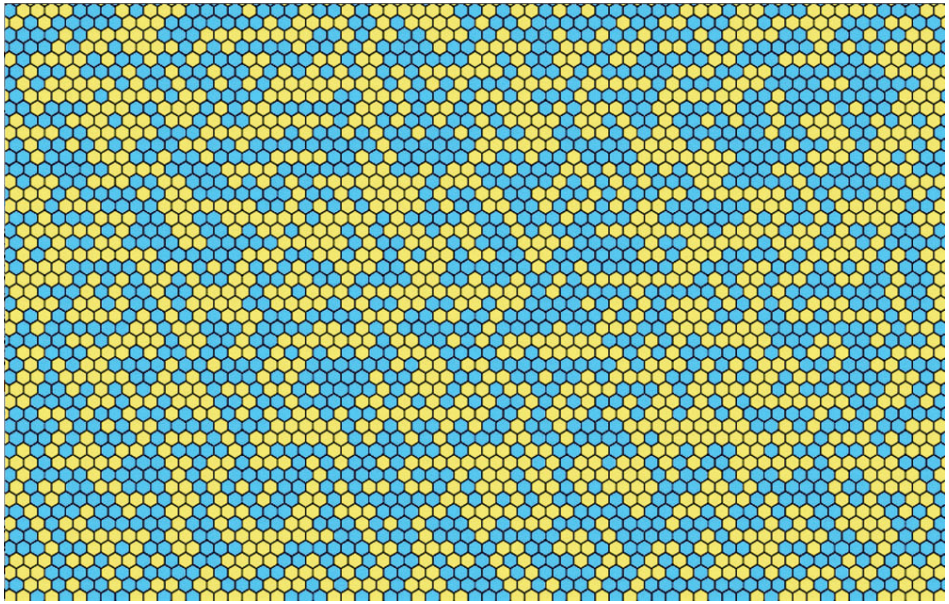
L'attrait des points-fixes en probabilités

Christophe Garban (université Lyon 1)



Qu'est ce qu'un "point fixe" ?

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*Un **point fixe** pour une fonction $f : \mathbb{X} \to \mathbb{X}$ est un "point" $x^\star$ tel que*

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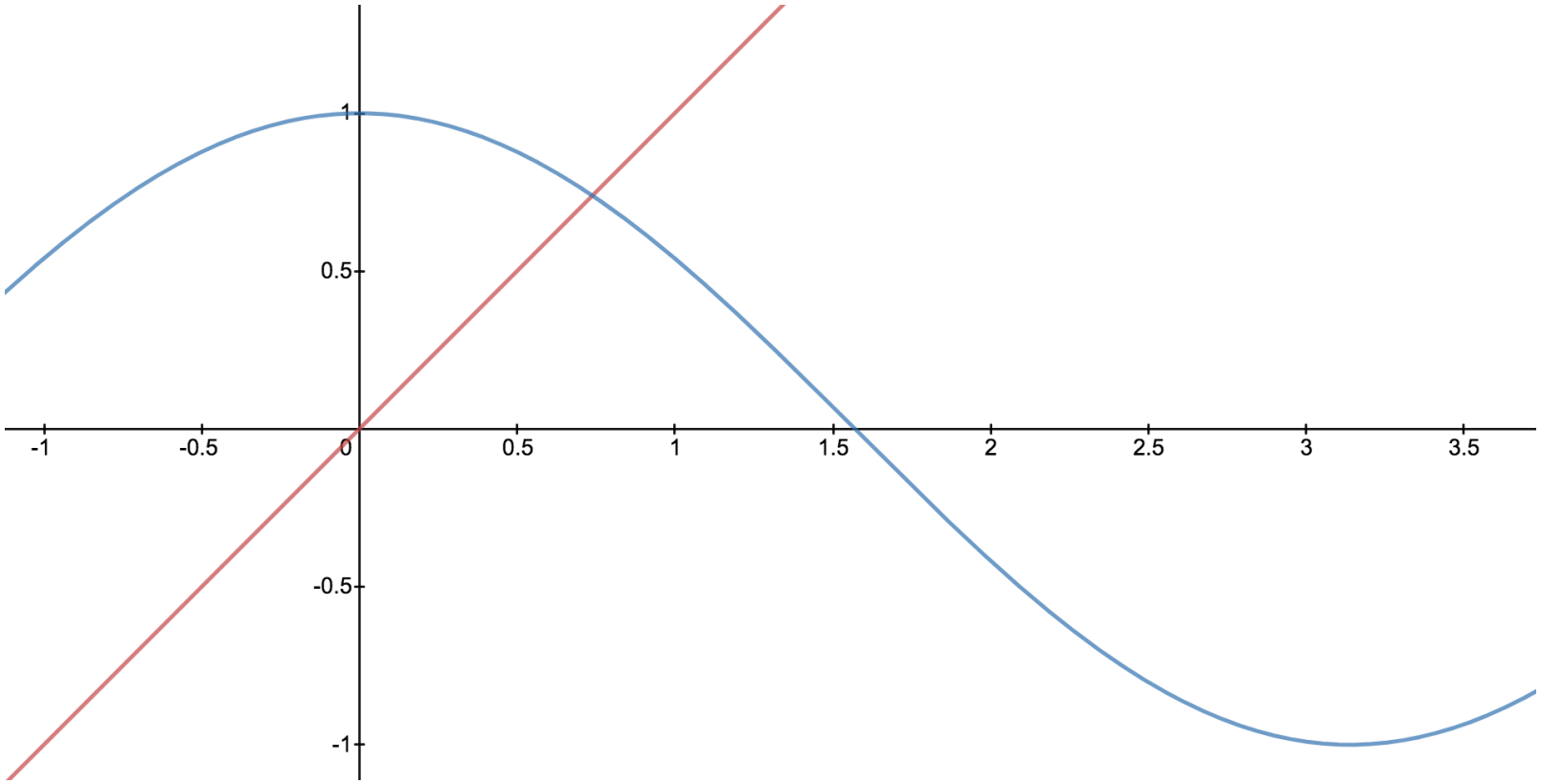
$f(x) = \cos(x)$



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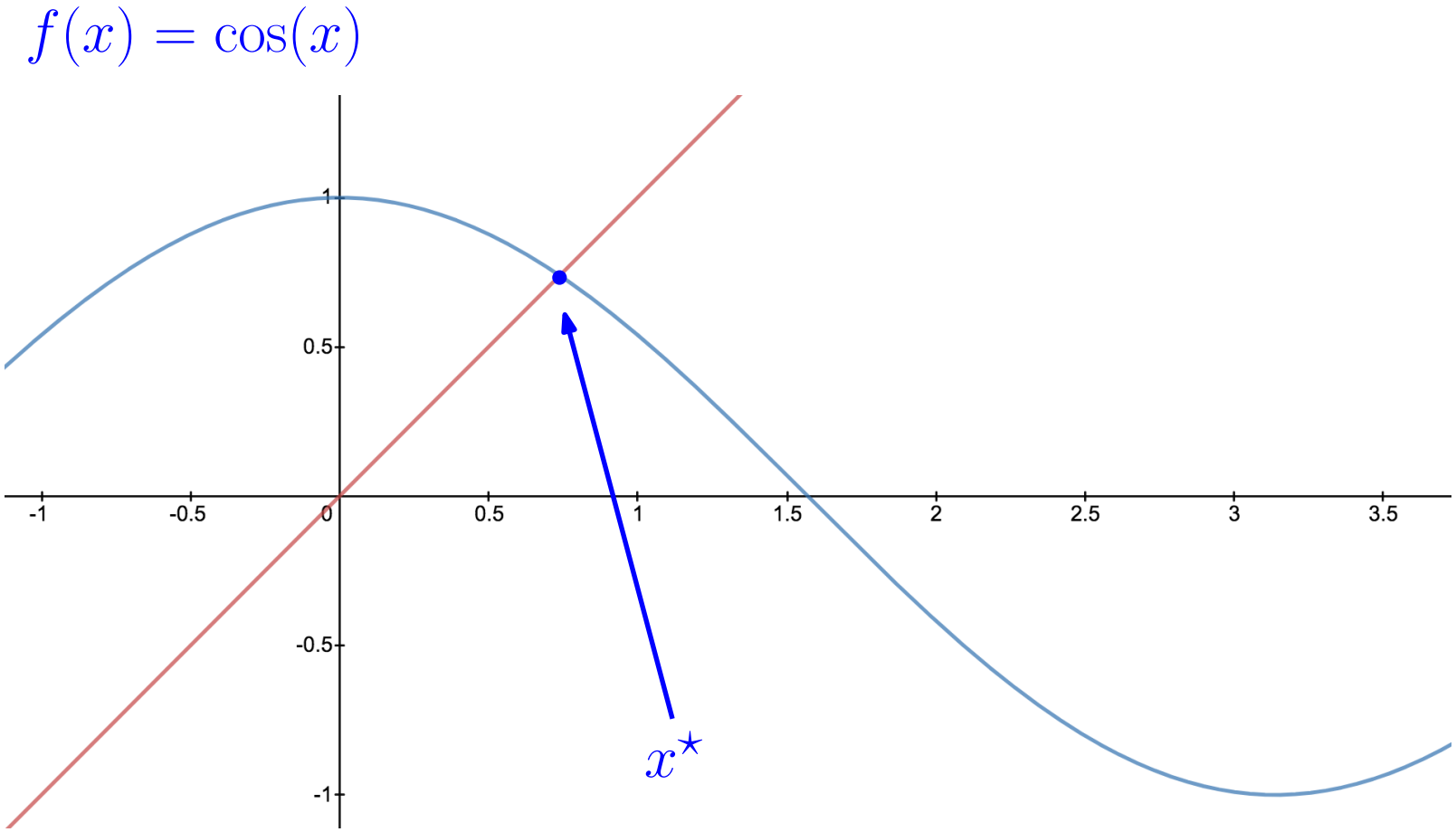

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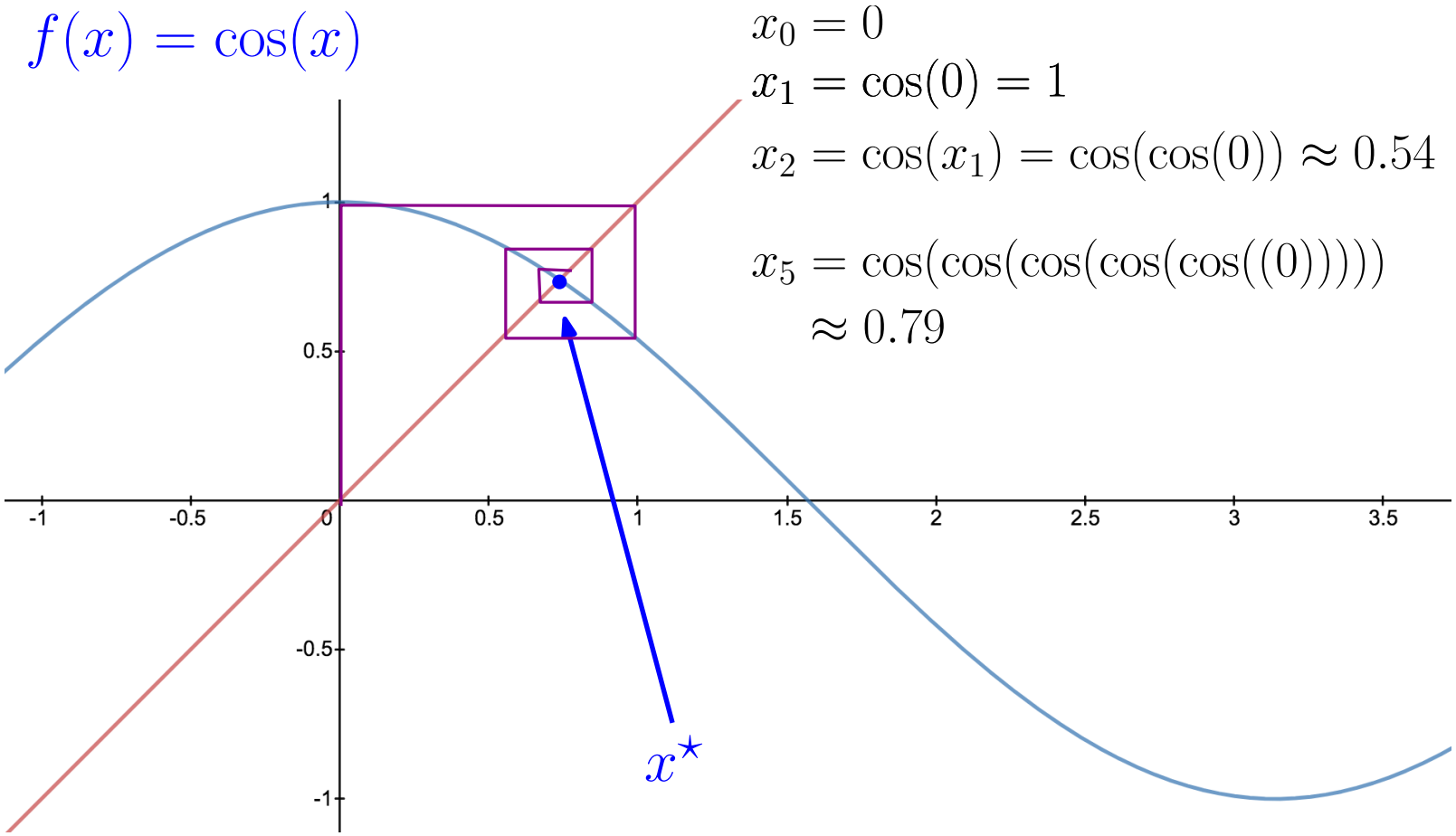

Un exemple gastronomique

$\mathbb{X} = \{\text{Ingredients}\}$

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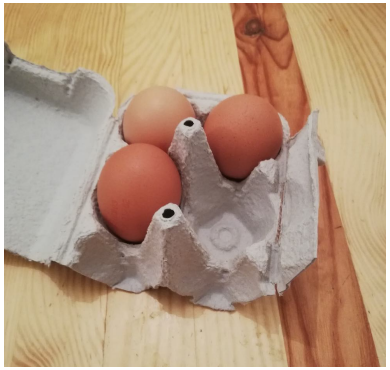
$x =$ 

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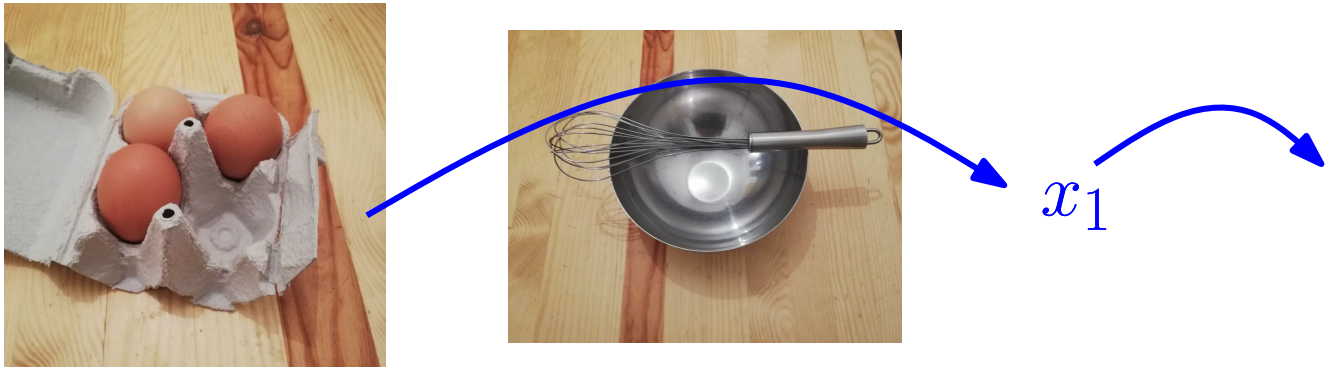

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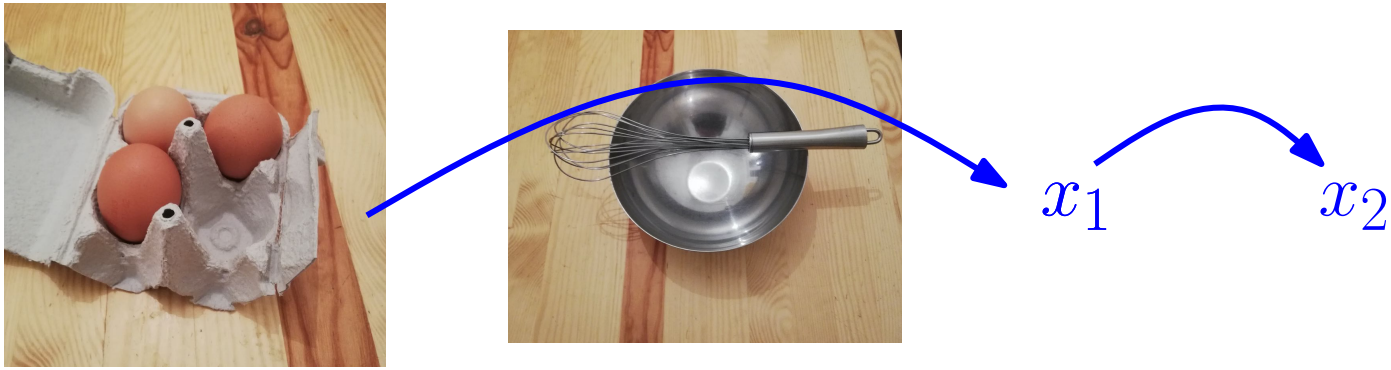

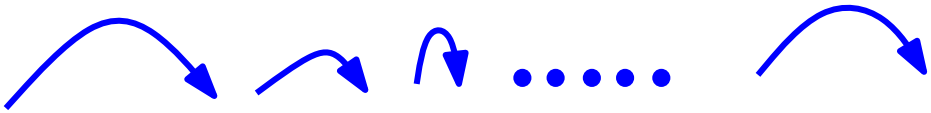

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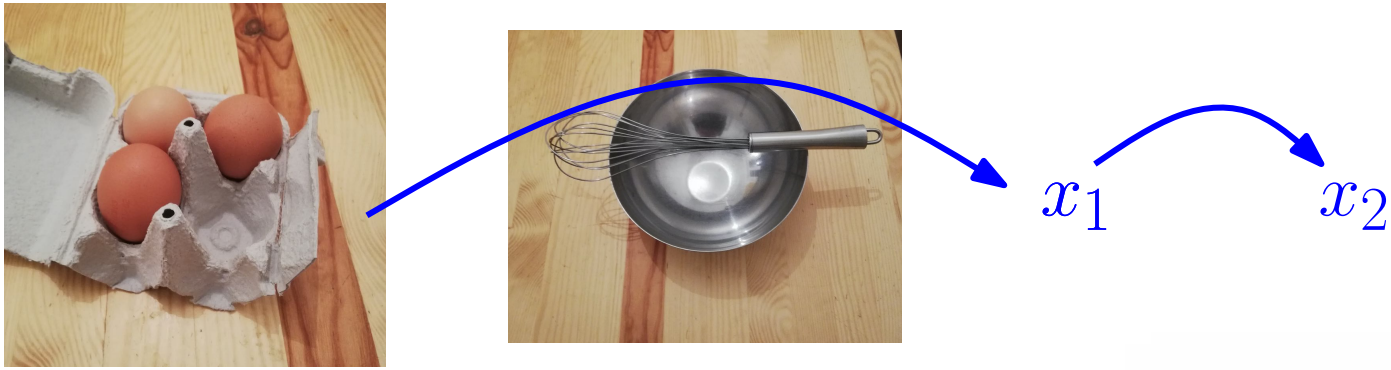

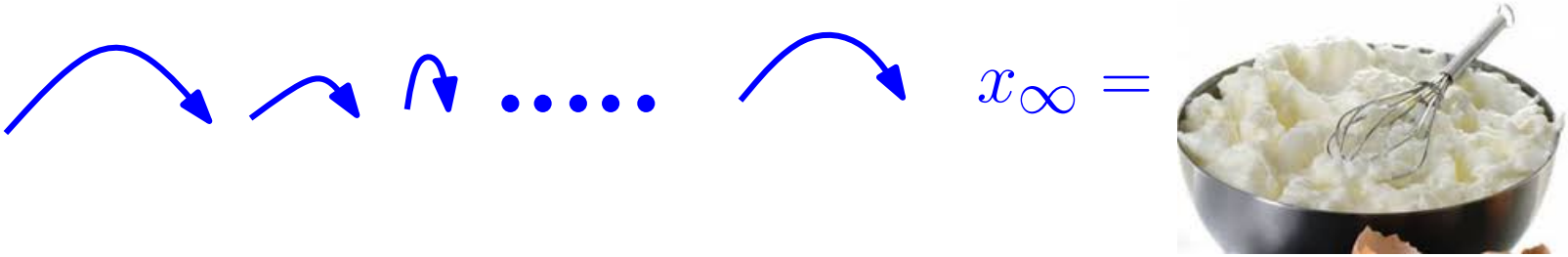

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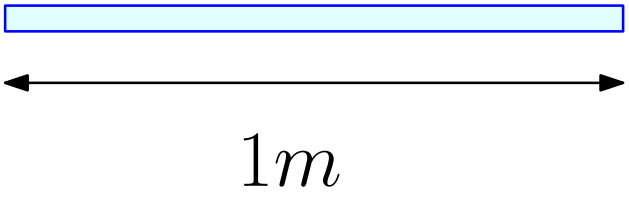

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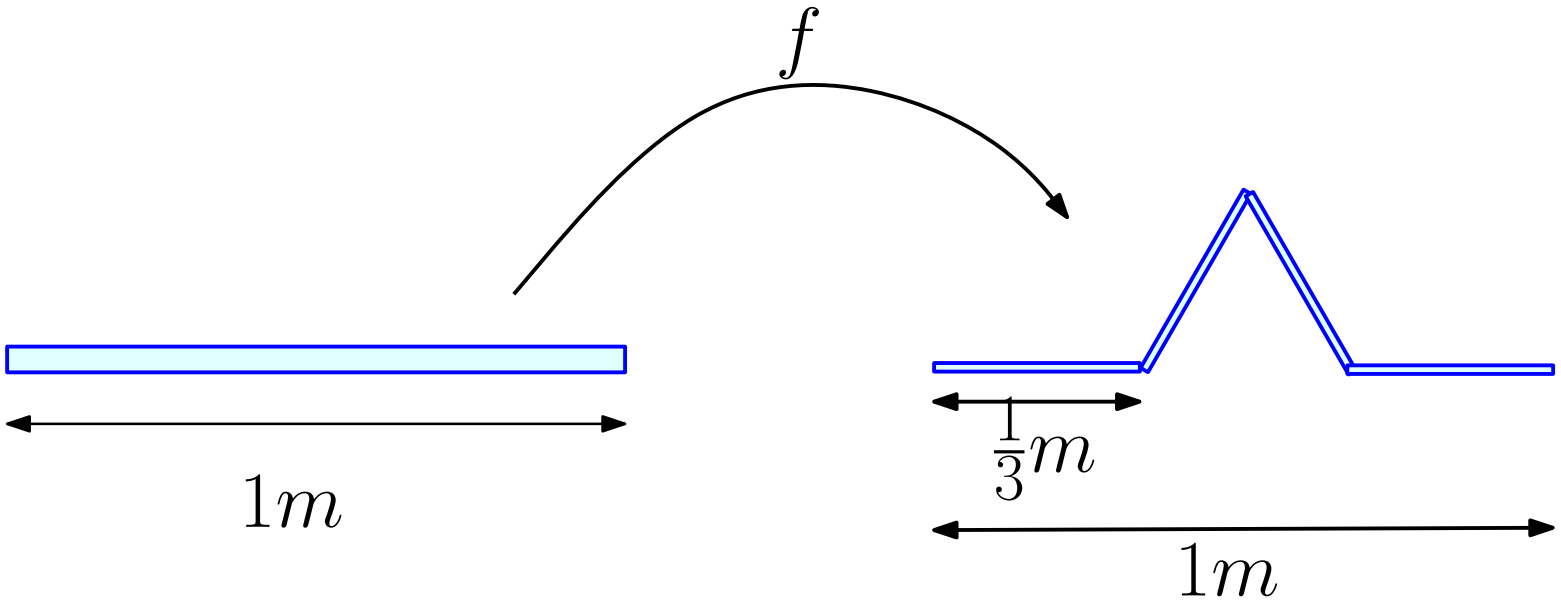

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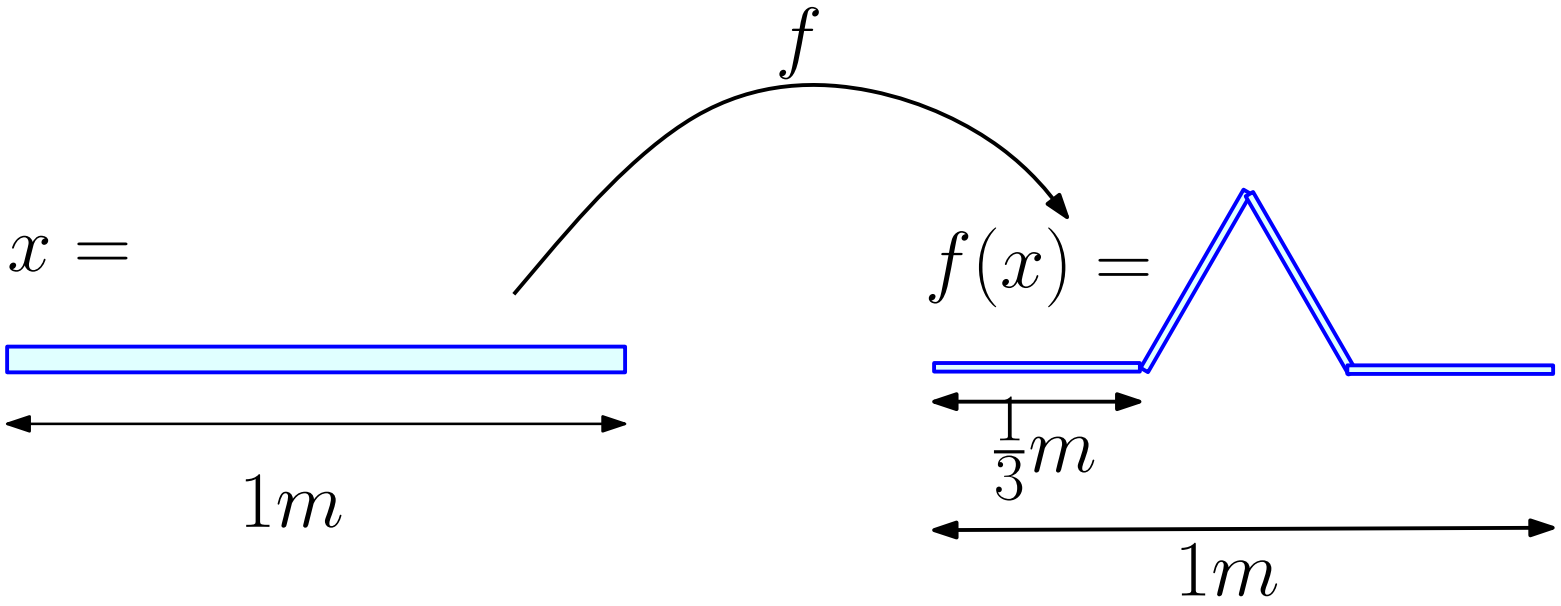

