s_2(r) := \sup\{t > 1/2, |\gamma(t) - b| = 2r\}$. As the set $\partial\Omega$ is a simple curve and is thus locally connected, we deduce that necessarily one has the following properties:

$$\begin{cases} s_2(r) - s_1(r) \to 0, \text{ as } r \to 0 \\ |\gamma(s_1(r)) - \gamma(s_2(r))| > 0 \text{ (obvious)} \\ \delta(r) := \sup\{|\gamma(u) - b|, u \in [s_1(r), s_2(r)]\} \to 0 \text{ as } r \to 0 \end{cases}$$

Using these properties, one can easily build a domain $(\Omega^{(r)}; a, b^{(r)})$ which satisfies the following properties (see Fig. 5.2):

i) It is r-flat near b, i.e. $\Omega^{(r)} \cap B(b,r)$ is identical to $B(b,r) \cap \mathbb{H}$ or $B(b,r) \cap (i\mathbb{H})$

ii) Ω and $\Omega^{(r)}$ are identical away from $B(b, \delta(r))$.

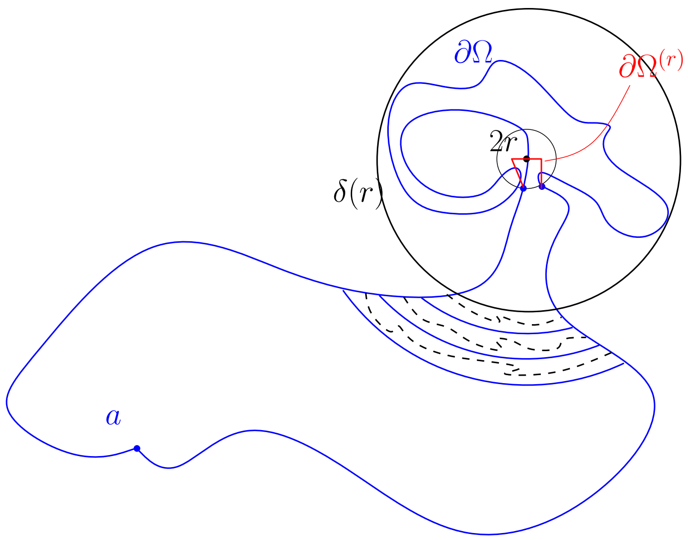

Figure 5.2

Now, using RSW theorem in annuli between radii $\delta(r)$ and $\sqrt{\delta(r)}$ (as in the proof of the strong mixing property of FK), we can couple, with arbitrary large probability (as $r \to 0$), the exploration processes (discrete or continuous) from a to b in Ω and from a to $b^{(r)}$ in $\Omega^{(r)}$ to coincide up to the first hitting time of $B(b, \sqrt{\delta(r)})$. Letting $r \to 0$, this concludes the proof of our main Theorem 1.1.

6 Detailed sketch of the convergence to radial SLE$(16/3, 16/3-6)$ and the one-arm exponent

The goal of this section is to give a detailed sketch of the different steps needed to adapt the proof in the chordal case to the radial case. This should not be considered as a complete proof, in particular in the case of item 4) below whose complete proof would require more topological details. In Subsection 6.2, we sketch how to derive Onsager's exponent 1/8 from the convergence of the radial exploration process.

6.1 On the convergence to radial SLE$(16/3, 16/3-6)$

One possible way to obtain the convergence to radial SLE$(16/3, 16/3-6)$ would be to design and analyse a discrete parafermionic observable well-adapted to a radial exploration process. This would be in some sense the approach followed in [KS15, KS16]. Once one has the convergence to chordal SLE$(16/3, 16/3-6)$, another natural route is to extract the convergence to radial SLE$(16/3, 16/3-6)$ using the fact that the chordal SLE$(16/3, 16/3-6)$ is **target-independent** (Lemma 2.5). Even tough very natural, this strategy does not come for free and the following steps need to be addressed in order to prove the convergence to radial SLE$(16/3, 16/3-6)$:

1. First, the powerful topological setup from [KS17] needs to be adapted to radial curves. In particular, one needs to show radial analogs of statements such as Theorem 2.7. It turns out that by following closely the proof from [KS17], there are no real difficulties on the way for this first item.

2. Then, one starts exploring the configuration inside and at the beginning one proceeds exactly as in the chordal case with stopping times $T_k^{n,\epsilon}$, $S_k^{n,\epsilon}$ etc. (The radial version of Proposition 3.7, i.e. Proposition 3.1 would be used to control the accumulation of dust).

3. We keep going until the first **disconnection times**: $D_1^n = D_1^n(\eta^n)$ and $D_1 = D_1(\eta)$. Here, one needs to show that for the coupled interfaces (η^n, η) one has $D_1^n \to D_1$ in probability, say. As explained above, this step requires some care as the use of a stopping time for the continuous curve η will ruin the spatial Markov property for η^n. The techniques we used in the chordal case (see Proposition 5.2) work in the same fashion in the radial setting except one needs to add the following step:

4. Similarly as in Proposition 4.1, we need to show in the radial setting the uniform convergence of discrete harmonic measures θ^n. More precisely, let $\eta^n = \phi^n(\gamma^n)$ and $\eta = \phi(\gamma)$ be the radial curves conformally mapped into $\mathbb{U}$ and parametrised by their disk–capacity. From the analog of Theorem 2.7 mentioned in item 1., we get that $\eta^n \to \eta$ and $W^n \to W$ locally uniformly. Let $t \mapsto \theta^n(t)$ (resp $\theta(t)$) denote the harmonic measure of the free arc on the right of $\eta^n[0,t]$ (resp. $\eta[0,t]$). With these notations, we need to show that for any fixed $T > 0$,

$$\|\theta^n - \theta\|_{\infty,T} = \sup_{t\in[0,T]} |\theta^n(t) - \theta(t)| \to 0\,. \tag{6.1}$$

   A slightly different proof as the one we used in Section 4 is needed here, as the proof in Section 4 relies specifically on the geometry of $\mathbb{H}$. One possible way to proceed is to divide the proof in the following two steps:

   a) Equicontinuity of the family $\{\theta^n\}_{n\geq 1}$. As pointed out to us by Avelio Sepúlveda, one can obtain the equicontinuity of $\{\theta^n\}_n$ by relying on $\eta^n \to \eta$ locally uniformly together with some harmonic measure considerations. Indeed $\eta^n \to \eta$ locally uniformly implies that for any $T > 0$ and $r > 0$, there exists $\delta > 0$, s.t. for n large enough, $\eta^n([t, t+\delta])$ remain inside the ball $B(\eta^n(t), r)$ for any $t \in [0, T]$. Together with some easy harmonic measure estimates, this implies the equicontinuity of $\{\theta^n\}_n$. As such it reduces the question to the following second step.

   b) Pointwise convergence of $\theta^n \to \theta$. Let us then fix some $t \geq 0$. Consider the δ-neighborhood O_n^δ of $\eta^n([0,t])$. Let $F_n^\delta \subset \partial O_n^\delta$ be the closed set of points on the boundary of O_n^δ which are at geodesic-distance-measured-in-$\mathbb{U} \setminus \eta^n([0,t])$ δ from the free arc of $\eta^n([0,t])$ and which are at Euclidean distance at least $\delta^{1/100}$ from the tip $\eta^n(t)$ as well as from the current force point (last disconnection vertex). We claim that with high probability (as $\delta \to 0$), all points in F_n^δ are at a geodesic-distance-measured-in-$\mathbb{U} \setminus \eta^n([0,t])$ at least $\delta^{1/2}$ from the wired arc of $\eta^n([0,t])$: otherwise, one could find a six-arm event for the FK percolation (three-arm event if near the boundary of $\partial\mathbb{U}$), in an annulus of inner radius $\delta^{1/2}$ and outer radius $\delta^{1/100}$. This can be shown to be of vanishing probability (vanishing in $\delta \to 0$, uniformly in n) using the fact that the six-arm exponent for critical FK-Ising percolation is > 2. See for example Section 4.3 in [BPW18].