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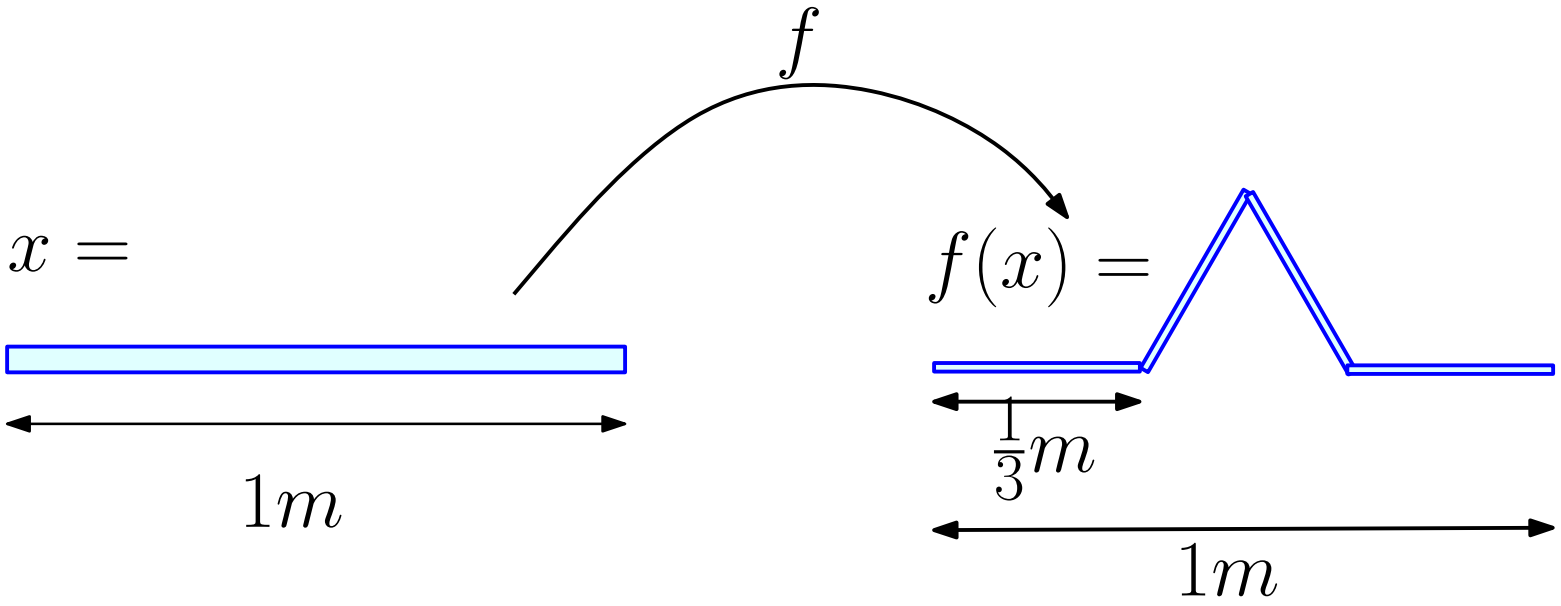

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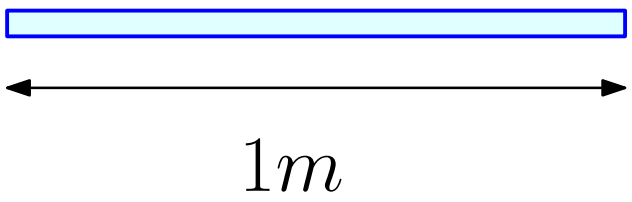

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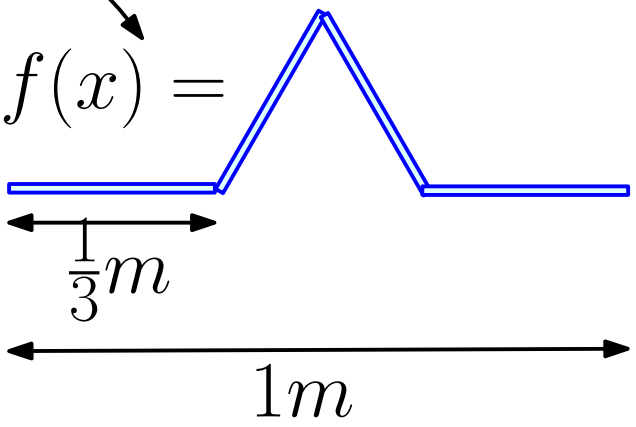

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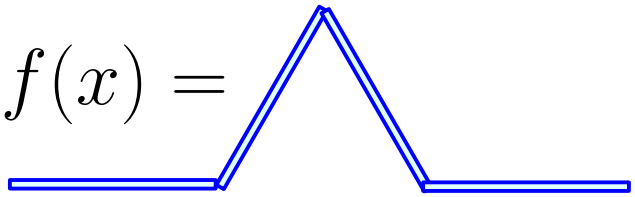
$f(x) =$ 

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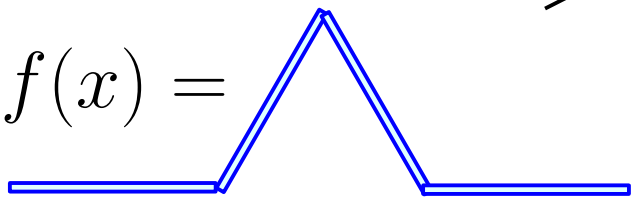
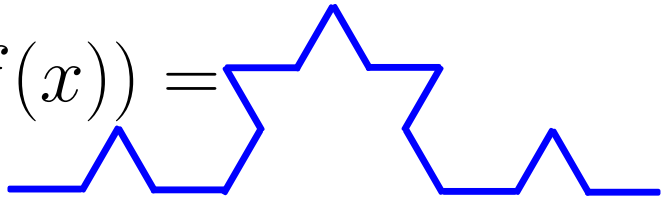
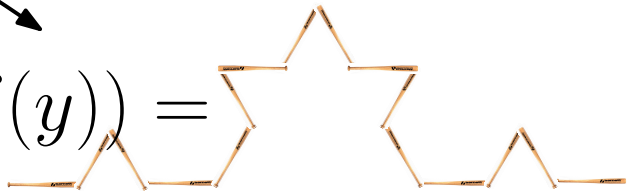
$f(x) =$  $f(f(x)) =$ 

$f(y) =$ $f(f(y)) =$ 

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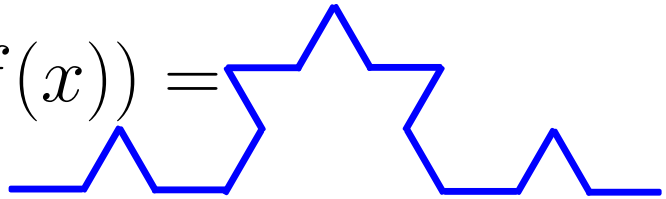
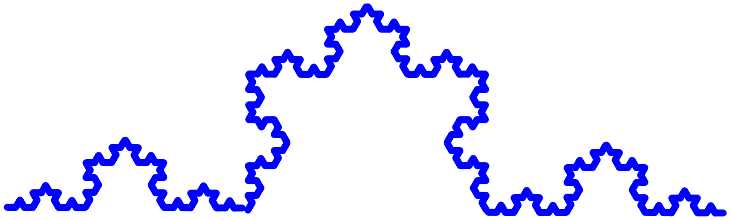
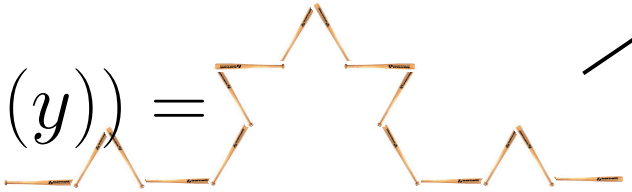
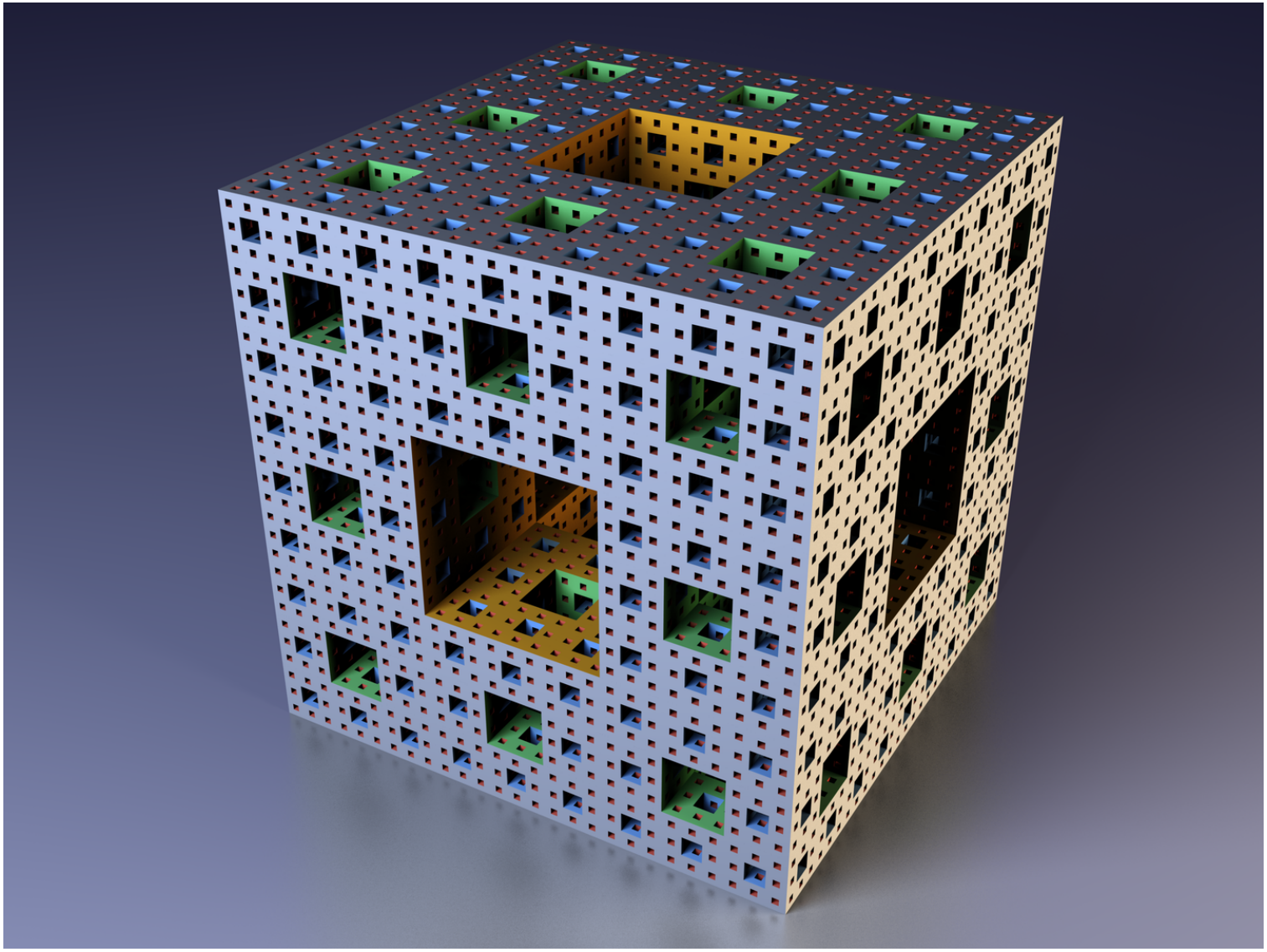
$f(f(x)) =$ 

$f(f(f(.....(x/y)))))$ 

$f(f(y)) =$ 

$$x^\star = f(x^\star) \ !!$$

Question : cet objet est-il un point fixe ?



Un point fixe probabiliste

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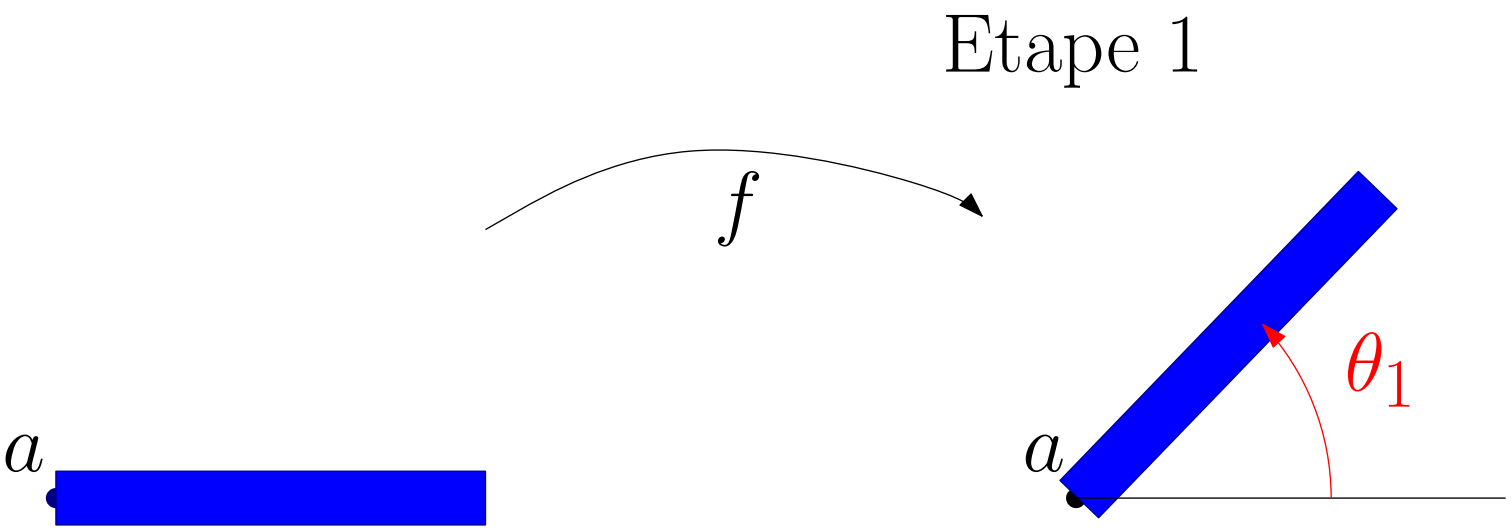
Le mouvement Brownien



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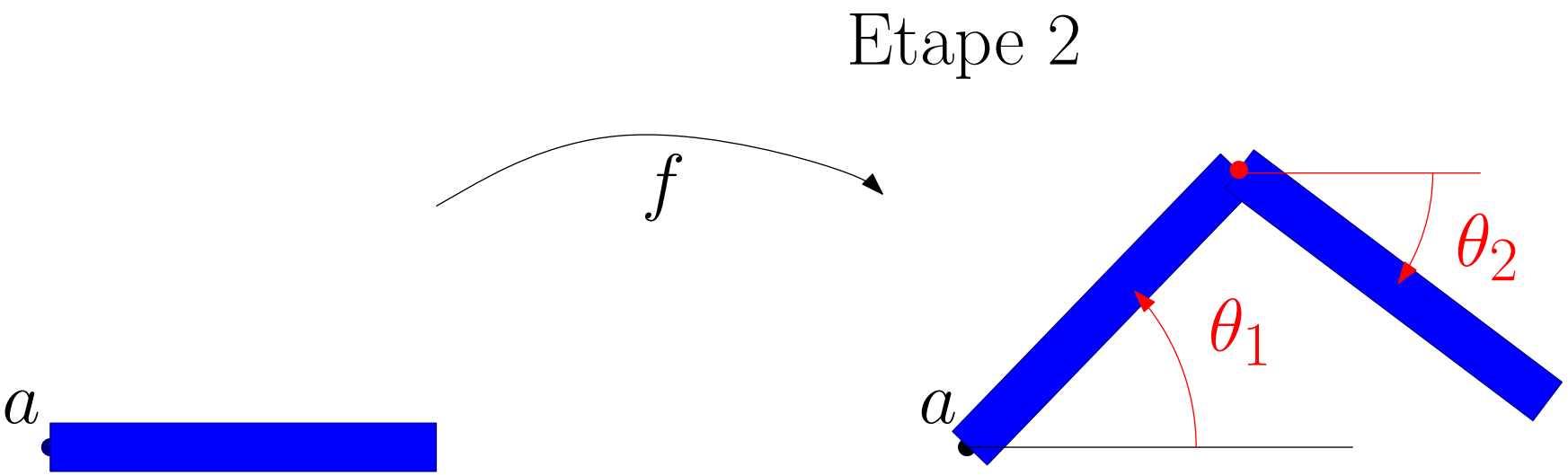

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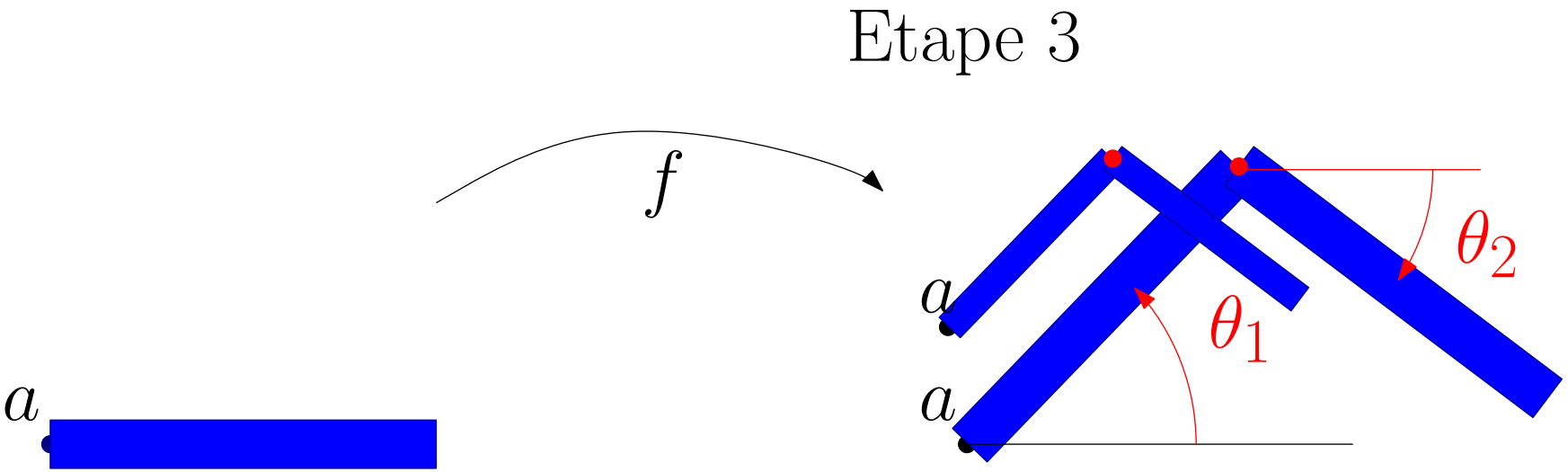

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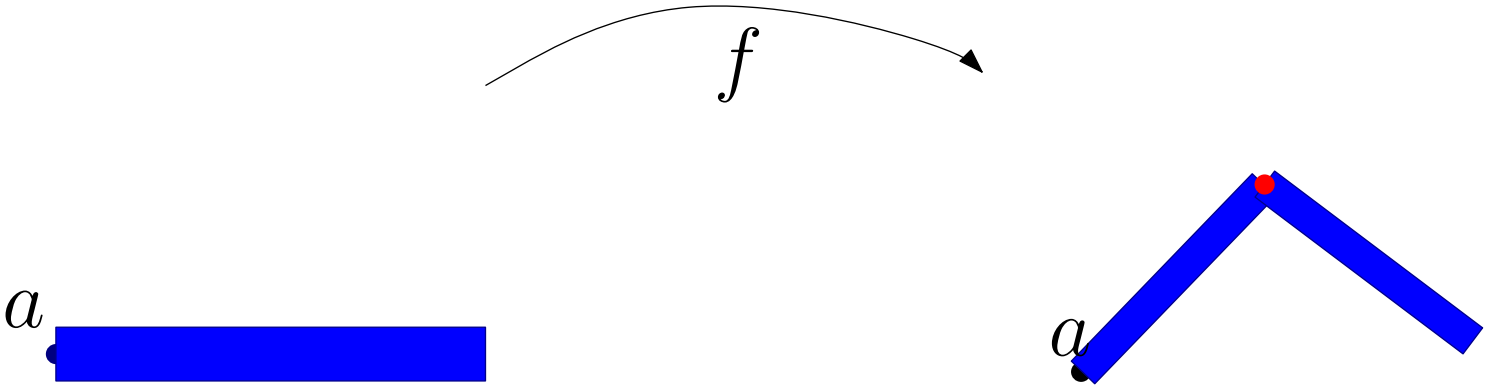

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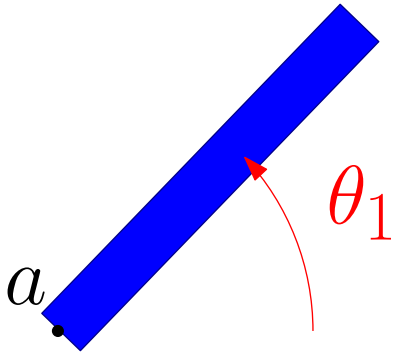
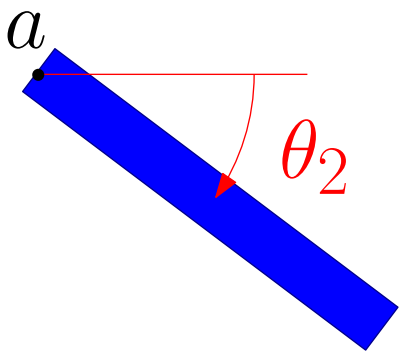


Tirage 2



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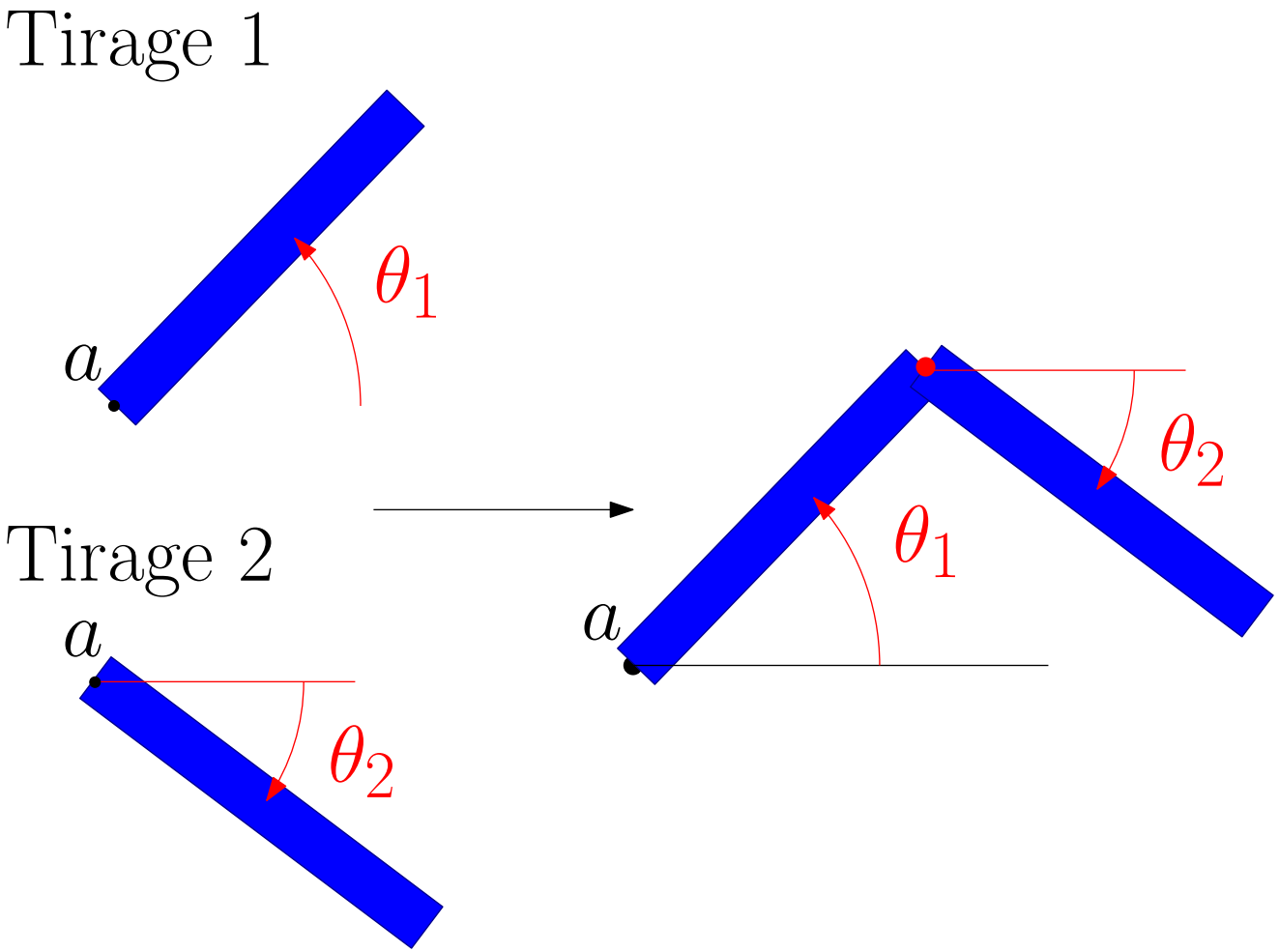

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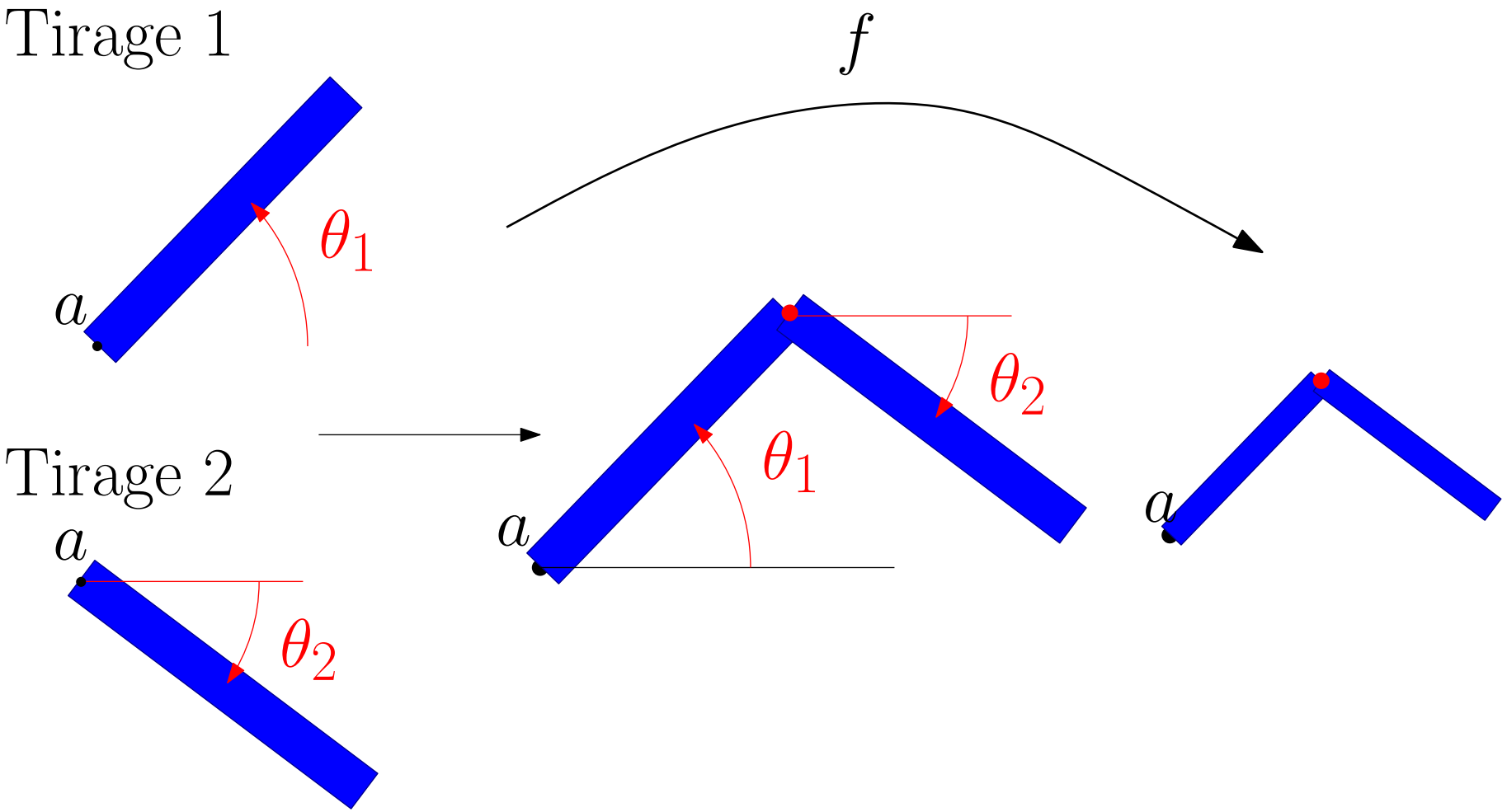

f(f( )) ?