Now, similarly as in Section 4, one can use Beurling's estimate to claim that once a Brownian motion in $\mathbb{U} \setminus O_n^\delta$ started at 0 will hit the set F_n^δ, it will hit with very high probability the free arcs of $\eta^n([0,t])$ as well as $\eta([0,t])$ before intersecting the respective wired arcs. One concludes by some easy harmonic measure considerations for the balls of radius $\delta^{1/100}$ around the tip and the force point. More topological details are certainly needed to turn this sketch into a formal proof.

5. As in the chordal case, one needs to justify limits of conditional expectations arising after say, the first disconnection time $t = D_1^n$. (I.e. the fact the spatial Markov property passes to the scaling limit definitively requires some justification). This step can be handled similarly as in Subsection 5.3.

6. Finally, one can extract the radial Loewner driving function W from the angle θ evolving like a cotan-Bessel process on $[0, 2\pi]$ by relying on a suitable radial analog of Lemma 2.4. It is not so immediate to generalize Lemma 2.4 to the radial setting, because the assumption $\mathrm{Leb}(\eta \cap \mathbb{R}) = 0$ does not suit the radial setting. One possible way is to compromise to an almost sure conclusion (instead of deterministic conclusion): in the chordal setting, one replaces the requirement $\mathrm{Leb}(\eta \cap \mathbb{R}) = 0$ by Condition C2, since $\mathrm{Leb}(\eta \cap \mathbb{R}) = 0$ holds almost surely under Condition C2 (see Lemma 2.8), then the conclusion holds almost surely. This compromised version of Lemma 2.4 in the chordal setting is easily generalized to the radial setting.

6.2 On Onsager's one-arm exponent (equal to $1/8$)

We briefly outline here what are the classical steps which enable to extract Onsager's celebrated exponent $\frac{1}{8}$ assuming the convergence to radial $\mathrm{SLE}(16/3, 16/3-6)$ is proved (i.e. assuming item 4 above). First let us point out that the convergence to radial $\mathrm{SLE}(16/3, 16/3-6)$ only implies a weaker result than Onsager's one: similarly to the one-arm exponent for critical percolation [LSW02], it implies that as $n \to \infty$,

$$\mathbb{P}^{\text{wired}}_{\Lambda_n, p_c(2)}\big[0 \leftrightarrow \partial\Lambda_n\big] = n^{-\frac{1}{8}+o(1)}\,,$$

while there are no sub-polynomial $o(1)$ corrections in Onsager's result. The main steps to prove this are as follows:

1. A computation of the exponent $1/8$ on the continuous level. This corresponds to the one-arm exponent $\tilde{\alpha}_1$ of radial $\mathrm{SLE}_\kappa(\kappa-6)$ which was calculated in [SSW09]:

$$\tilde{\alpha}_1 = \frac{(8-\kappa)(3\kappa-8)}{32\kappa}.$$

   Note that $\tilde{\alpha}_1 = 1/8$ when $\kappa = 16/3$.

2. Second, one needs to carefully argue that this continuous one-arm exponent $\tilde{\alpha}_1$ is the same as the exponent describing the crossing probability, for critical FK-Ising percolation, of large macroscopic annuli $\Lambda_n \setminus \Lambda_{rn}$ as $n \to \infty$ and $r \in (0,1)$. This step can be made rigorous in essentially two steps: a) first by showing similarly as in Proposition 5.2 that the discrete disconnection times D_k^n for the radial exploration process converge to the continuous ones D_k. And b) by showing via some separation types of lemmas that the probability of not disconnecting (i.e. θ not reaching 0) is up to constant the same as connecting $\partial\Lambda_n$ to $\partial\Lambda_{rn}$.

3. Finally as for critical percolation ($q = 1$), one relies on the **quasi-multiplicativity** of the discrete one-arm event to conclude. see [LSW02] or [SW01, Section 4.2] in the case $q = 1$. In fact this quasi-multiplicativity of the one-arm event is even rigorously known for all critical random-cluster models with $q \in [1, 4]$ thanks to the recent Russo-Seymour-Welsh Theorem proved in [DCST17, Theorem 7].

References

[BPW18] Vincent Beffara, Eveliina Peltola and Hao Wu. On the uniqueness of global multiple SLEs. *arXiv:1801.07699*, 2018.