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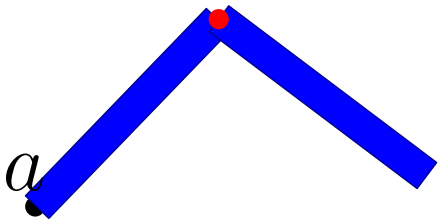

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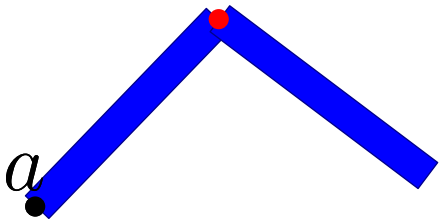

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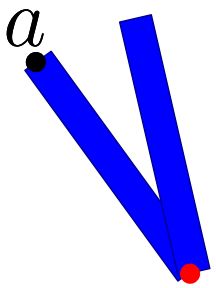

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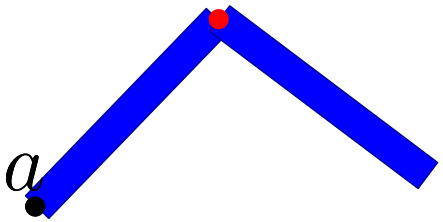

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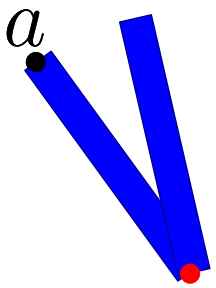

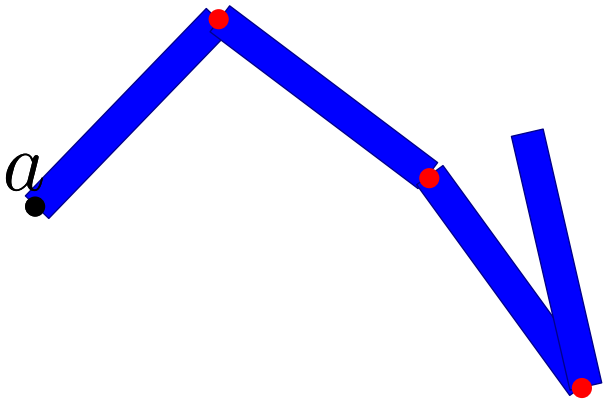

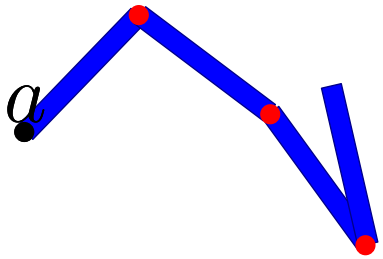

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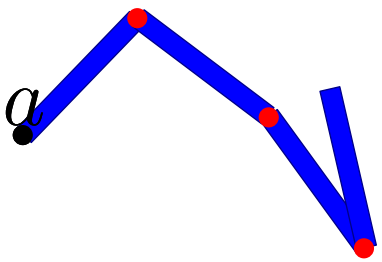

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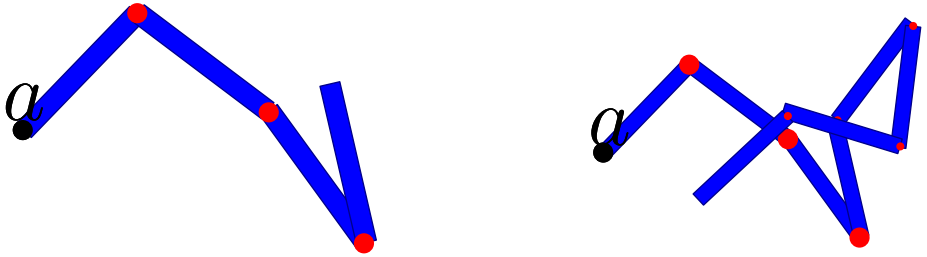

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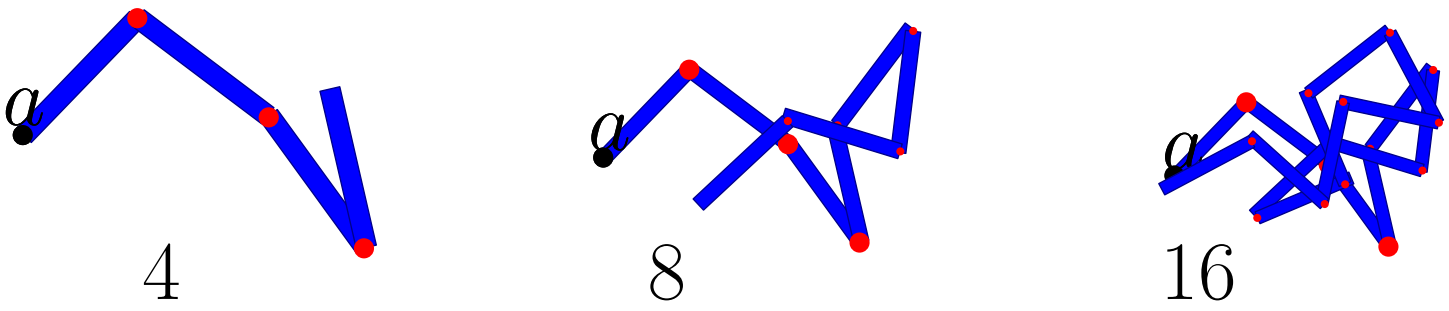

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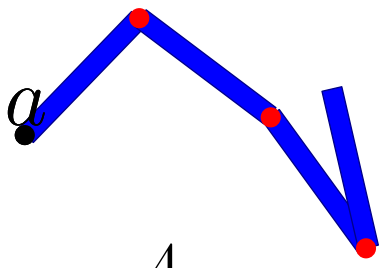

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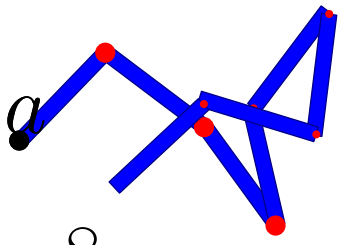

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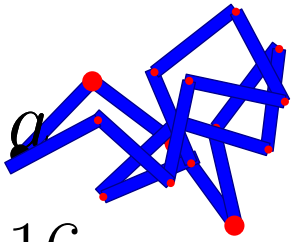

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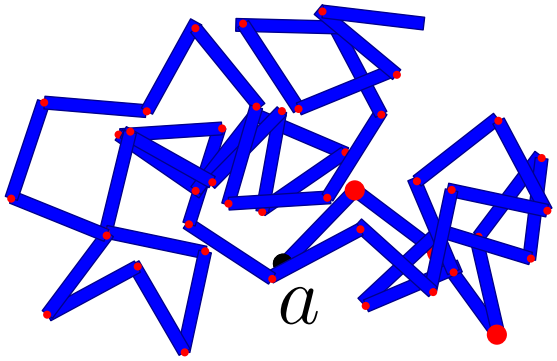

beaucoup...

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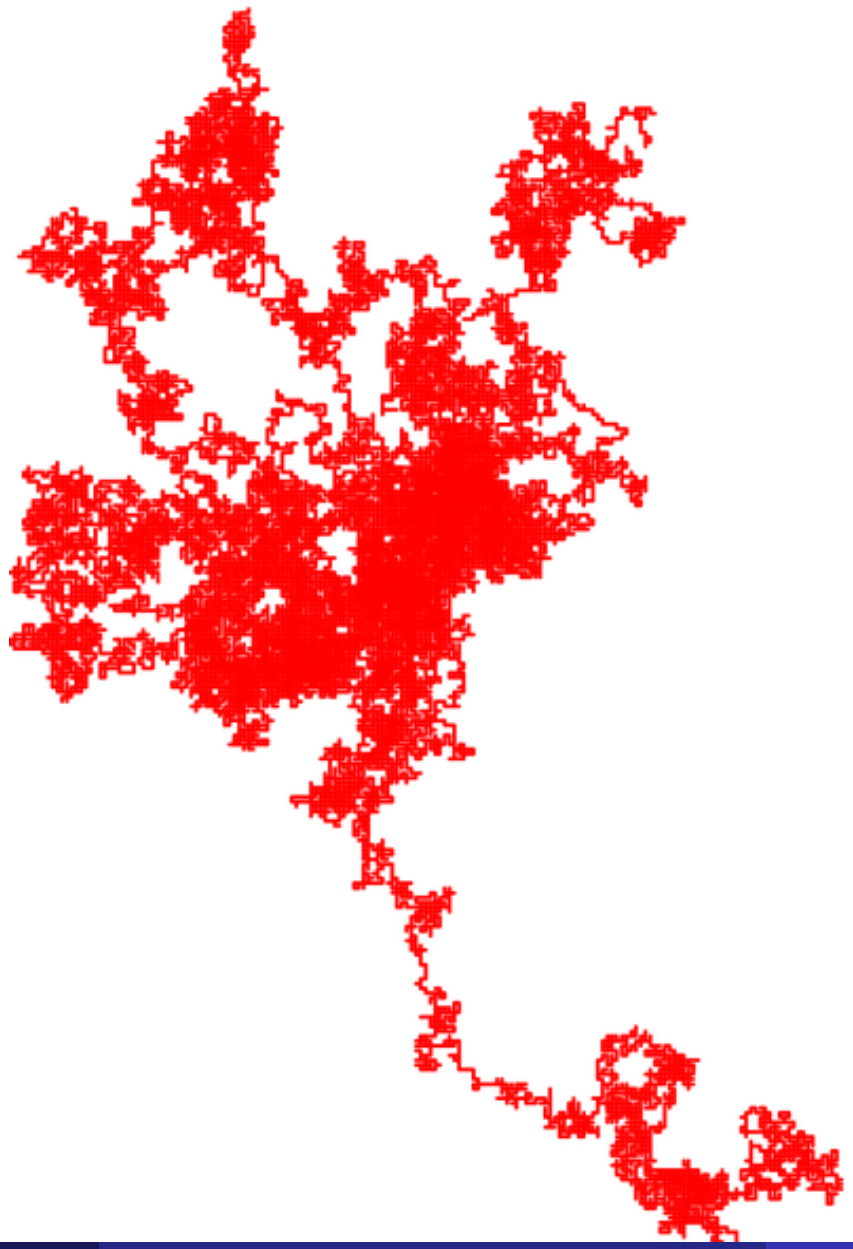

Un point fixe plus récent : la percolation

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$f : \mathbb{X} \to \mathbb{X}$

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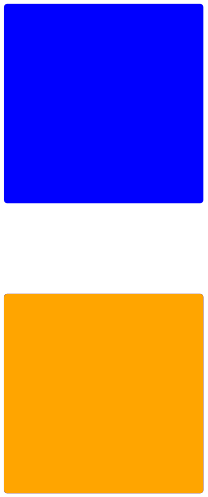
$f : \mathbb{X} \to \mathbb{X}$

$x_0 = \mu_0(p) =$



probabilité p
(exemple $p = 0.8$)

probabilité $1 - p$
(exemple $1 - p = 0.2$)

Un point fixe plus récent : la percolation

$\mathbb{X} = \{\text{Coloriages (aléatoires) du plan}\}$

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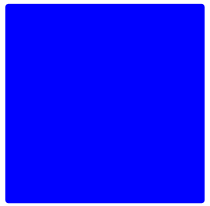

$p = 0.8$

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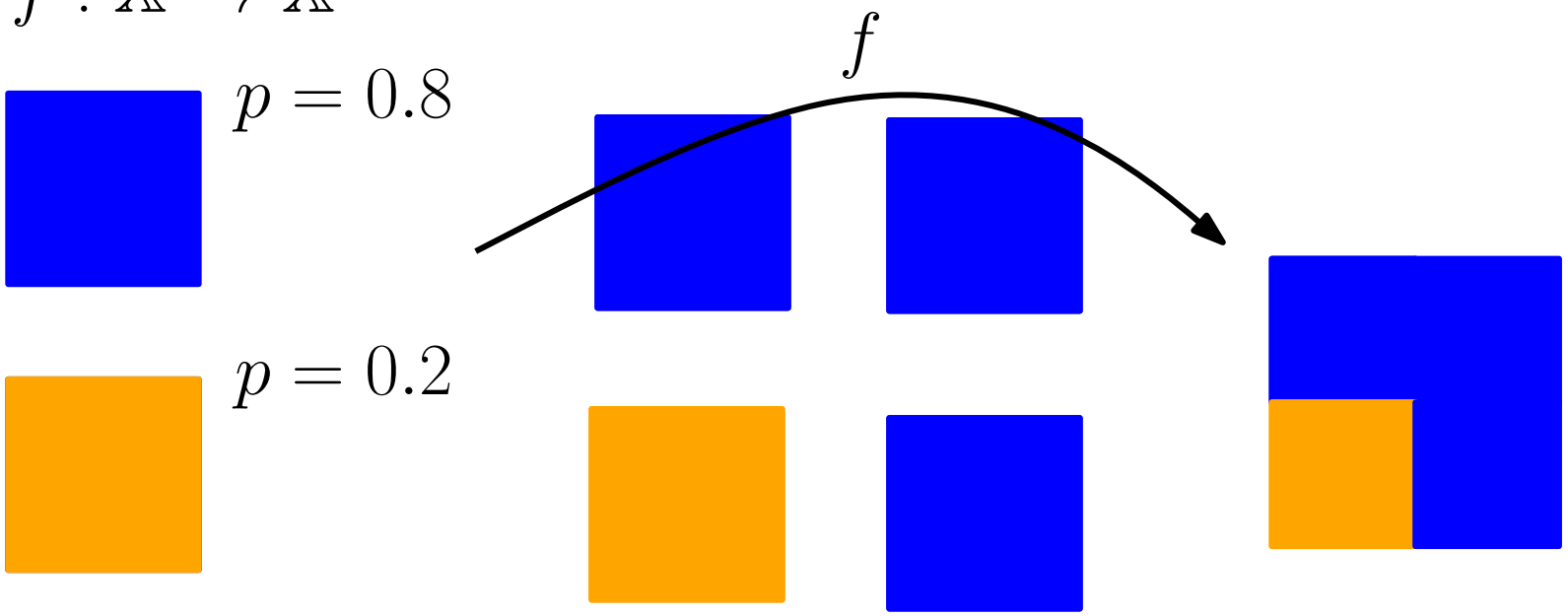

Un point fixe plus récent : la percolation

$f(f())?$

Un point fixe plus récent : la percolation

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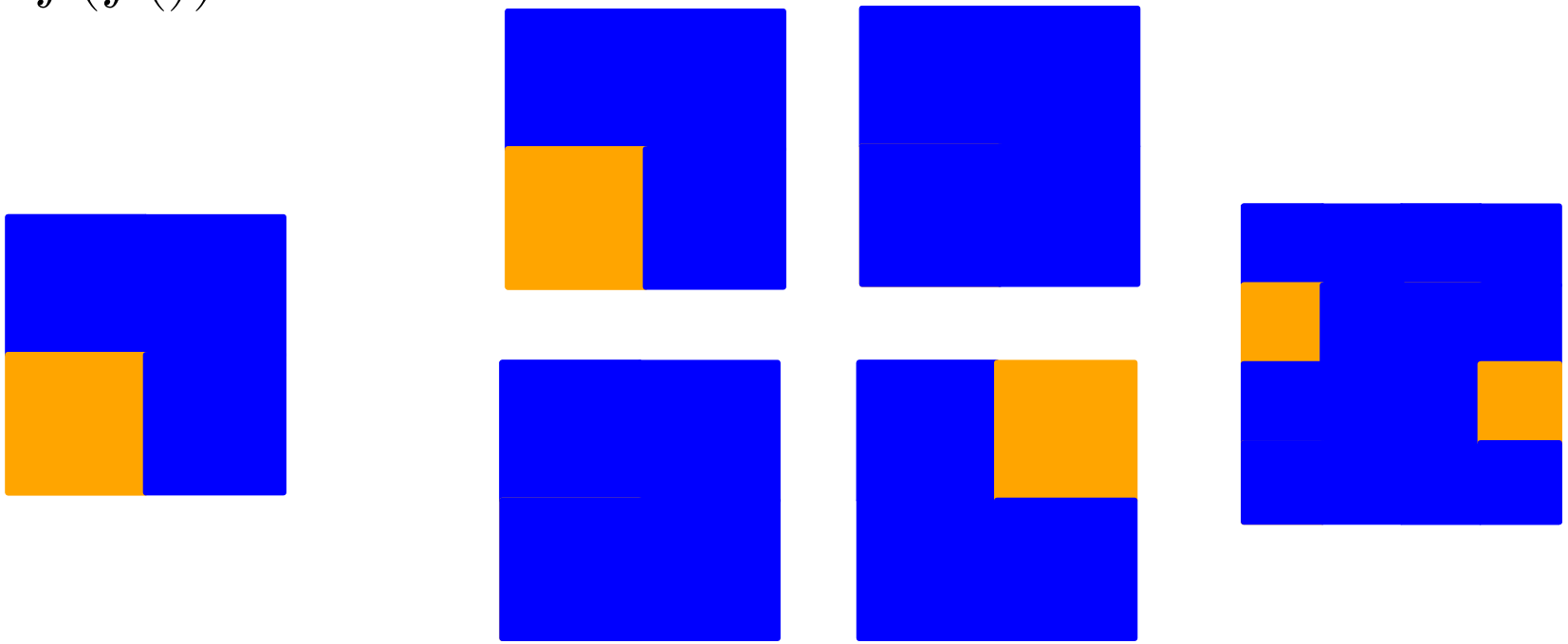
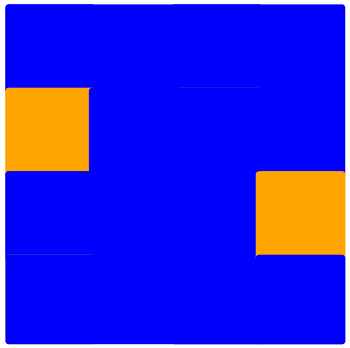
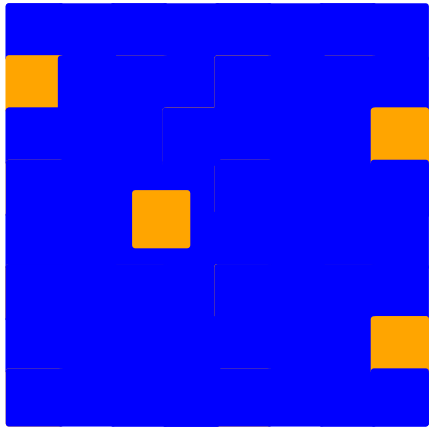


Un point fixe plus récent : la percolation

$f(f(f(\ )))?$





Un point fixe plus récent : la percolation

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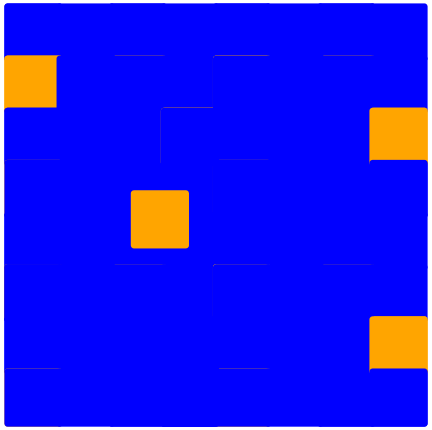

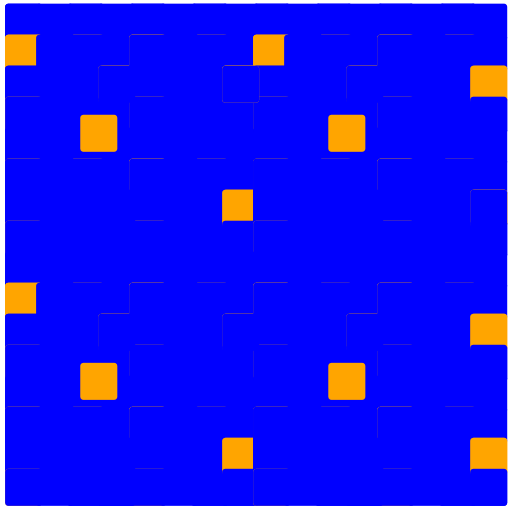

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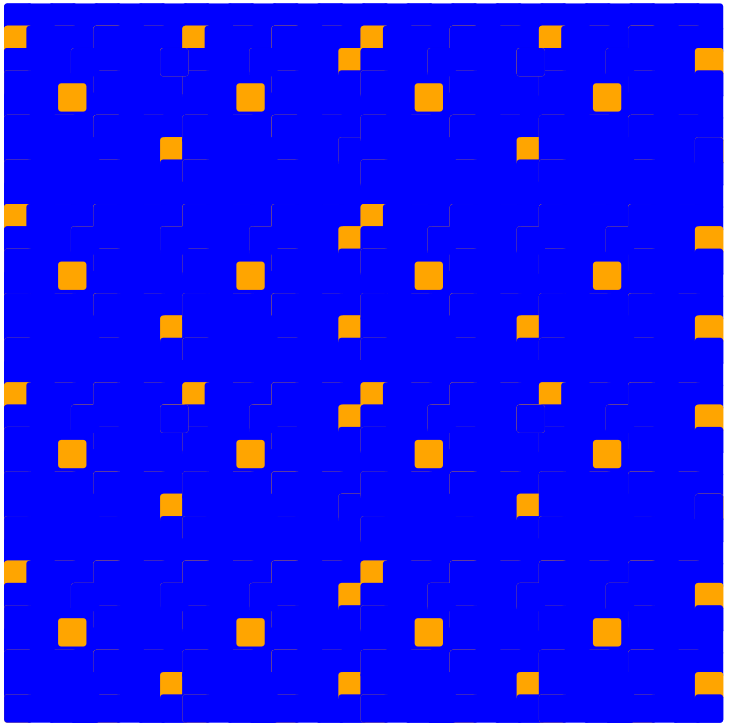
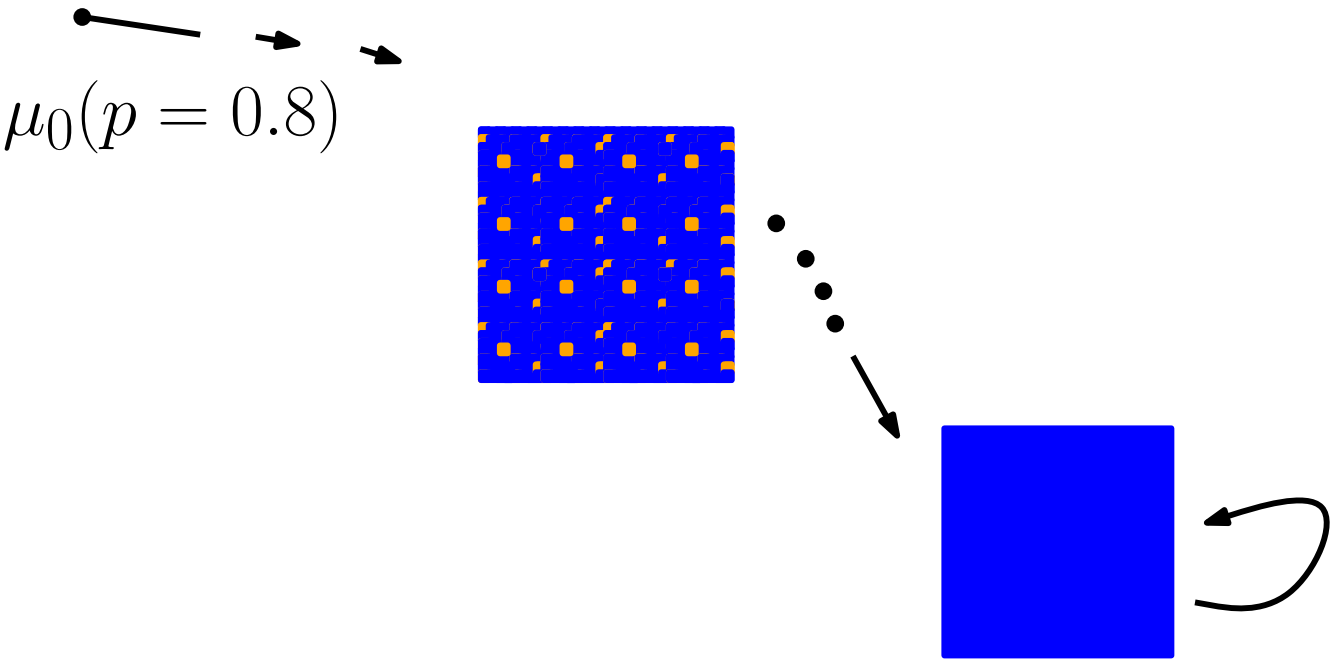


Un point fixe plus récent : la percolation

Espace des phases



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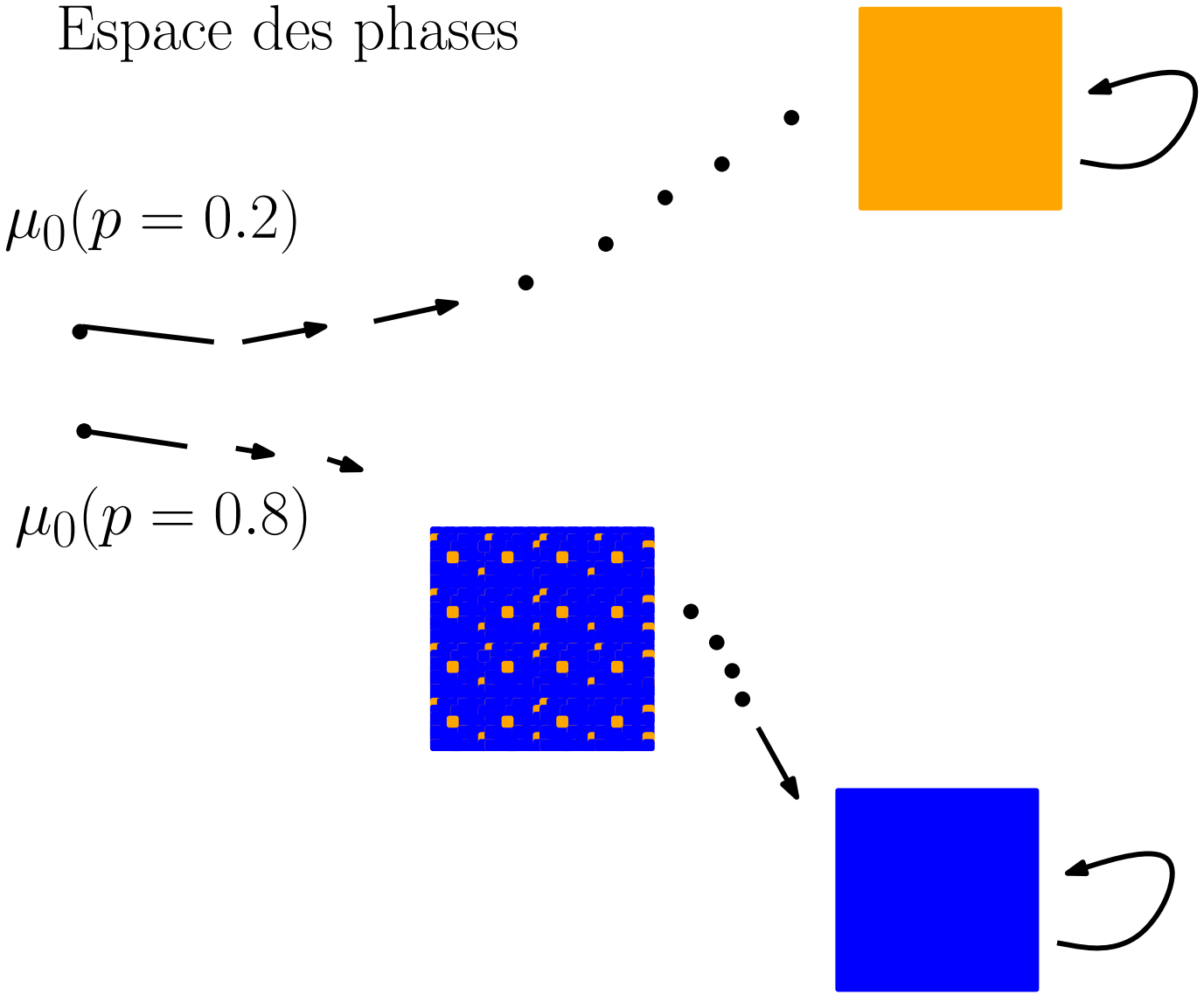

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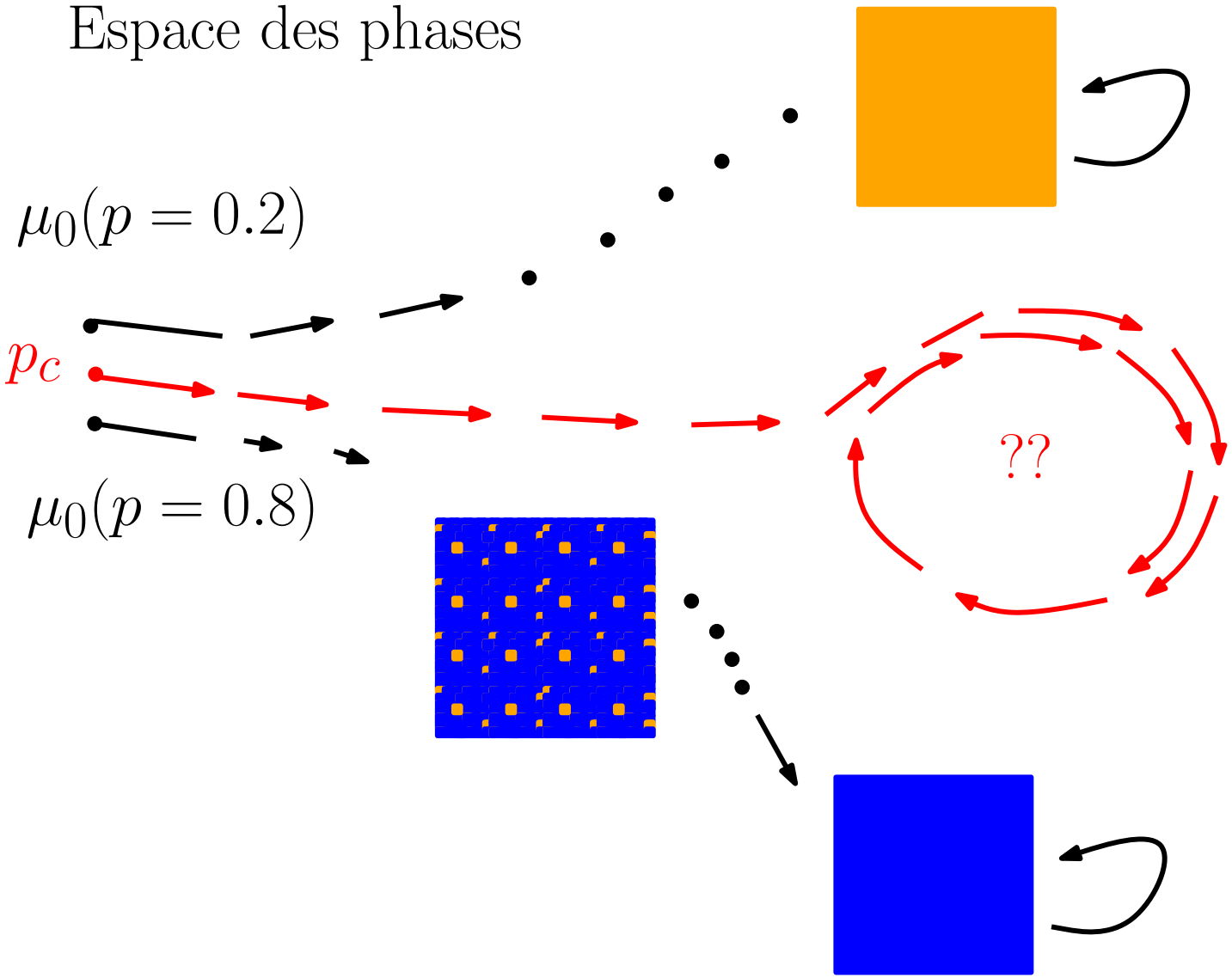

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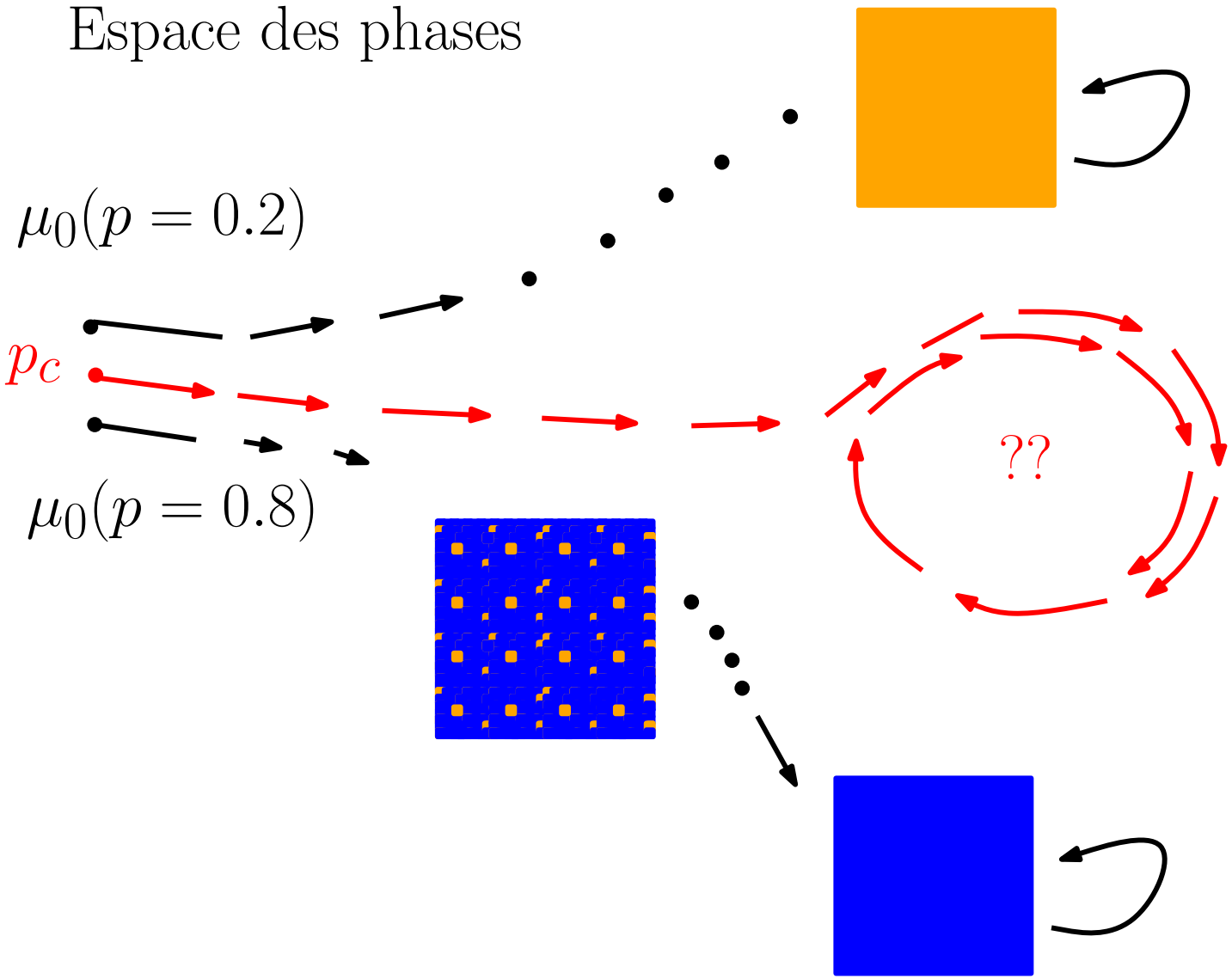
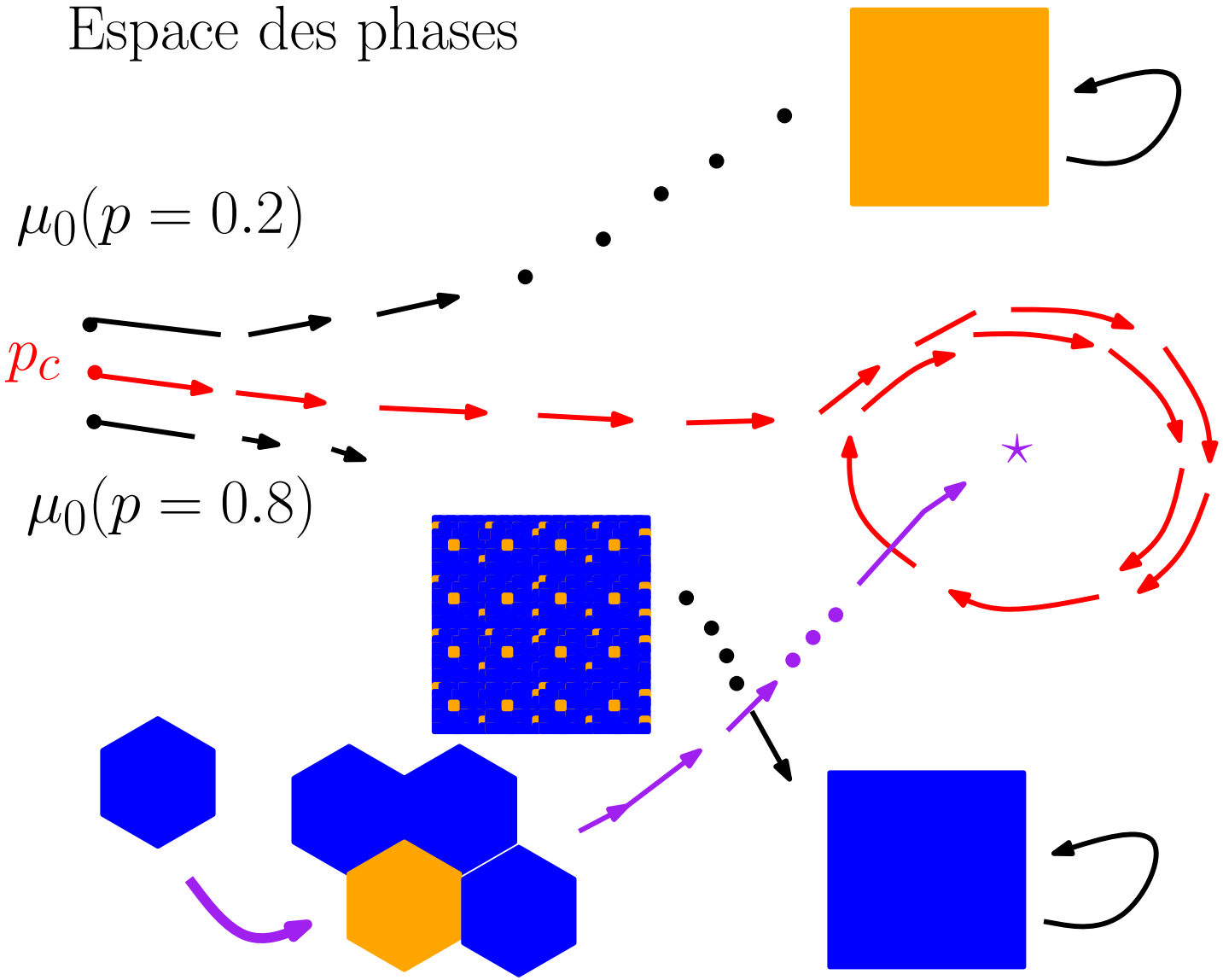
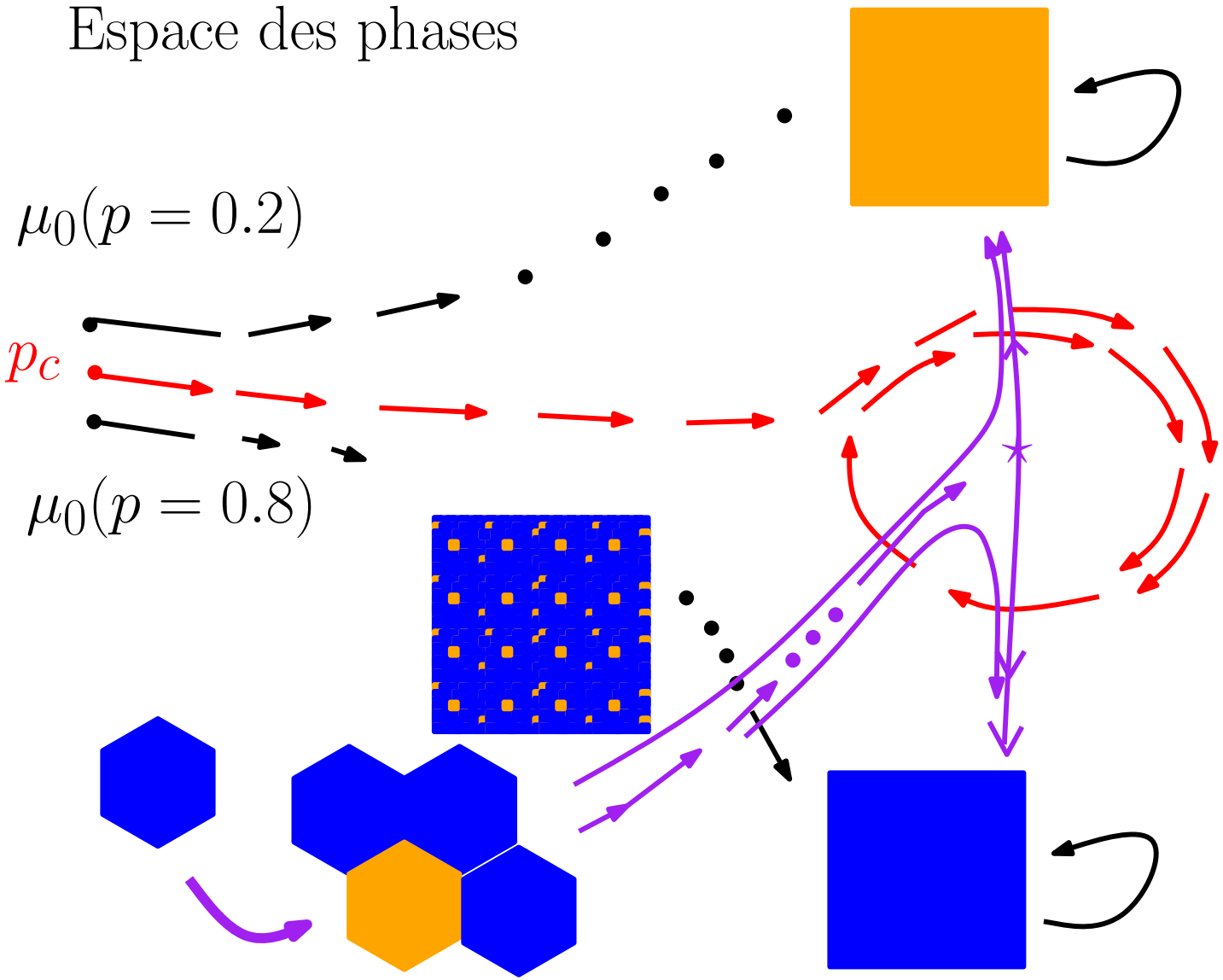


Un point fixe plus récent : la percolation



Un point fixe plus récent : la percolation



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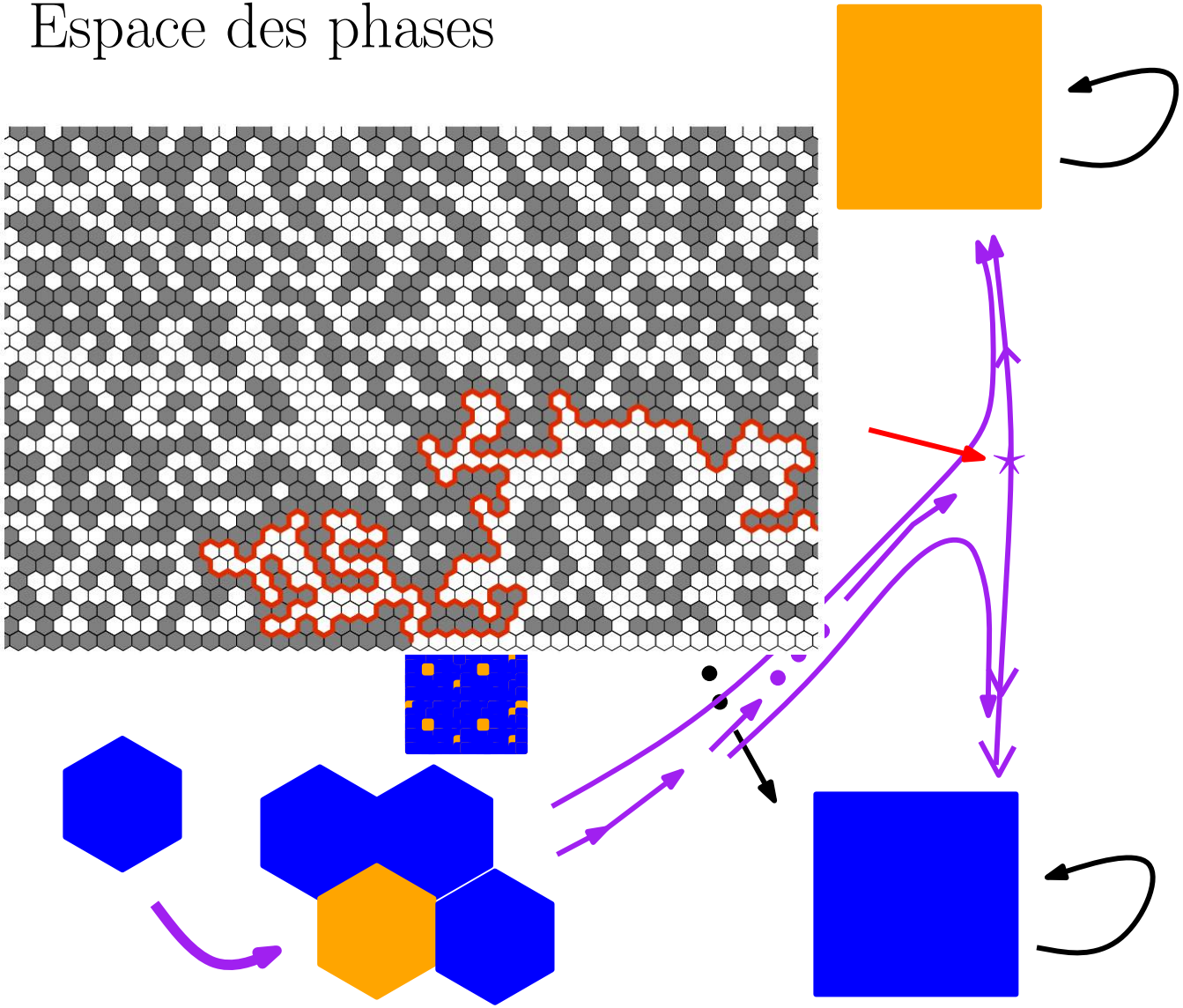
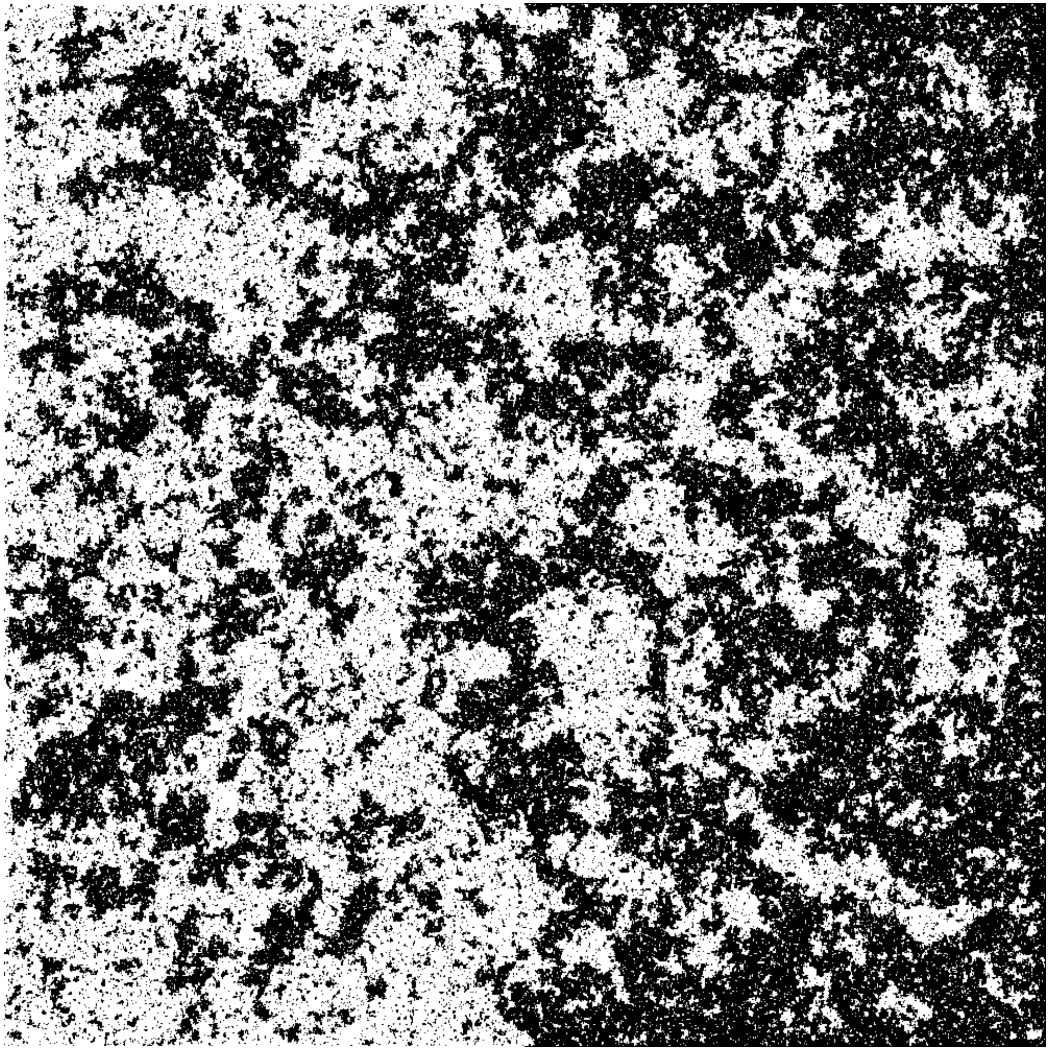