[BC16] Stéphane Benoist and Clément Hongler. The scaling limit of critical Ising interfaces is CLE(3). *arXiv:1604.06975*, 2016

[CDCH+14] Dmitry Chelkak, Hugo Duminil-Copin, Clément Hongler, Antti Kemppainen, and Stanislav Smirnov. Convergence of Ising interfaces to Schramms SLE curves. *Comptes Rendus Mathematique*, 352(2):157–161, 2014.

[CDCH16] Dmitry Chelkak, Hugo Duminil-Copin, and Clément Hongler. Crossing probabilities in topological rectangles for the critical planar FK-Ising model. *Electron. J. Probab.*, 21:Paper No. 5, 28, 2016.

[Che16] Dmitry Chelkak. Robust discrete complex analysis: A toolbox. *Ann. Probab.*, 44(1):628–683, 2016.

[CS12] Dmitry Chelkak and Stanislav Smirnov. Universality in the 2D Ising model and conformal invariance of fermionic observables. *Invent. Math.*, 189(3):515–580, 2012.

[DC13] Hugo Duminil-Copin. Parafermionic observables and their applications to planar statistical physics models. *Ensaios Matematicos*, 25:1–371, 2013.

[DCST17] Hugo Duminil-Copin, Vladas Sidoravicius, and Vincent Tassion. Continuity of the phase transition for planar random-cluster and Potts models with $1 \leq q \leq 4$. *Comm. Math. Phys.*, 349(1):47–107, 2017.

[Gog94] Eimear M. Goggin. Convergence in distribution of conditional expectations. *Ann. Probab.*, 22(2):1097–1114, 1994.

[GW18] Christophe Garban and Hao Wu. On the convergence of FK-Ising Percolation to SLE$(16/3, 16/3 - 6)$. *arXiv:1802.03939*, 2018.

[KS15] Antti Kemppainen and Stanislav Smirnov. Conformal invariance of boundary touching loops of FK Ising model. *arXiv:1509.08858*, 2015.

[KS16] Antti Kemppainen and Stanislav Smirnov. Conformal invariance in random cluster models. II. full scaling limit as a branching SLE. *arXiv:1609.08527*, 2016.

[KS17] Antti Kemppainen and Stanislav Smirnov. Random curves, scaling limits and Loewner evolutions. *Ann. Probab.*, 45(2):698–779, 03 2017.

[Law05] Gregory F. Lawler. *Conformally invariant processes in the plane*, volume 114 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2005.

[LSW02] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. One-arm exponent for critical 2D percolation. *Electron. J. Probab.*, 7:no. 2, 13 pp. (electronic), 2002.

[LSW04] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, 32(1B):939–995, 2004.

[MS16a] Jason Miller and Scott Sheffield. Imaginary geometry I: interacting SLEs. *Probab. Theory Related Fields*, 164(3-4):553–705, 2016.

[MS16b] Jason Miller and Scott Sheffield. Imaginary geometry II: reversibility of $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ for $\kappa \in (0,4)$. *Ann. Probab.*, 44(3):1647–1722, 2016.

[New92] M. H. A. Newman. *Elements of the topology of plane sets of points.* Dover Publications, Inc., New York, second edition, 1992.

[Pom92] Ch. Pommerenke. *Boundary behaviour of conformal maps*, volume 299 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1992.

[RY94] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, second edition, 1994.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[SSW09] Oded Schramm, Scott Sheffield, and David B. Wilson. Conformal radii for conformal loop ensembles. *Comm. Math. Phys.*, 288(1):43–53, 2009.

[SW05] Oded Schramm and David B. Wilson. SLE coordinate changes. *New York J. Math.*, 11:659–669 (electronic), 2005.

[She09] Scott Sheffield. Exploration trees and conformal loop ensembles. *Duke Math. J.*, 147(1):79–129, 2009.

[Smi10] Stanislav Smirnov. Conformal invariance in random cluster models. I. Holmorphic fermions in the Ising model. *Annals of mathematics*, pp.1435–1467, 2010.

[SW01] Stanislav Smirnov and Wendelin Werner. Critical exponents for two-dimensional percolation. *Math. Res. Lett.*, 8(5-6):729–744, 2001.

[Wer07] Wendelin Werner. Lectures on two-dimensional critical percolation. 2007.

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