Une feuille métallique à $T = T_{Curie}$



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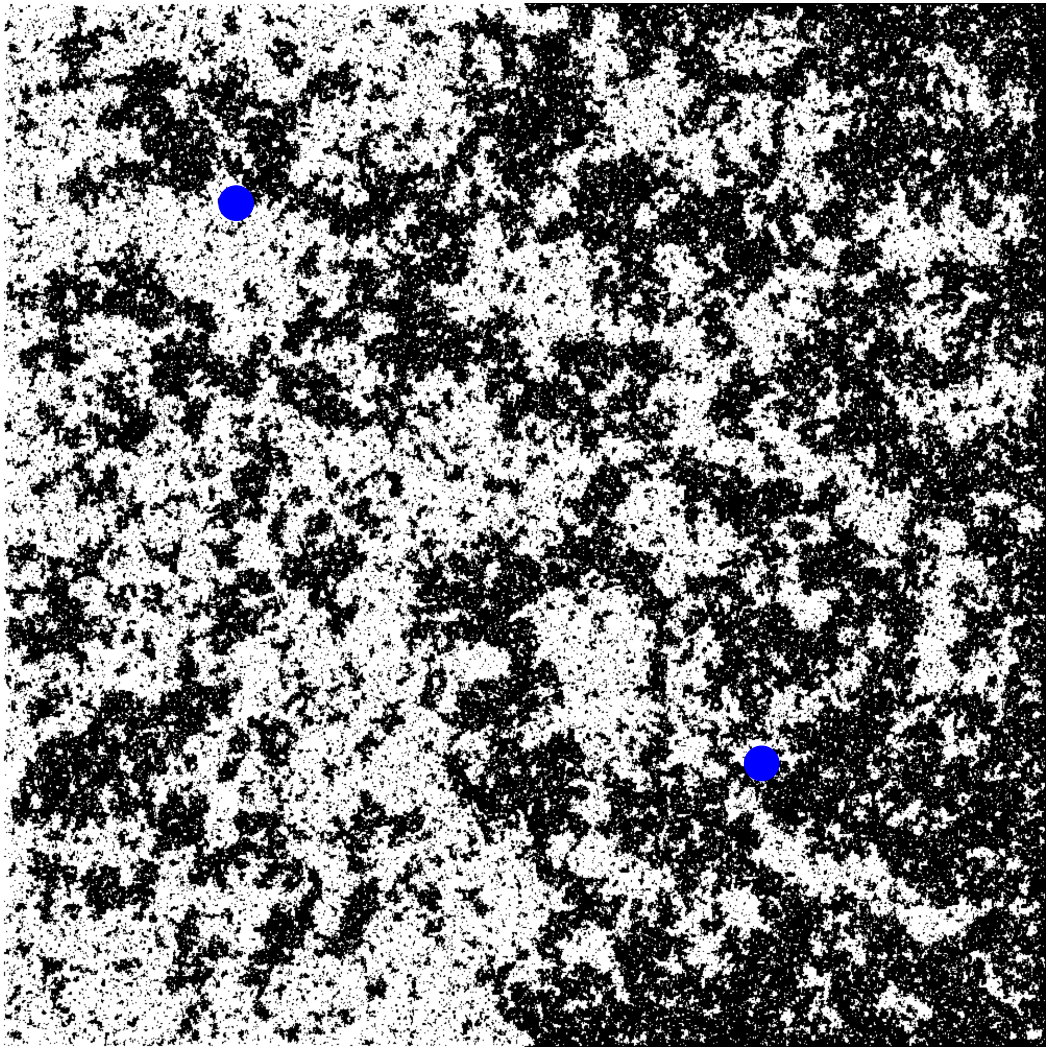
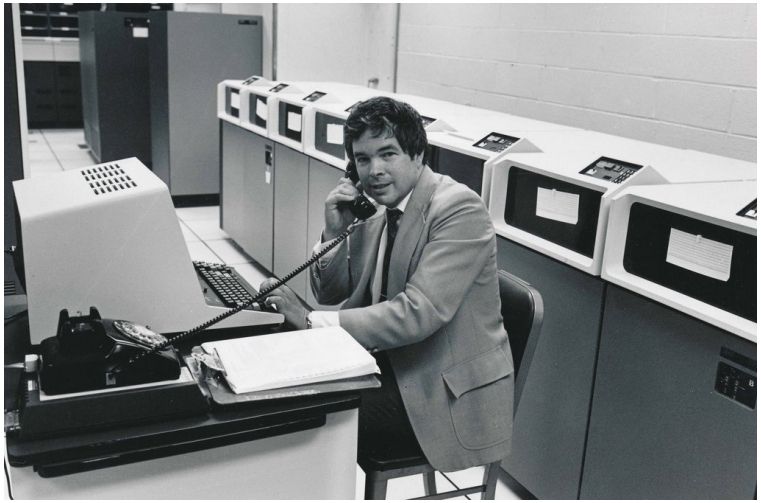


$$\mathbb{E}[\sigma_x \sigma_y] \approx \frac{1}{\|x-y\|^{1/4}}$$

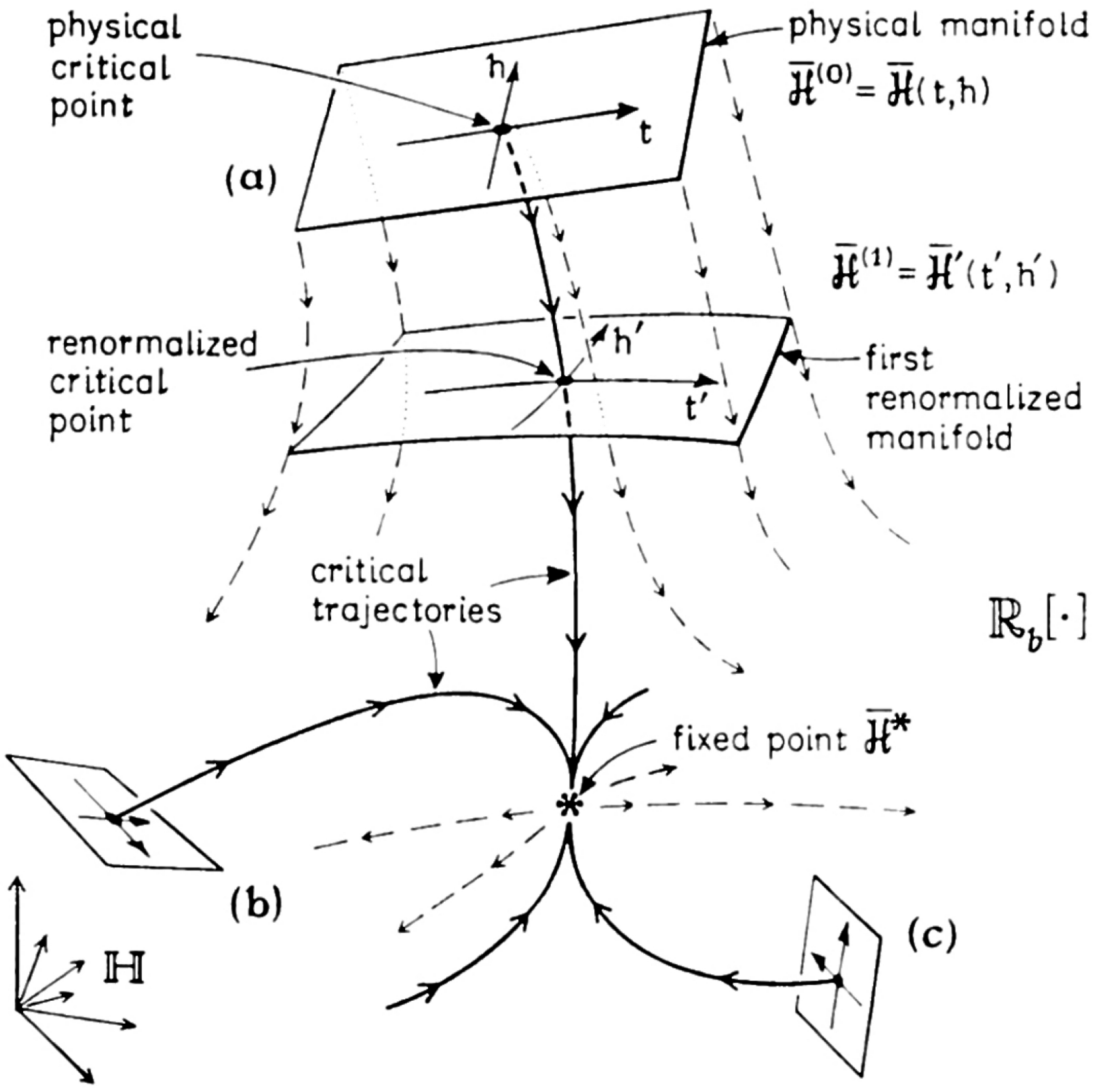
Kenneth Wilson et le “Groupe de Renormalization”



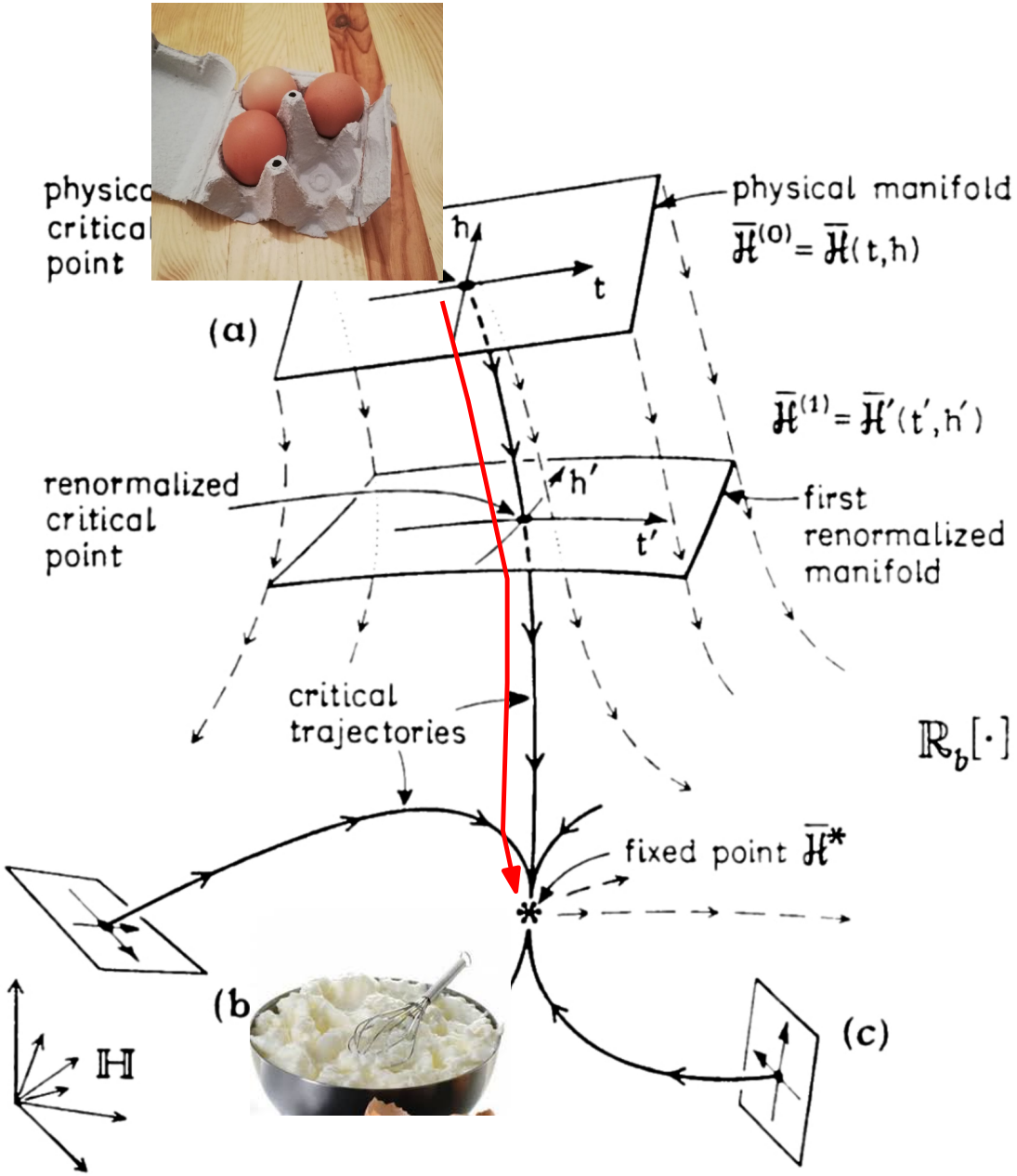
Kenneth Wislon, prix Nobel de physique en 1982.

1. Il a réussi à "calculer" la transition de phase des aimants en $d = 3$ en **trouvant à l'aide de l'ordinateur (70's!!)** un **point fixe non trivial** en dimension
$$d = 4 - \varepsilon \text{ !!!}$$
2. Sa théorie permet d'étudier d'étudier "numériquement" **Yang-Mills** (chromodynamique quantique, quarks, etc...)

Points fixes et RG group



Points fixes et RG group



Un théorème récent

Théorème (M. Aizenman, H. Duminil-Copin (2019))

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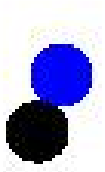
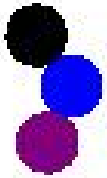
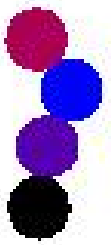
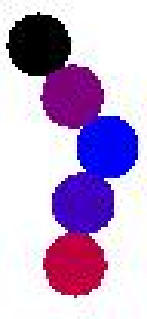
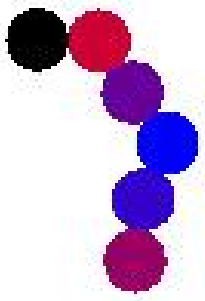
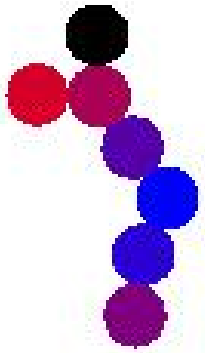
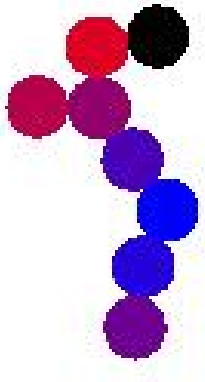
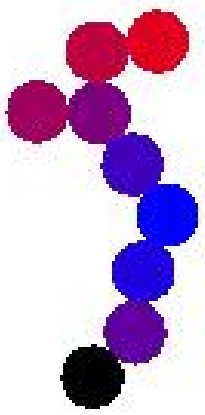
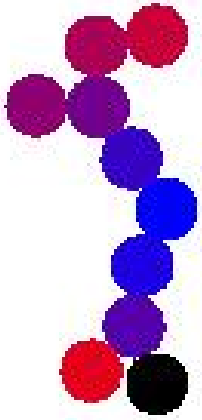
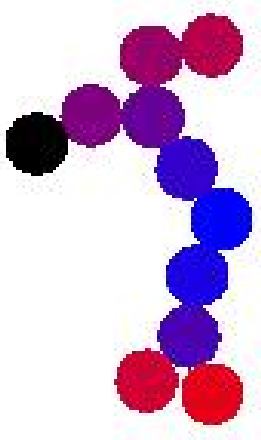
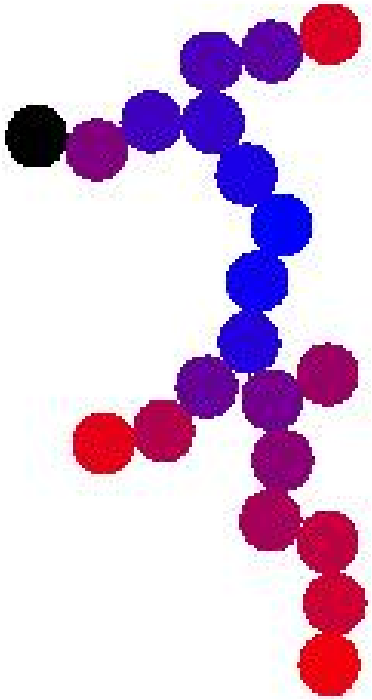
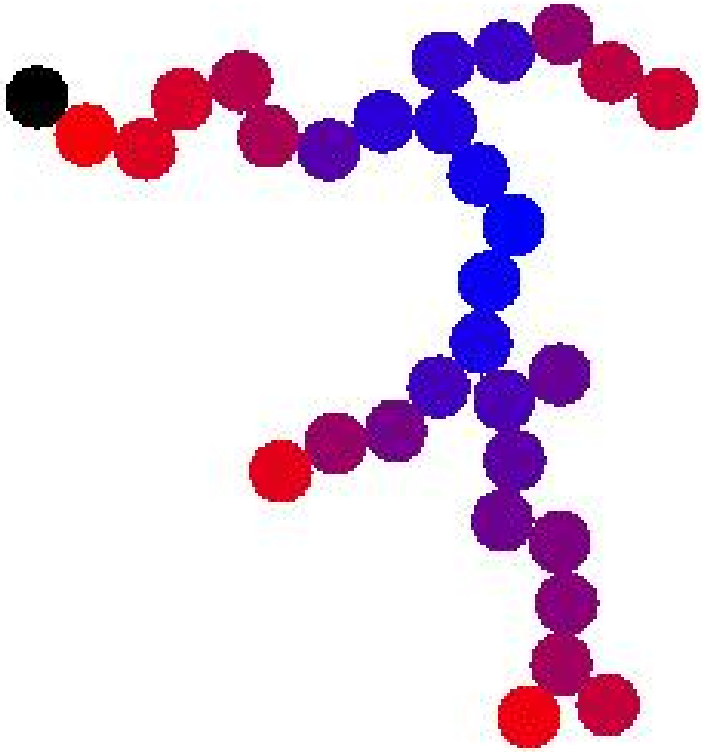
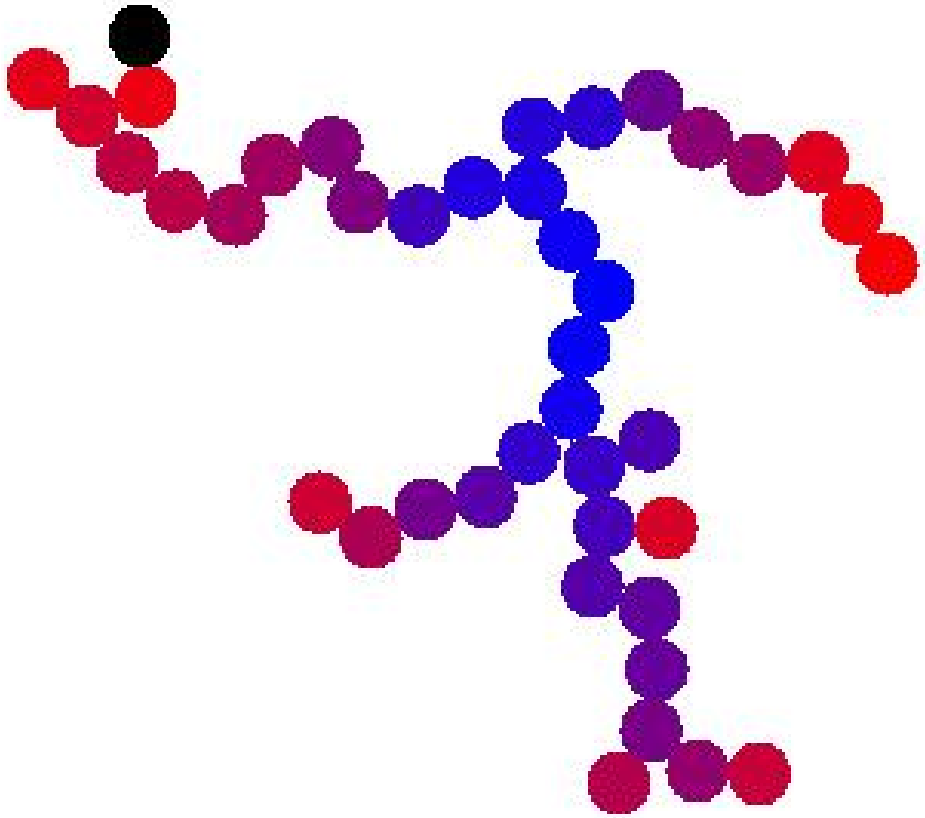
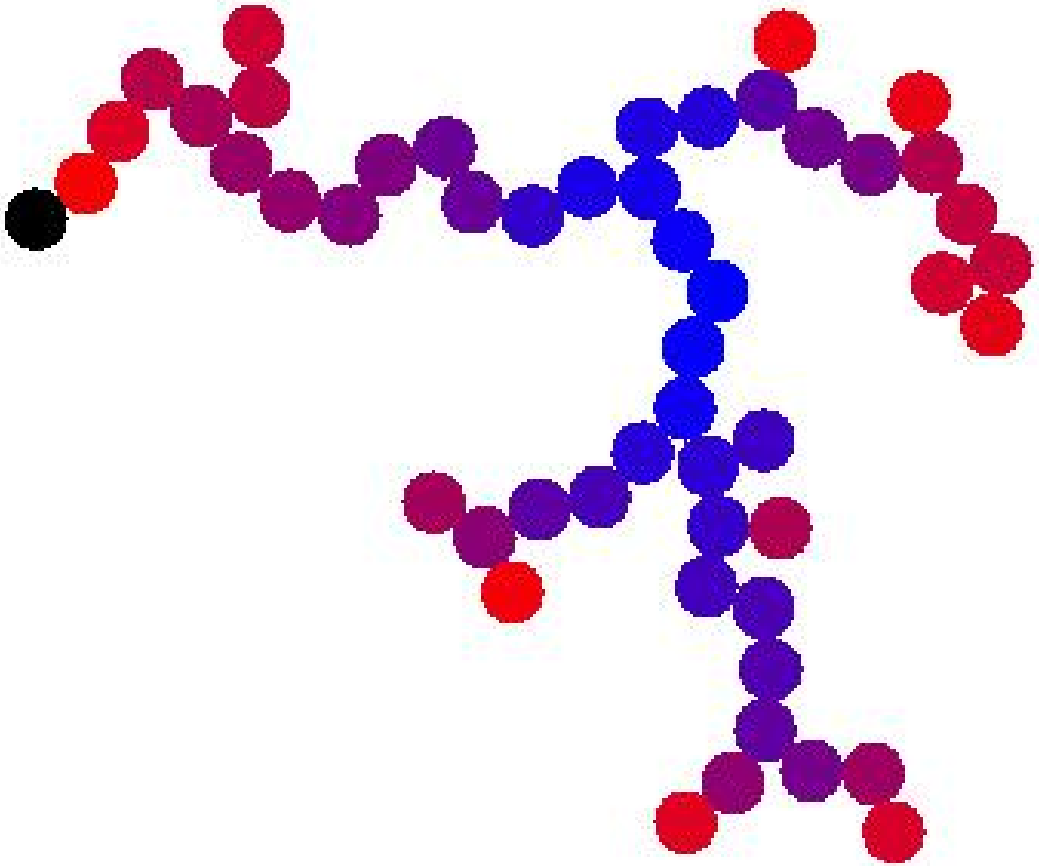
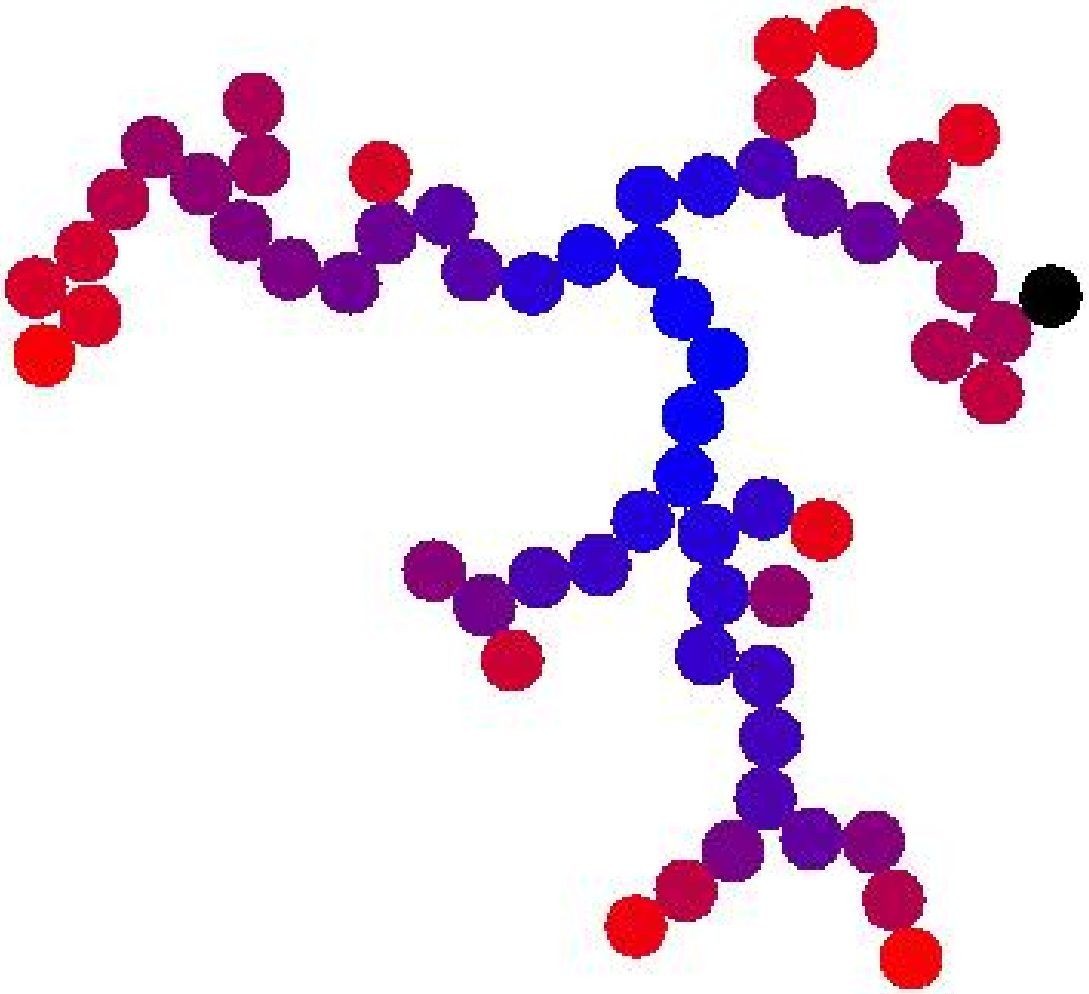
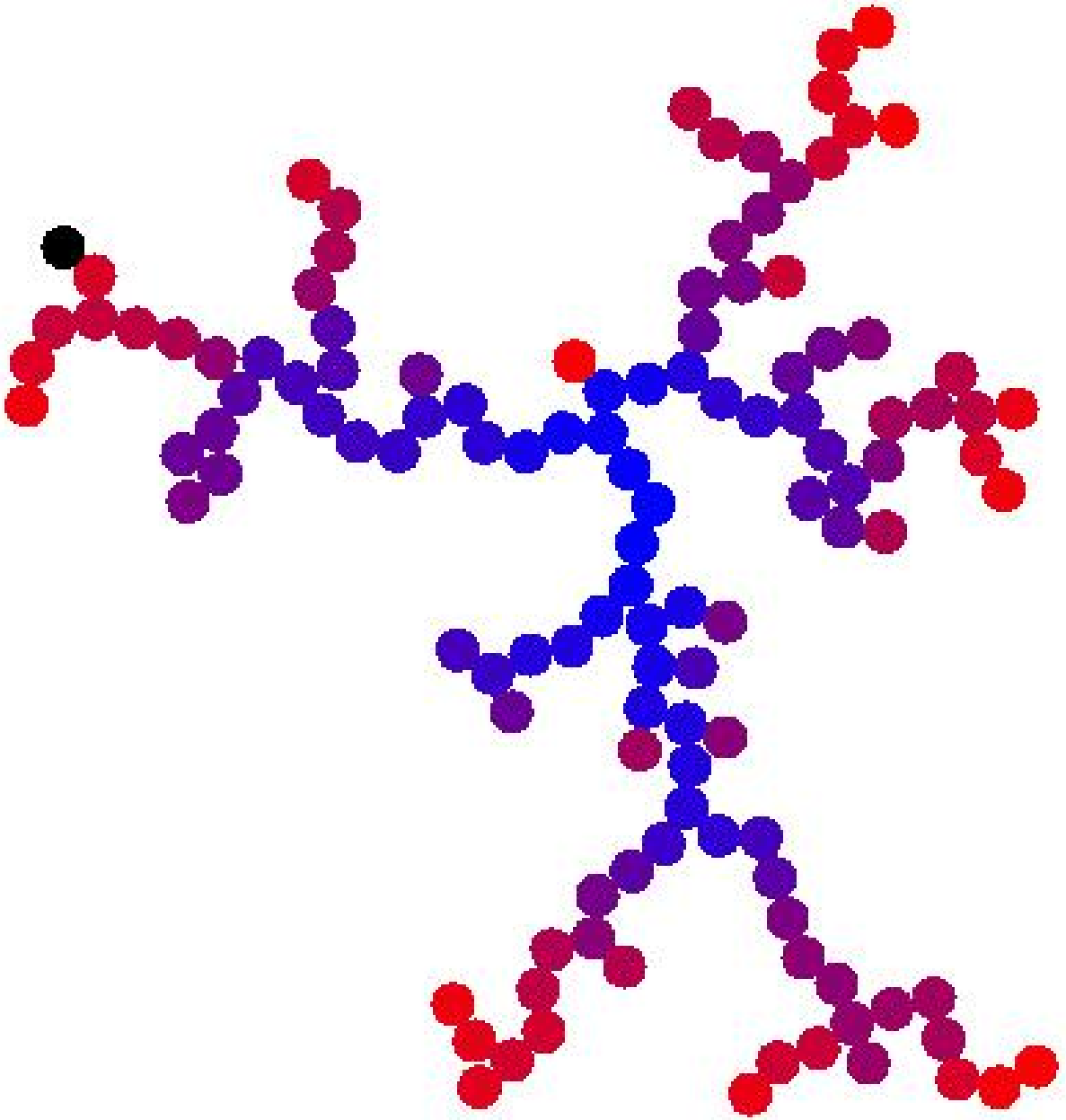
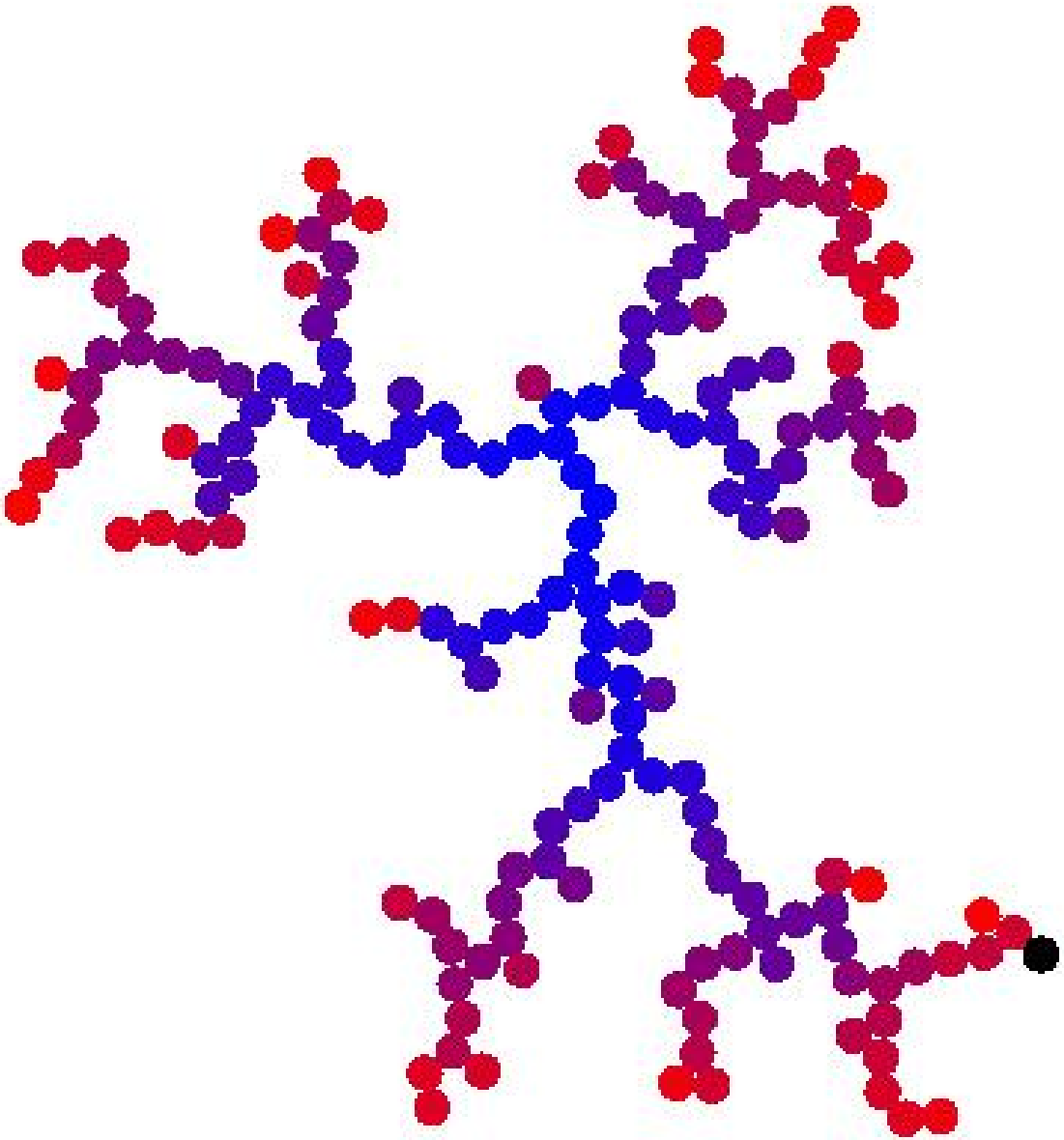
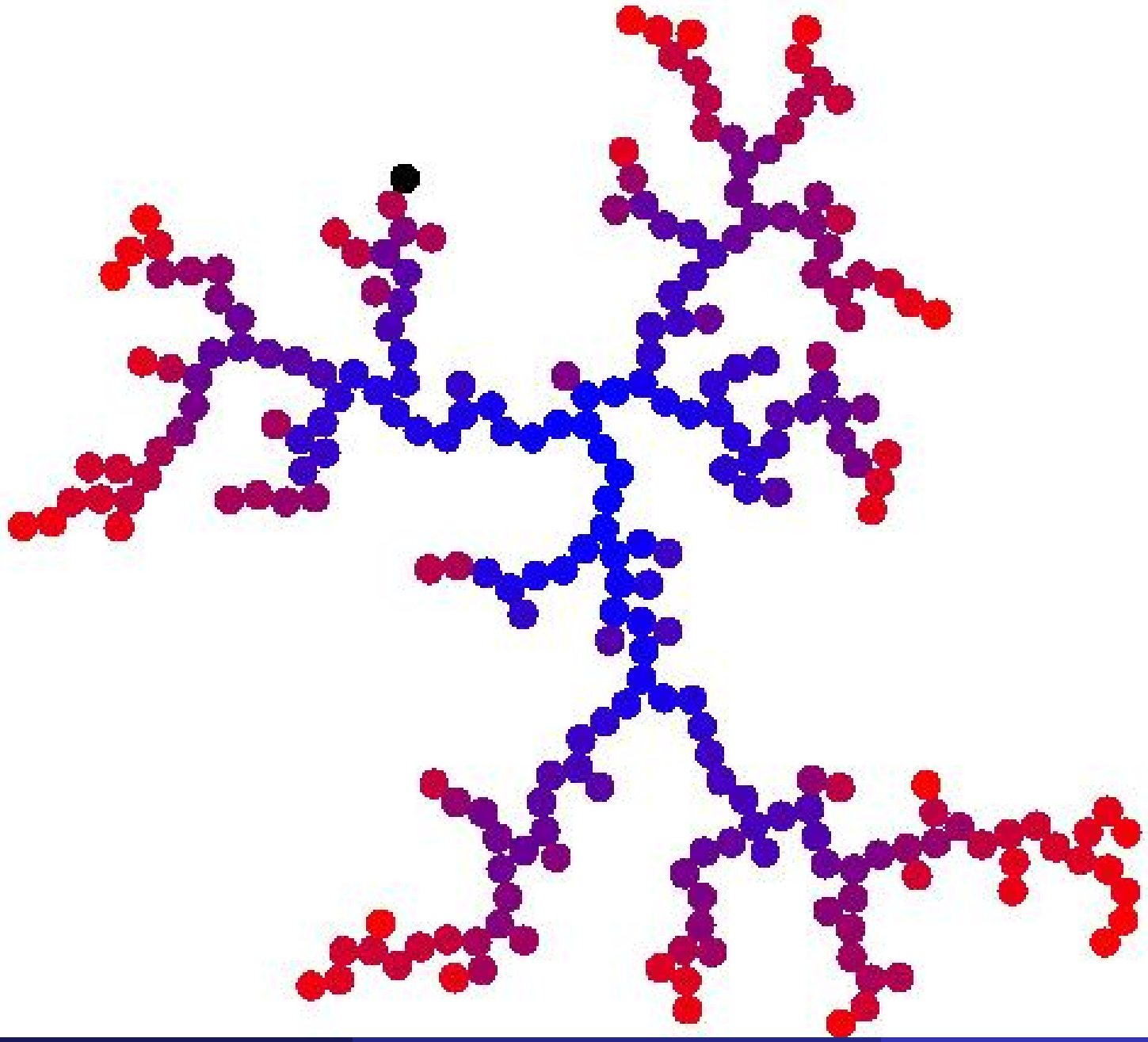
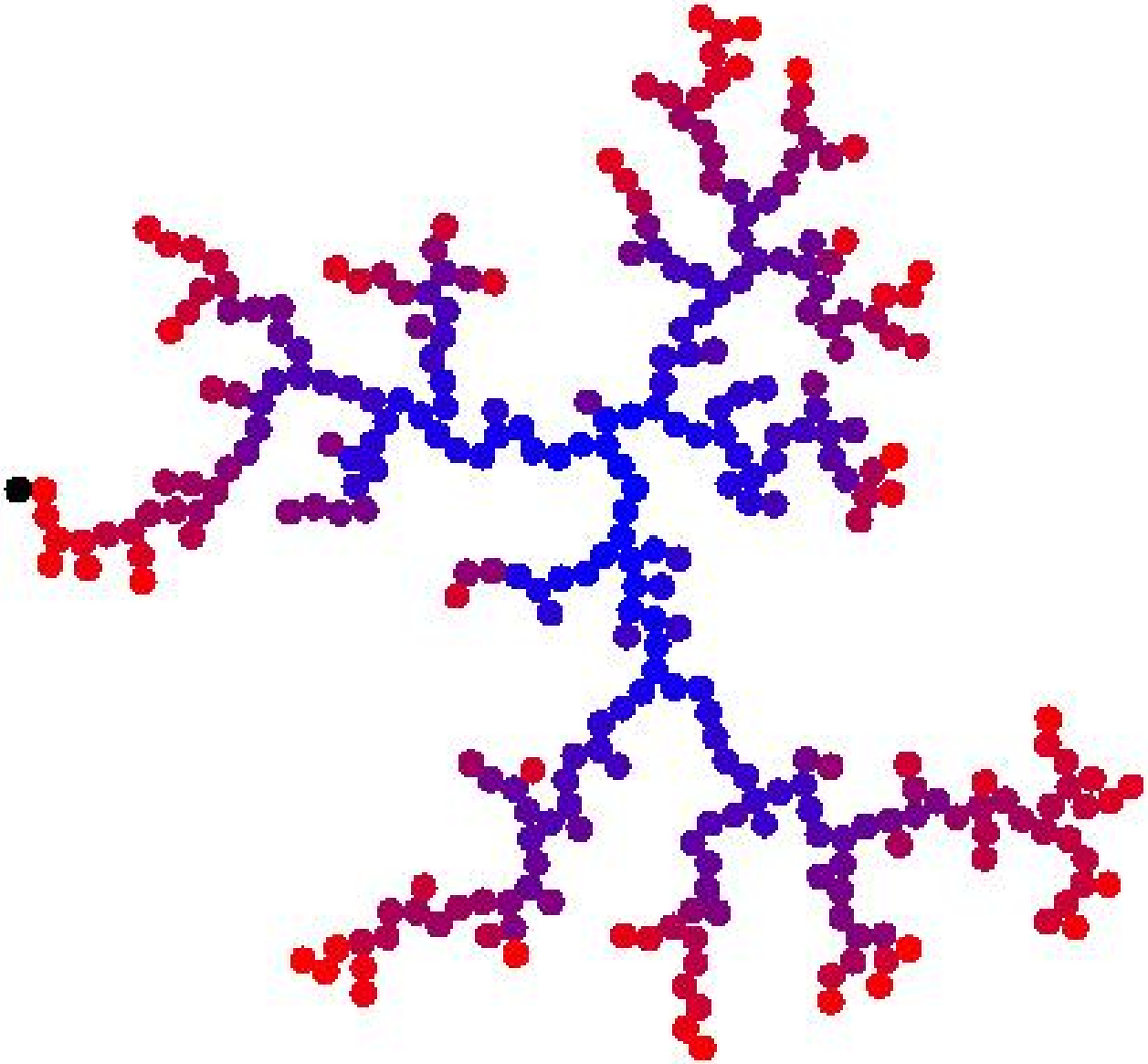
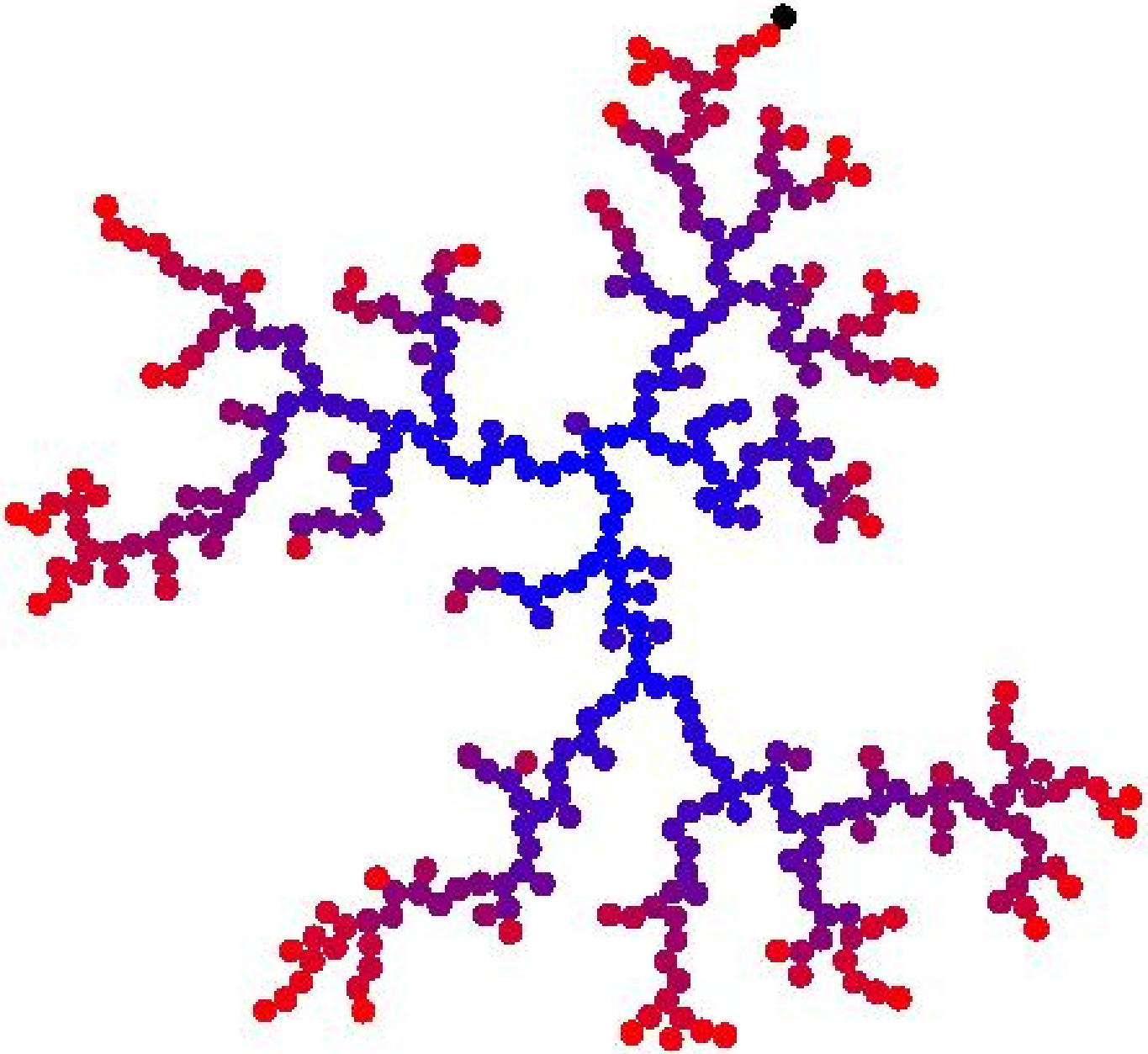
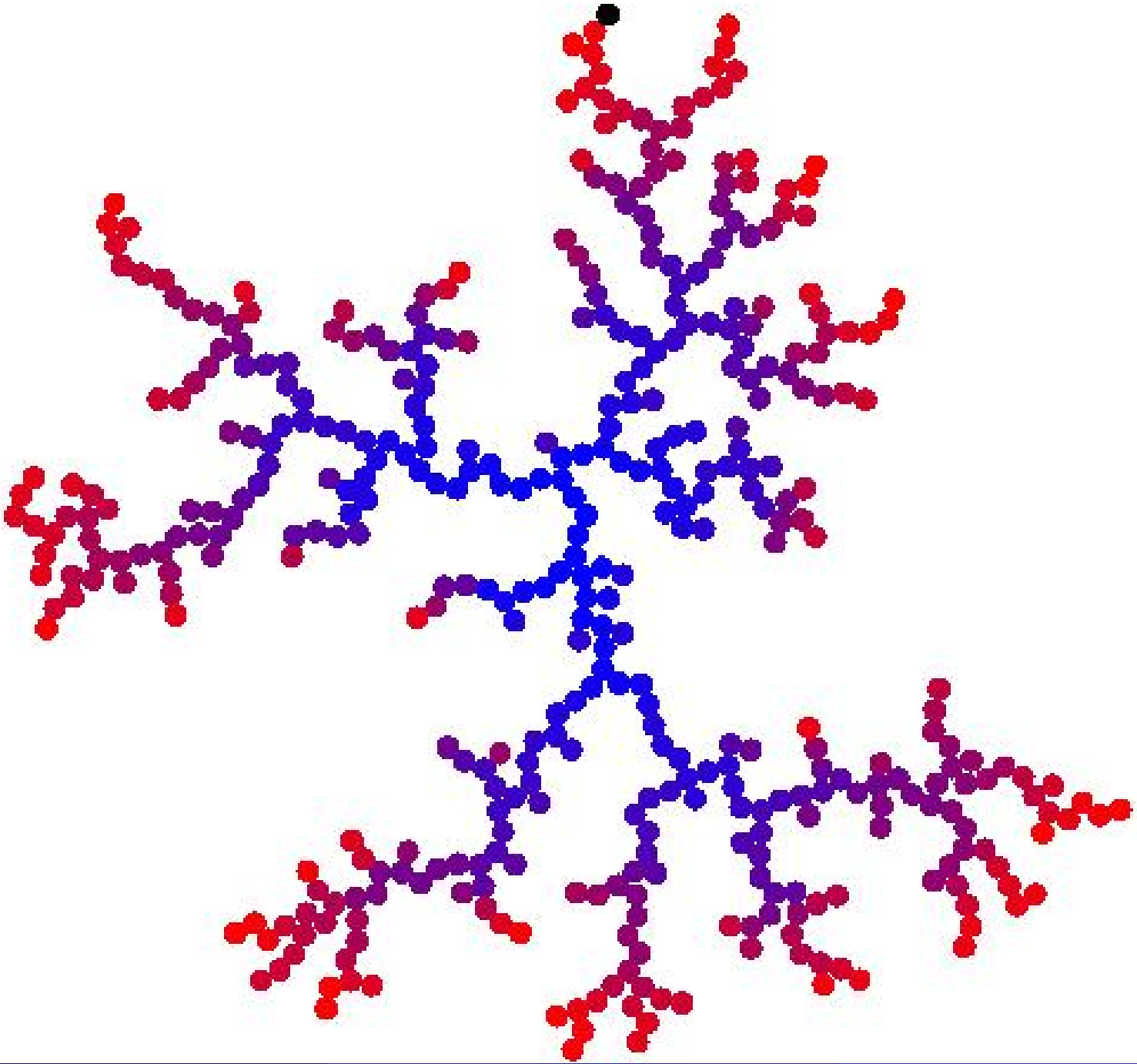
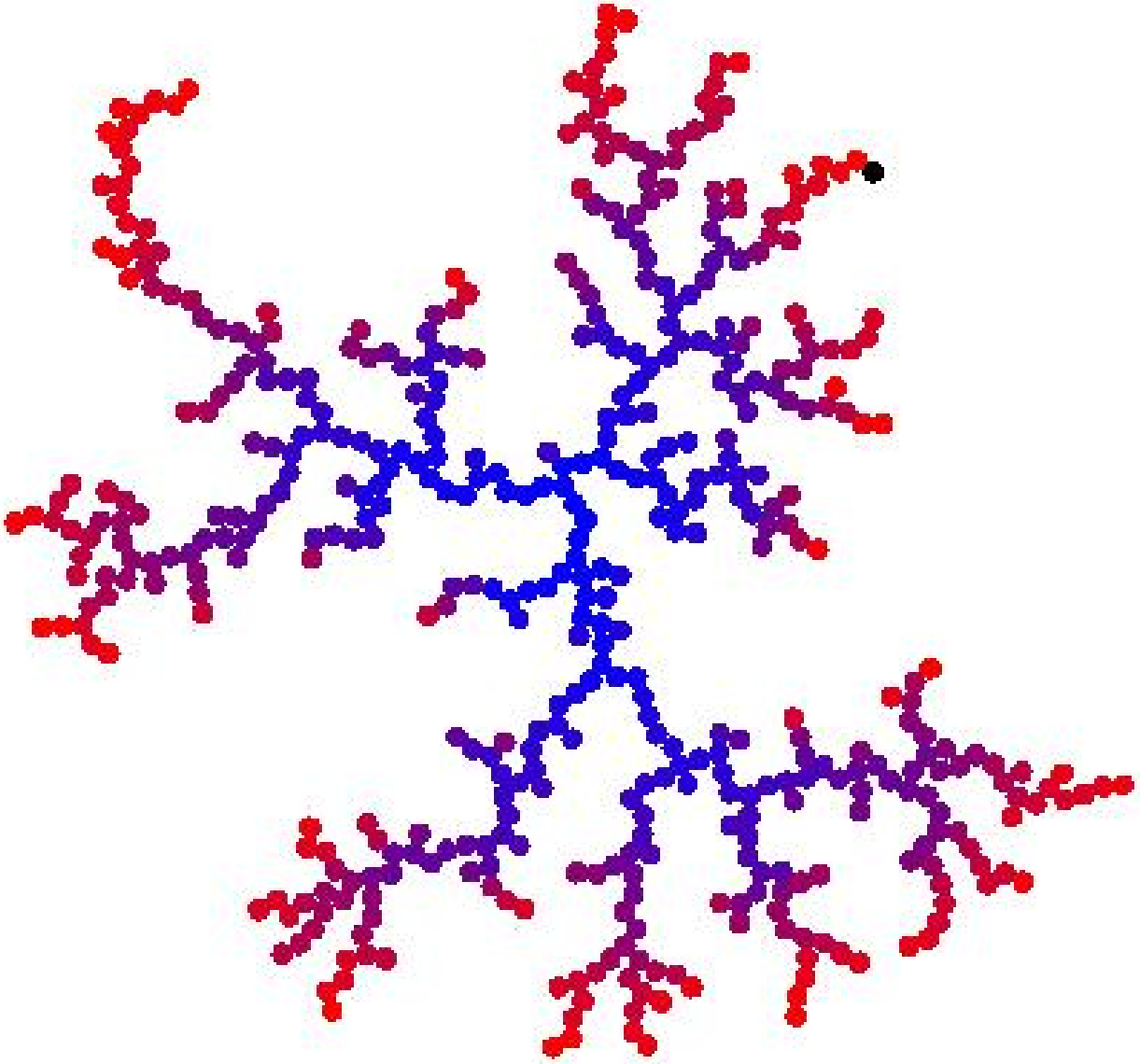
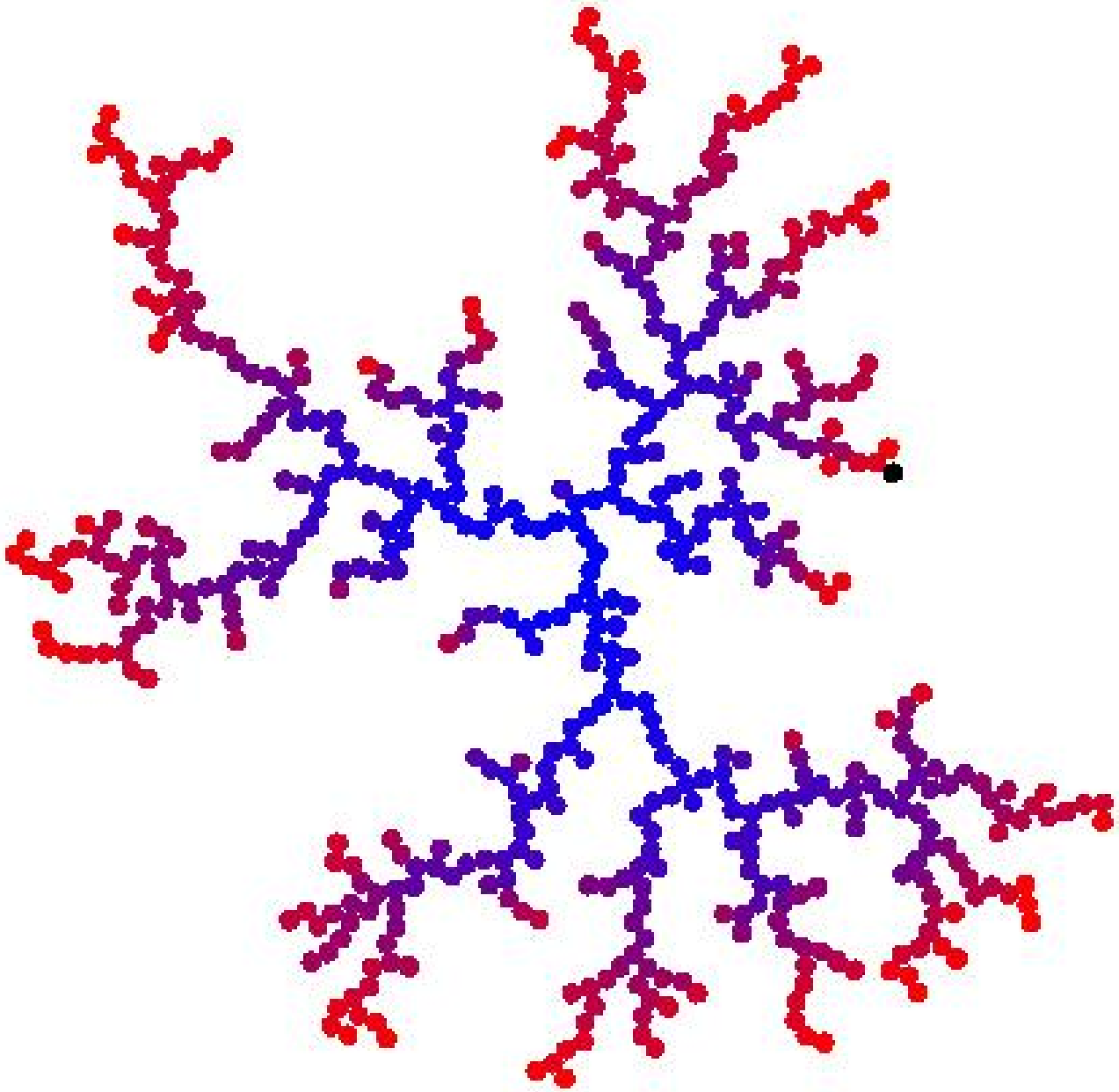
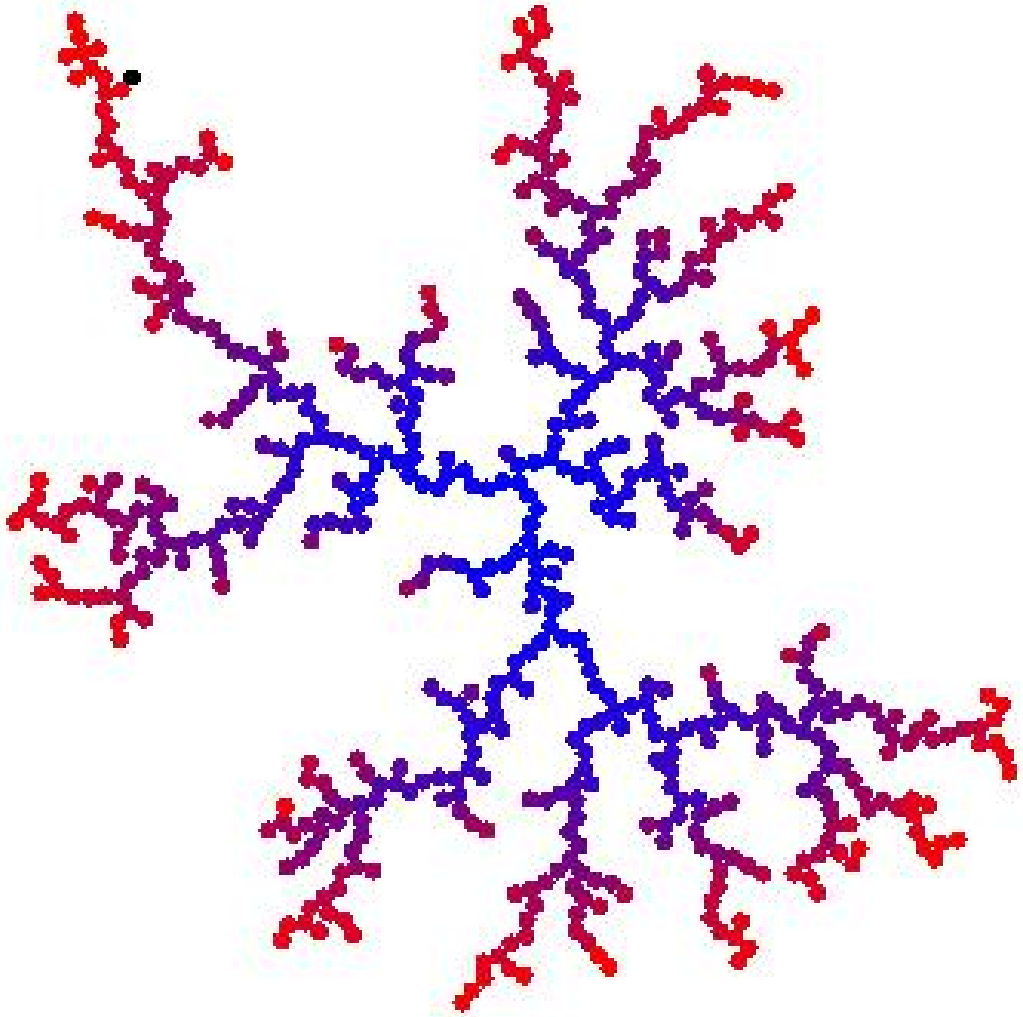
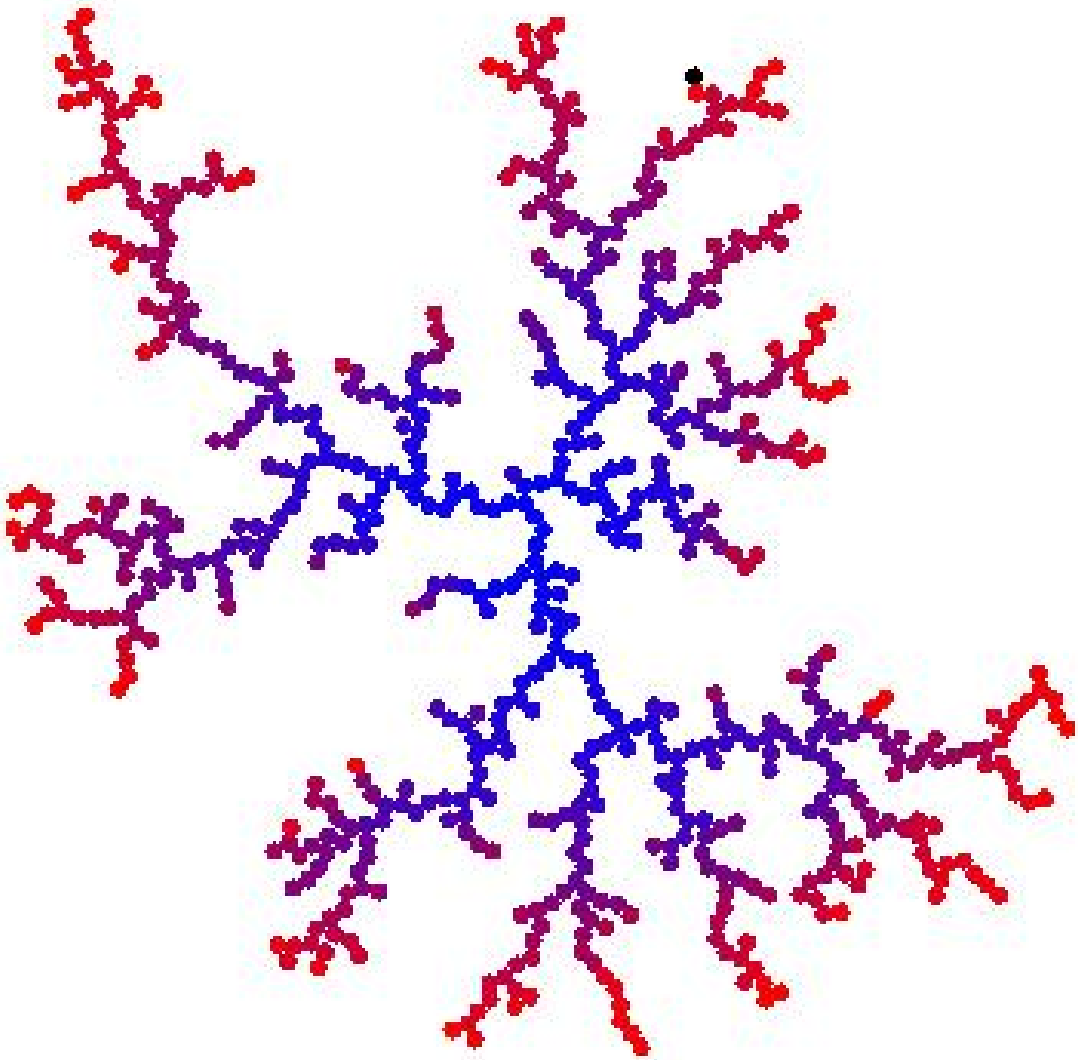
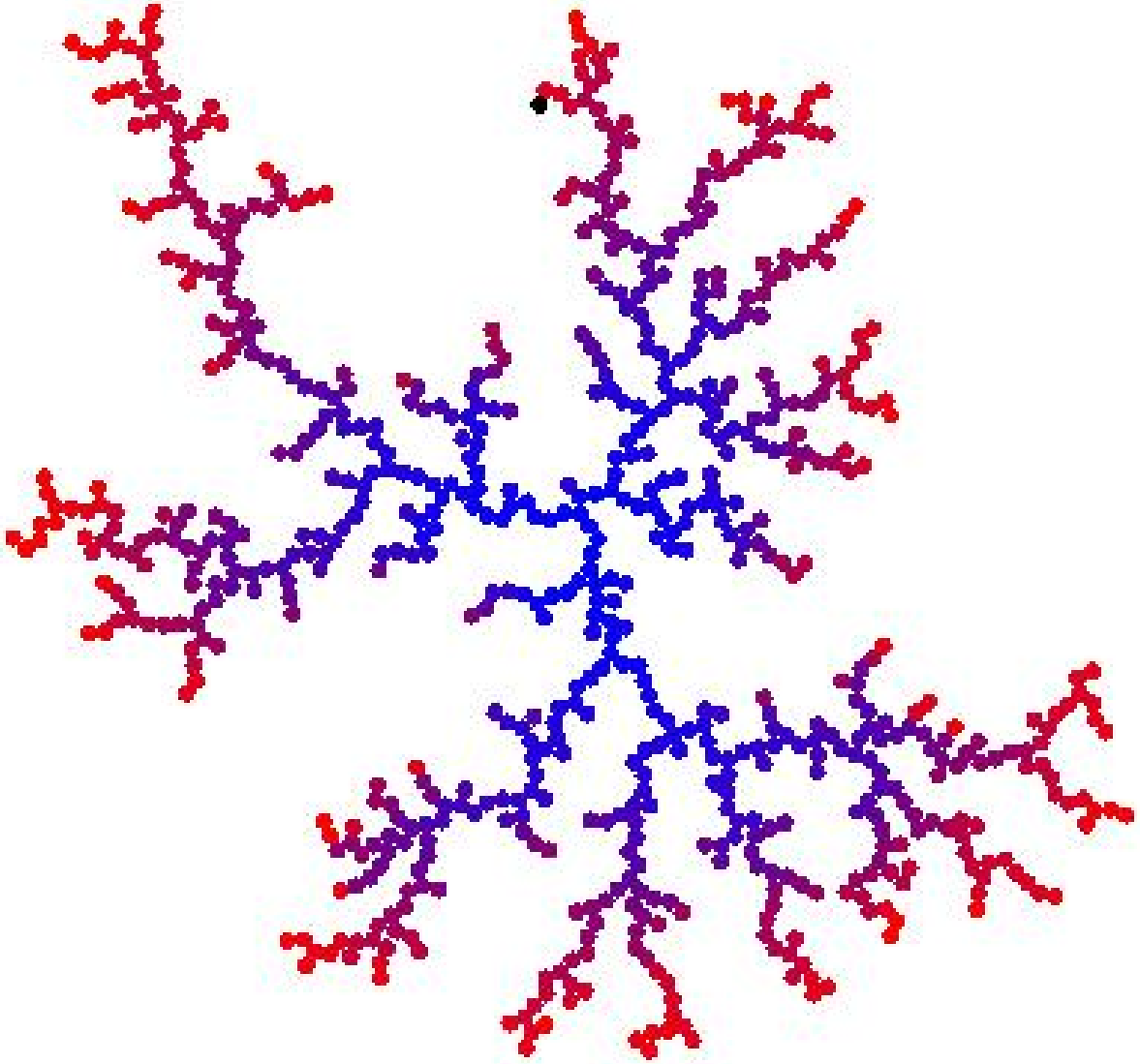
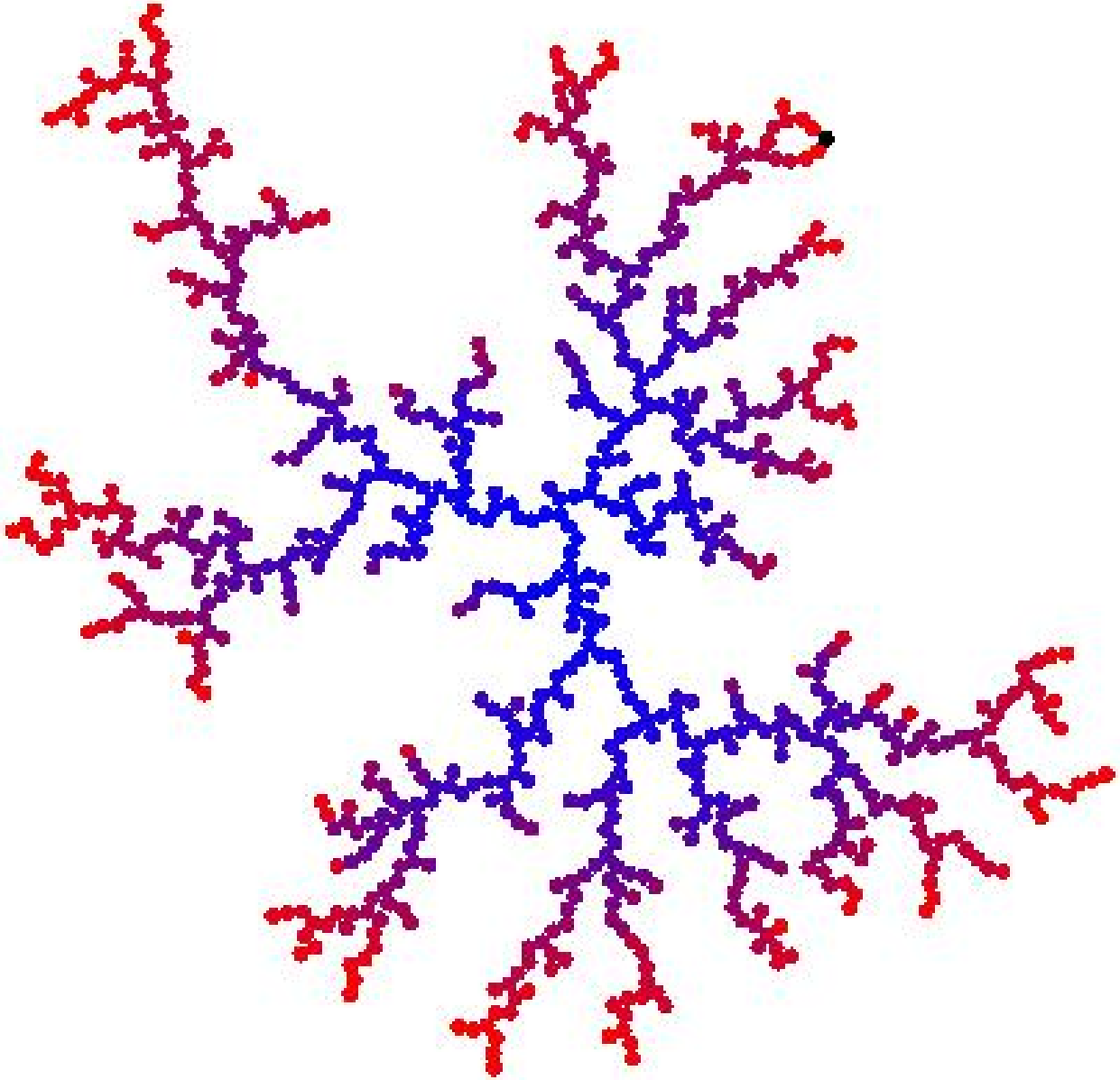
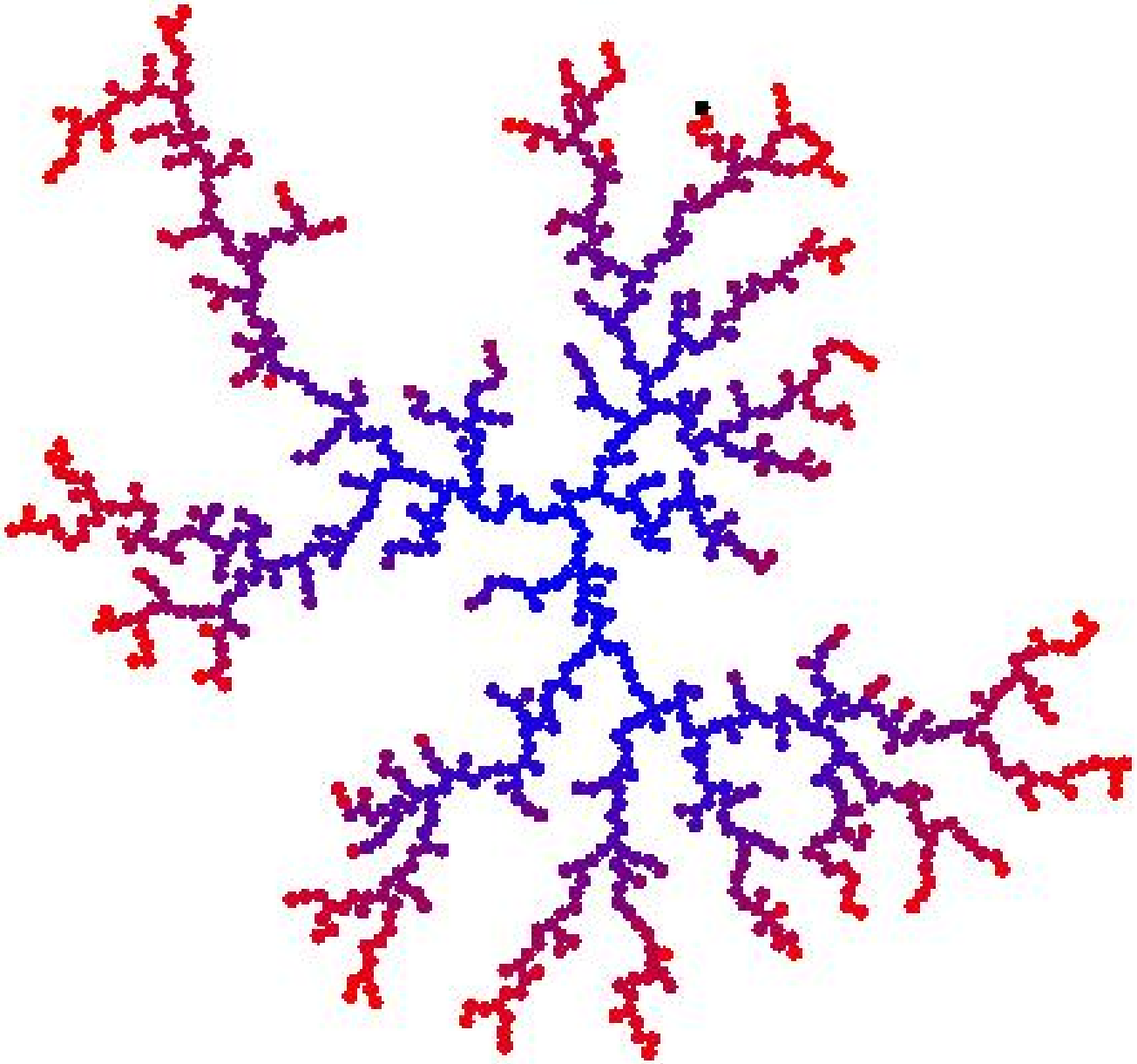
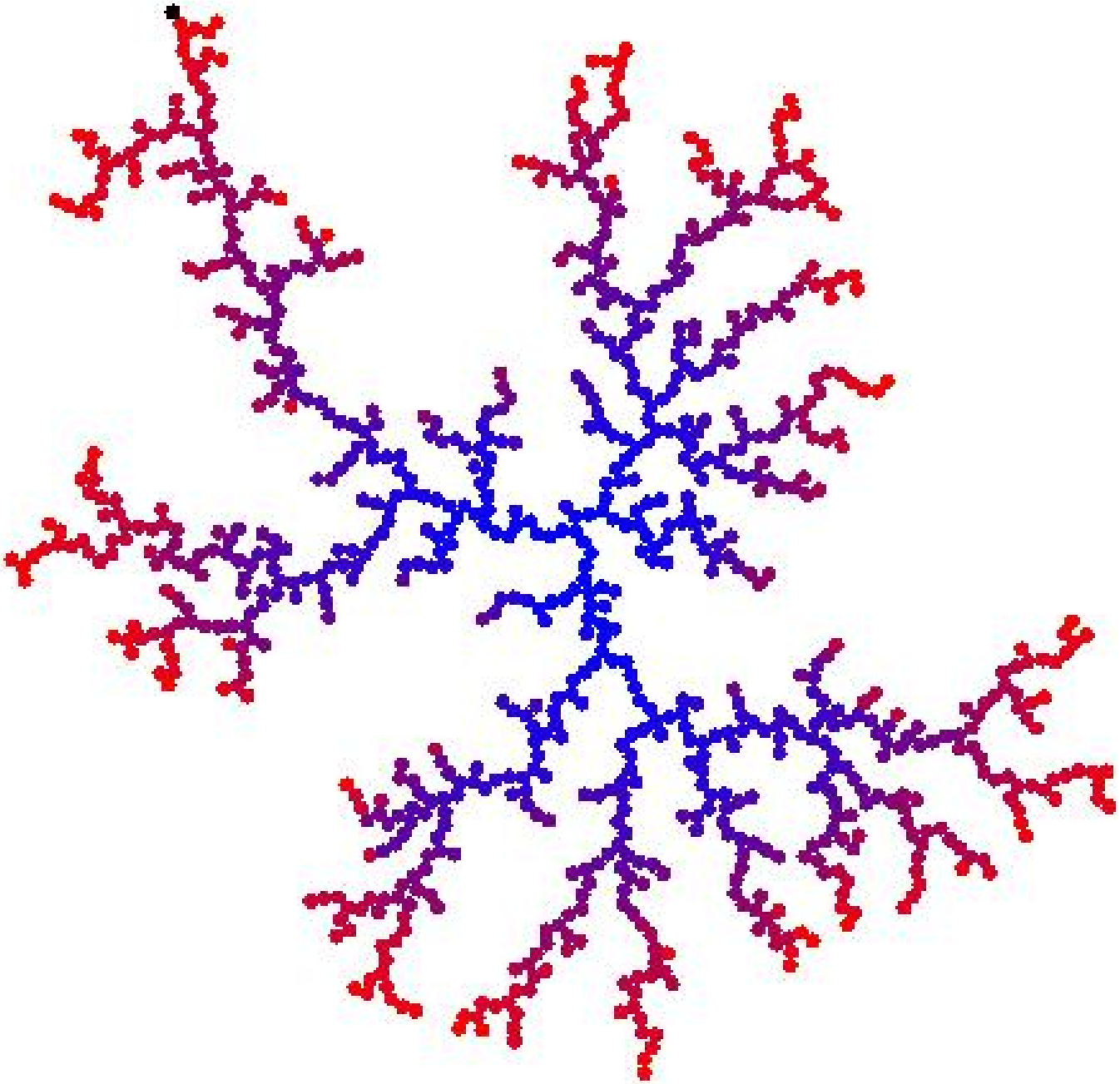
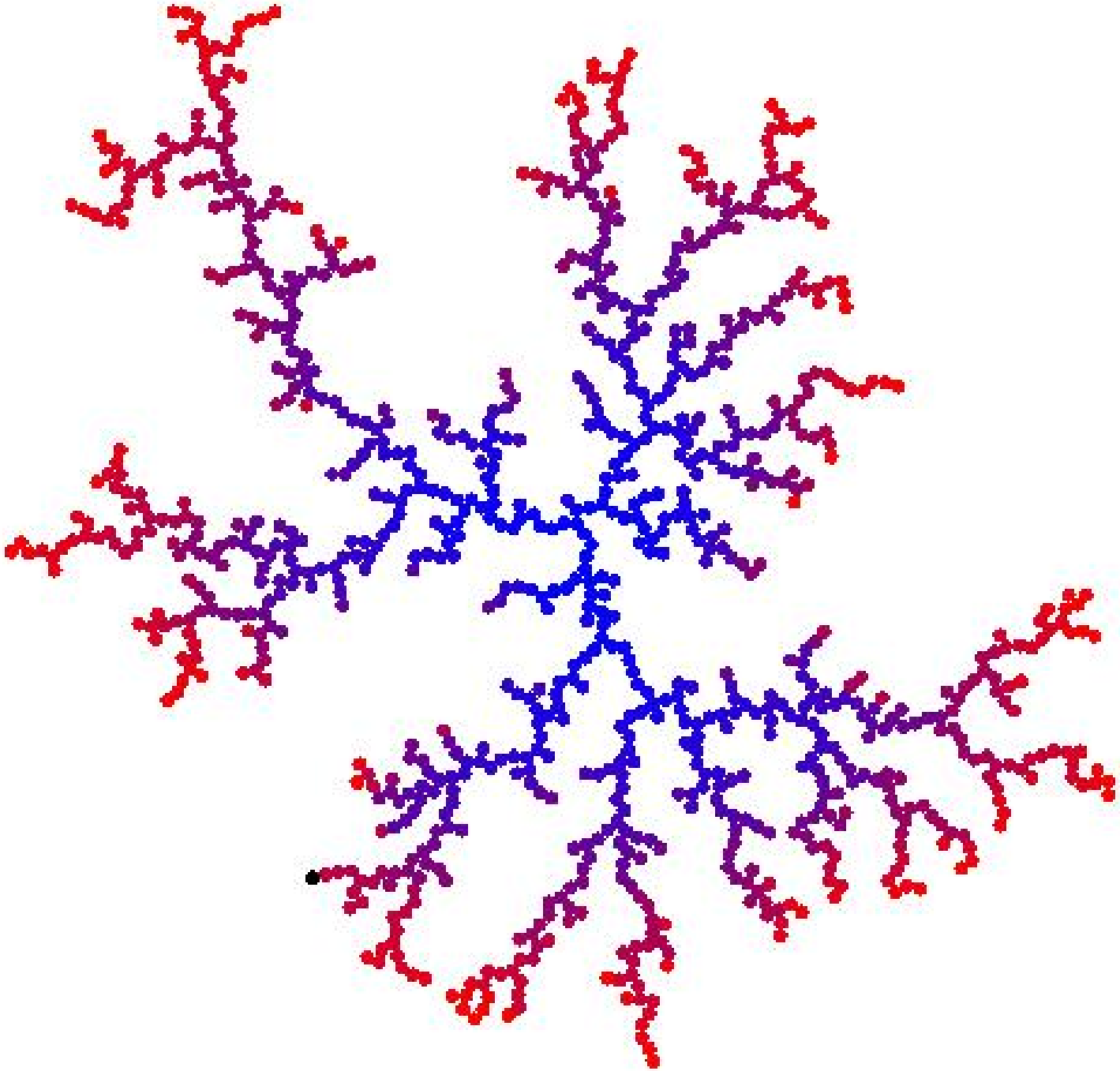
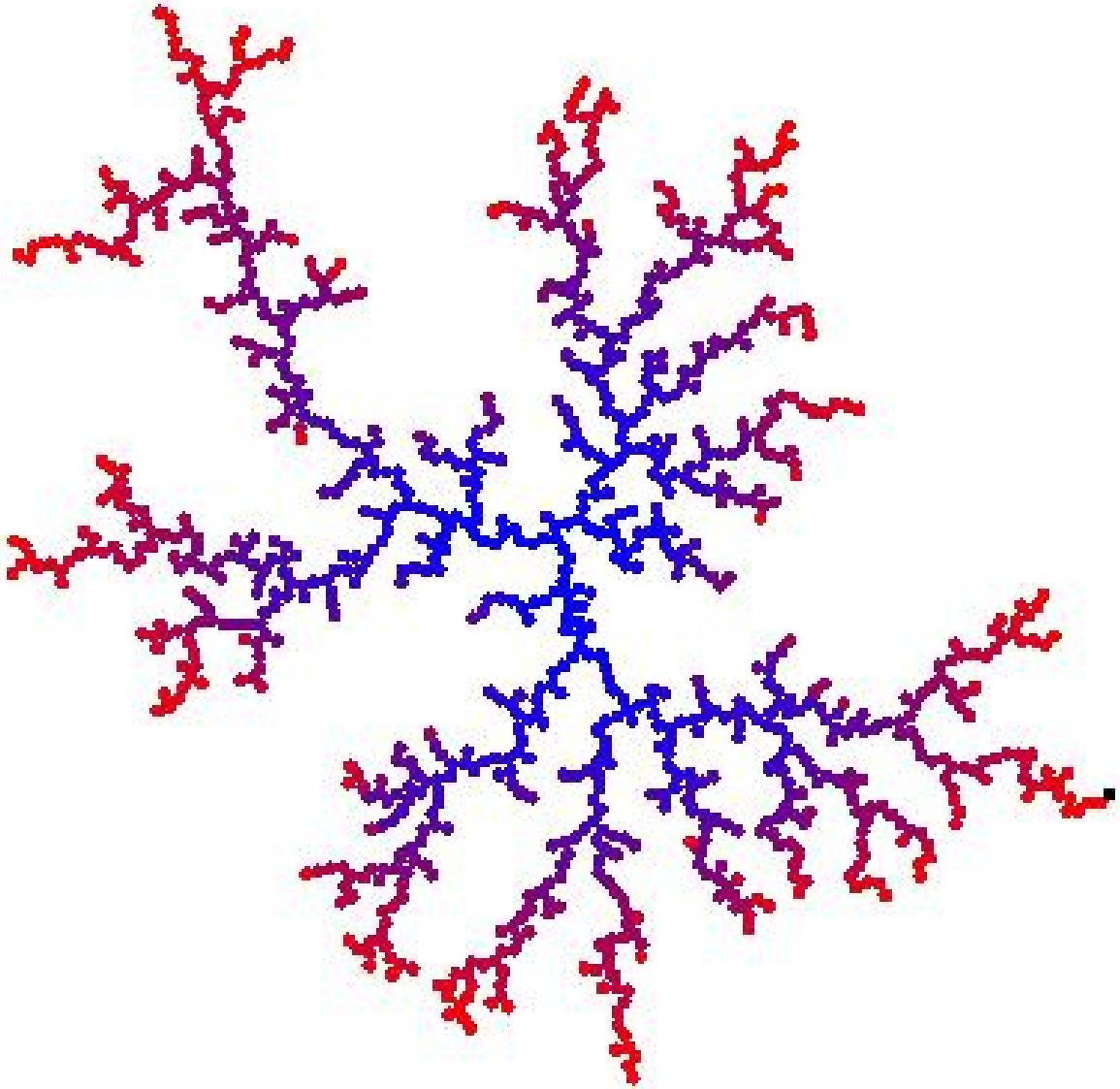
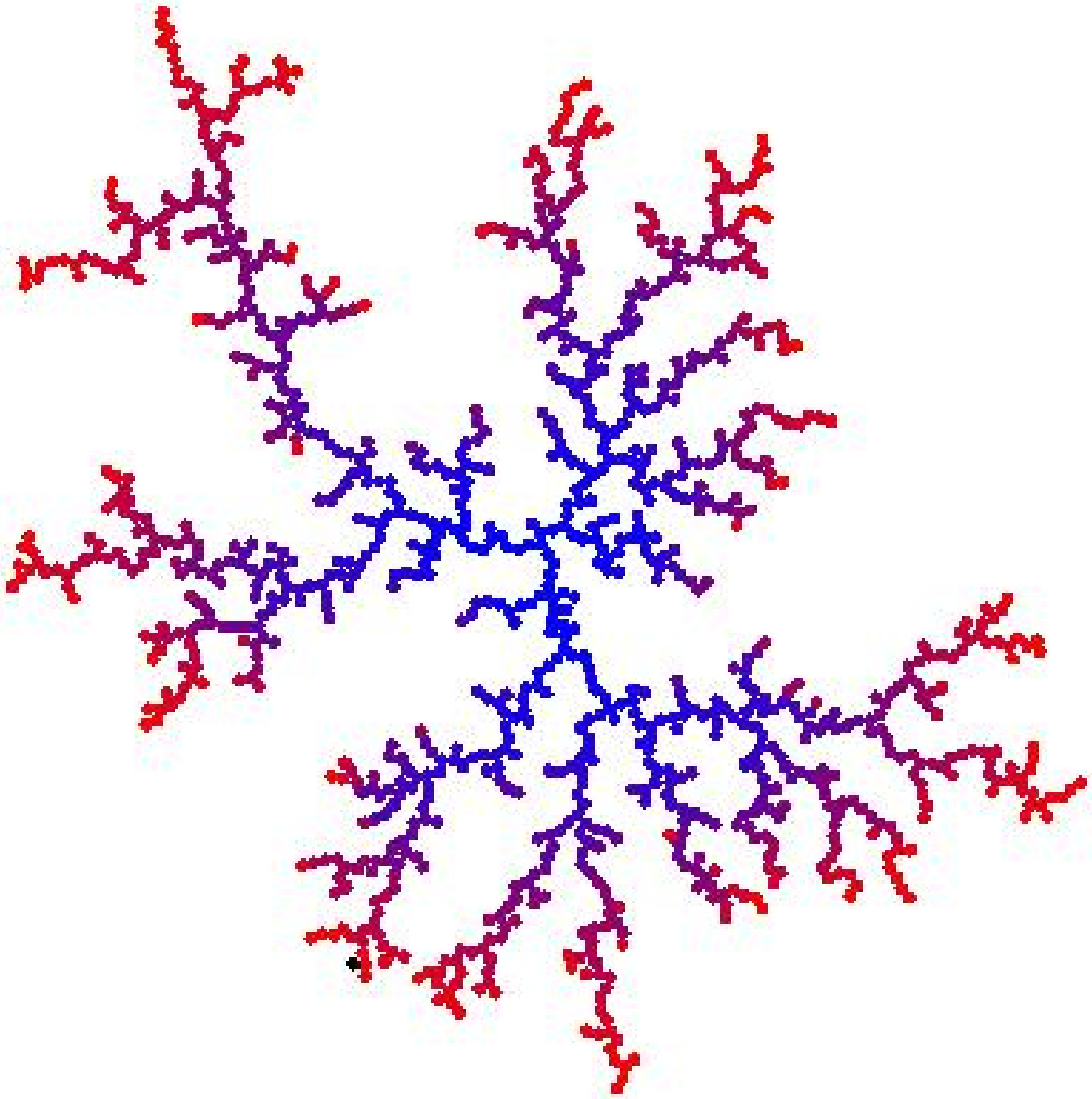
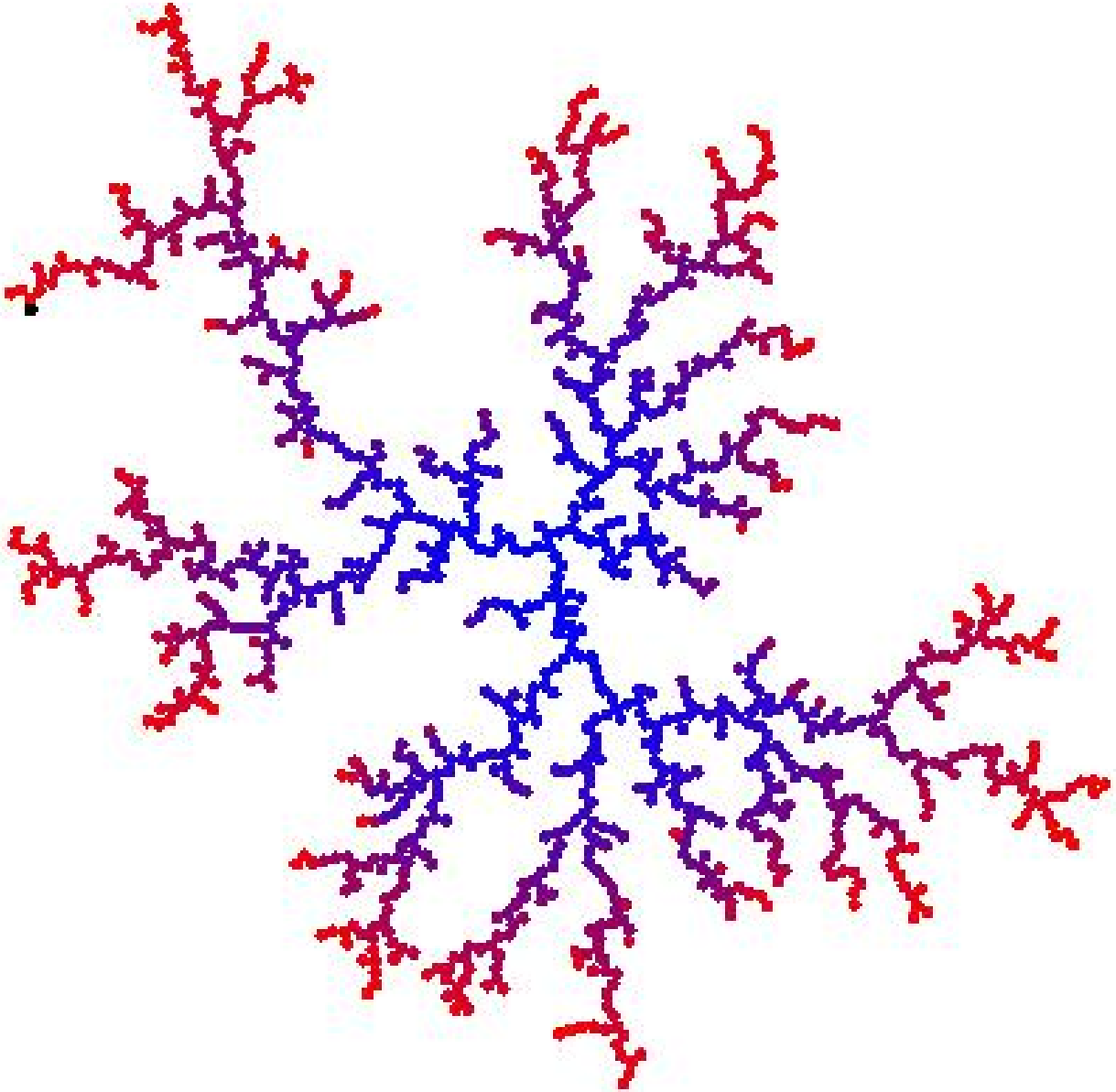
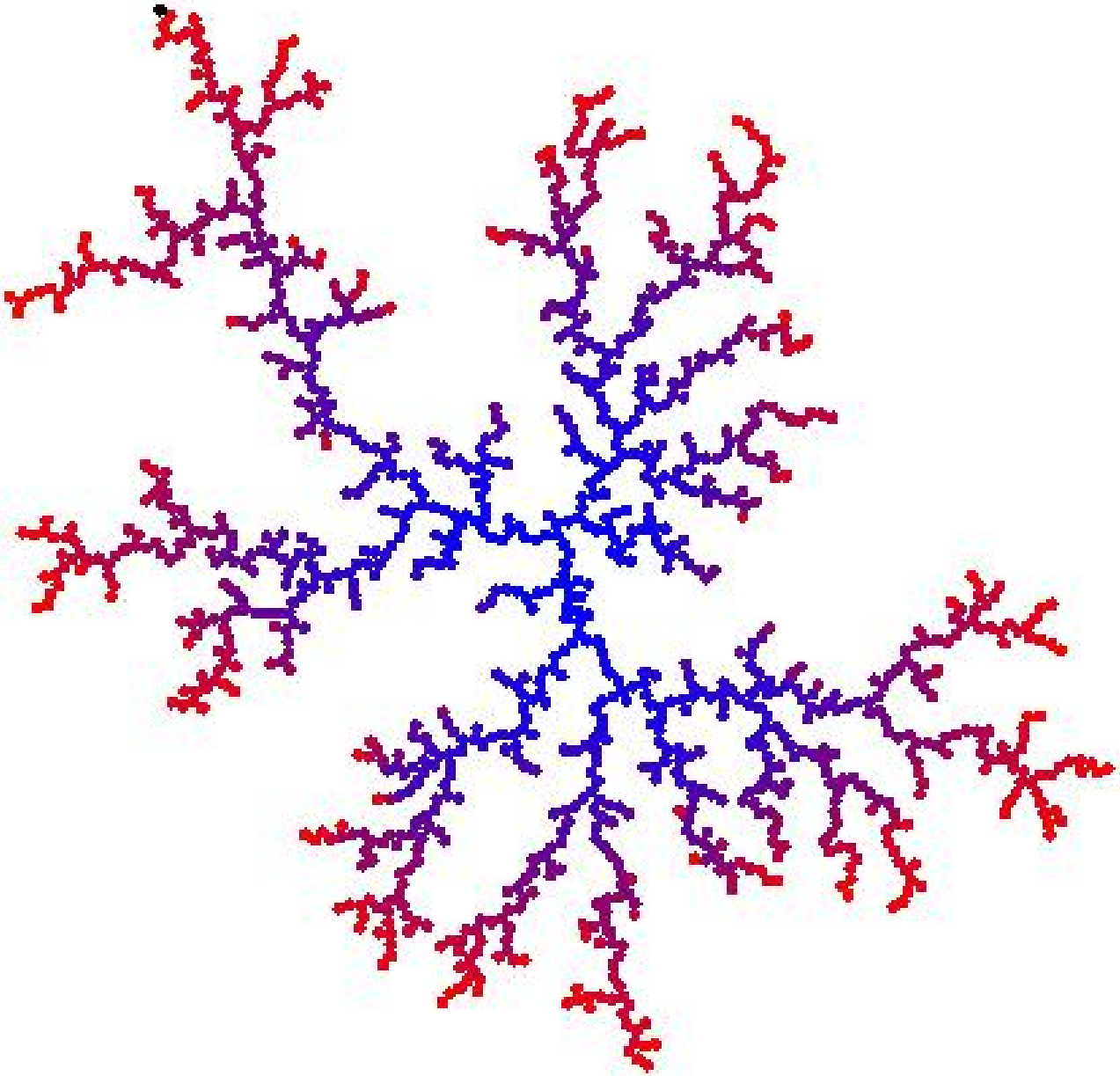
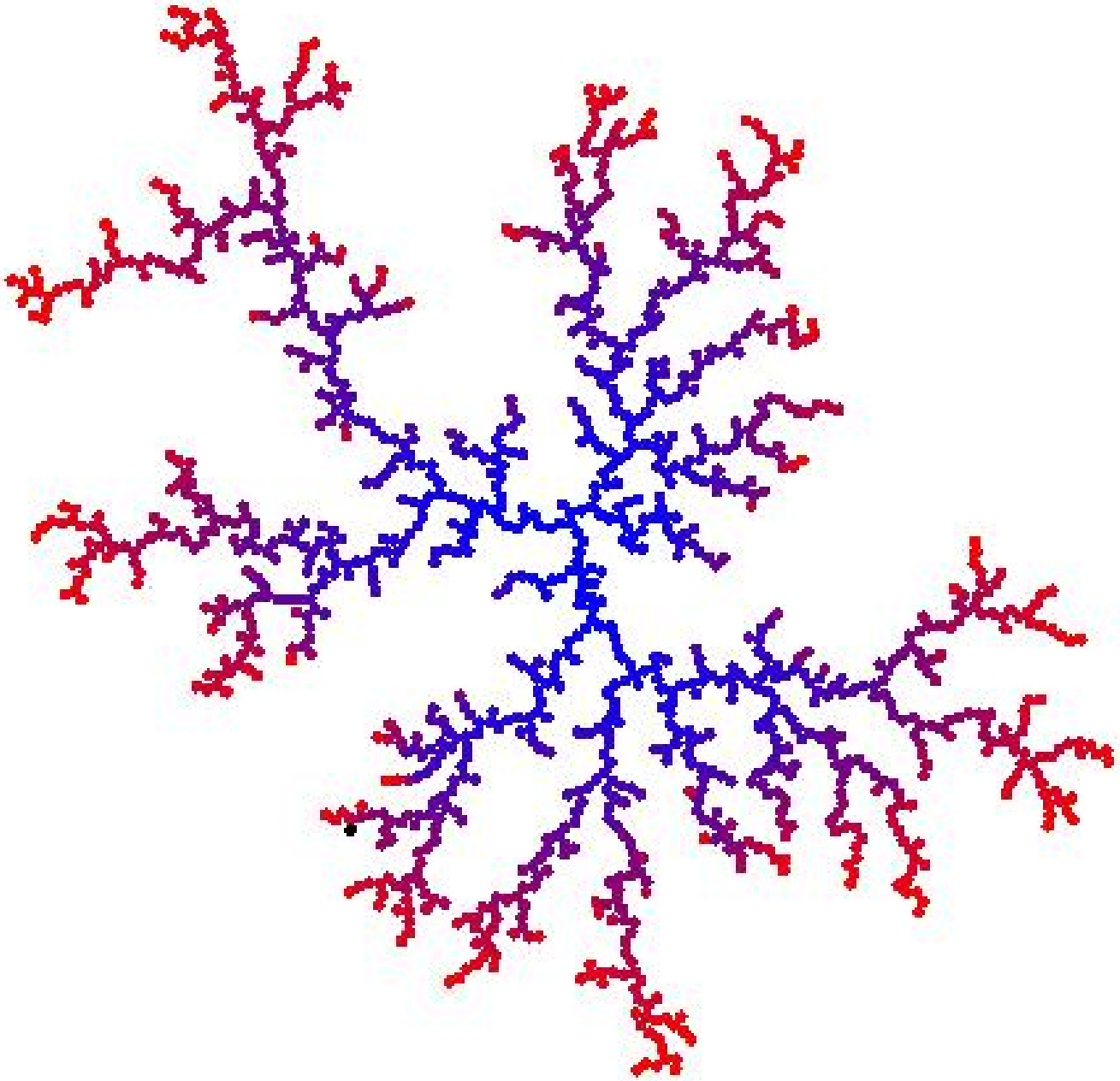
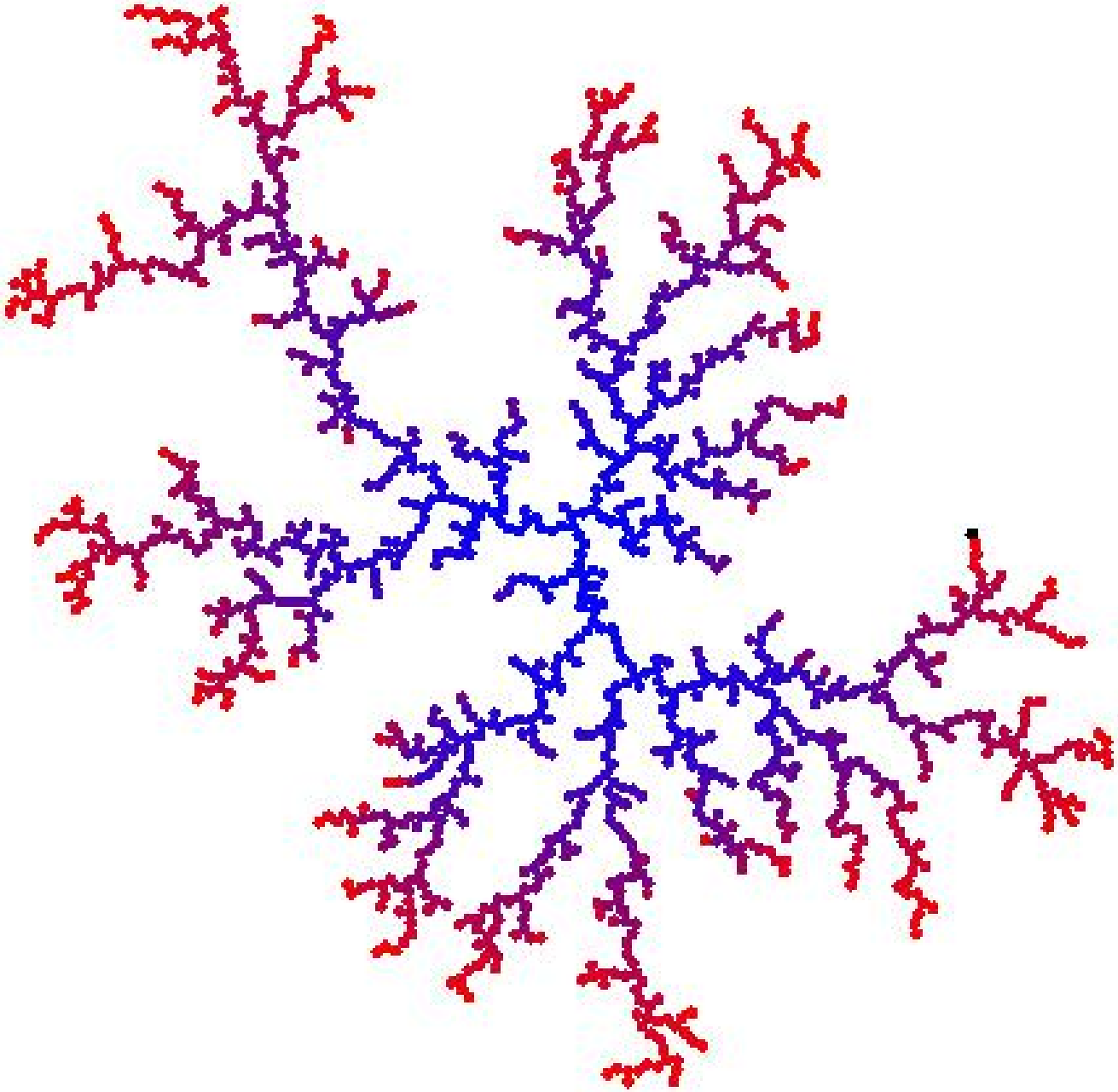
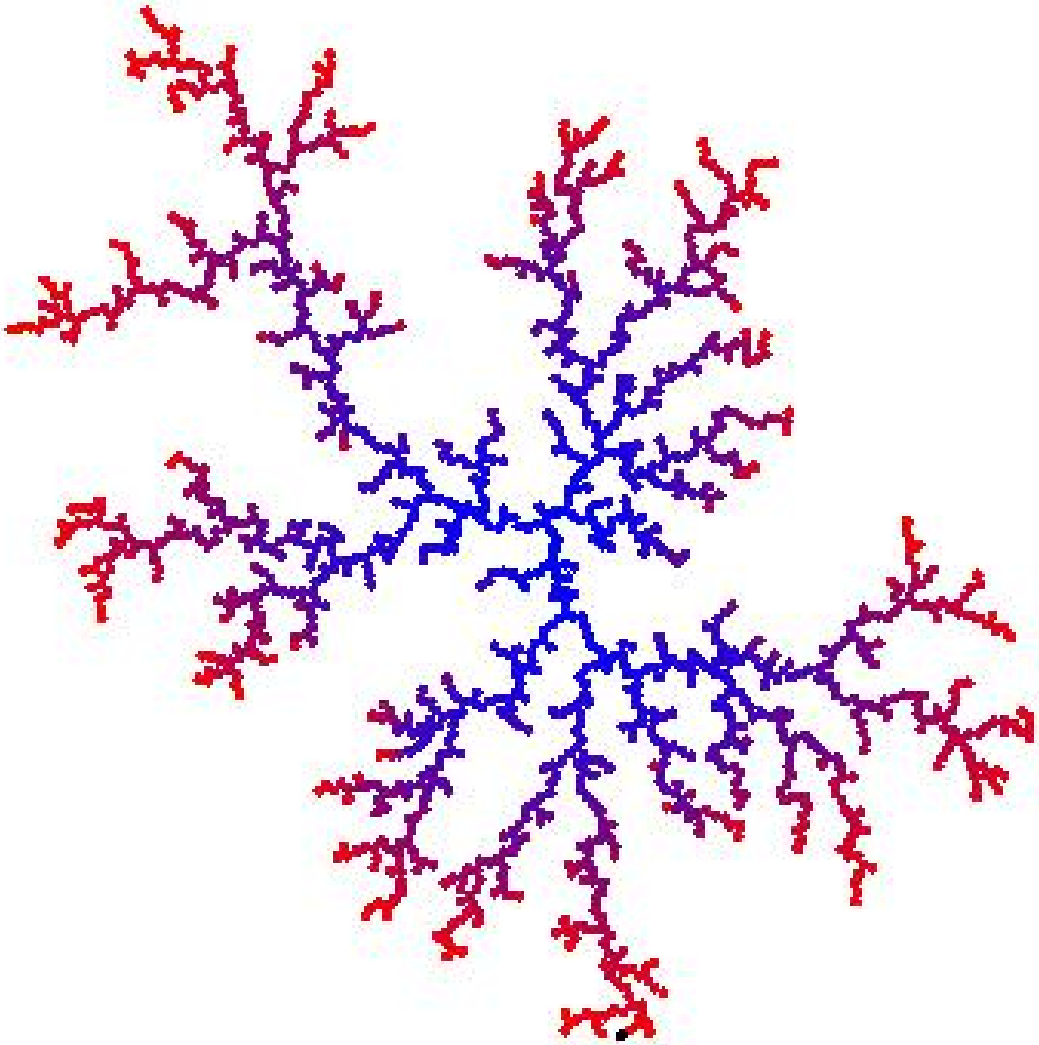
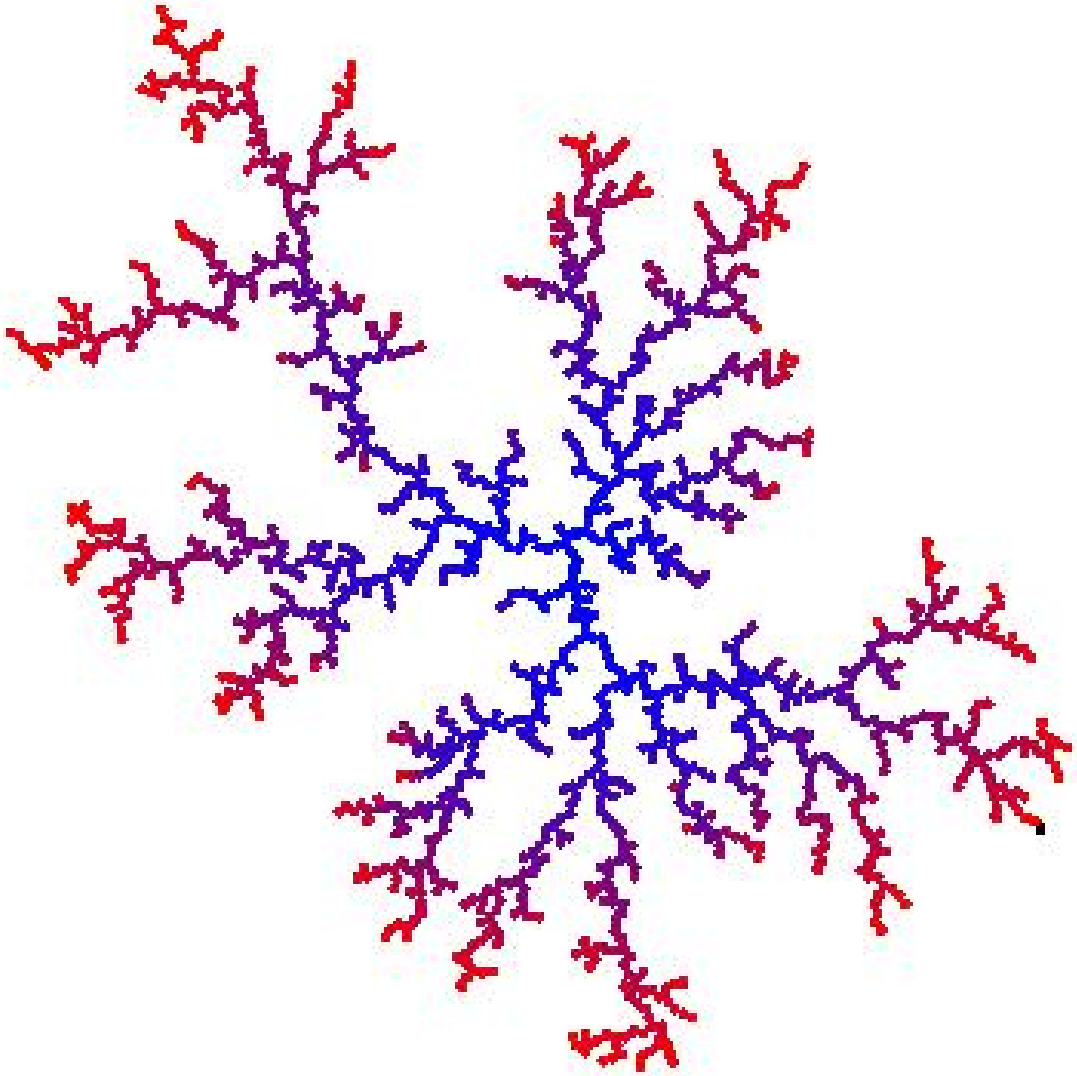
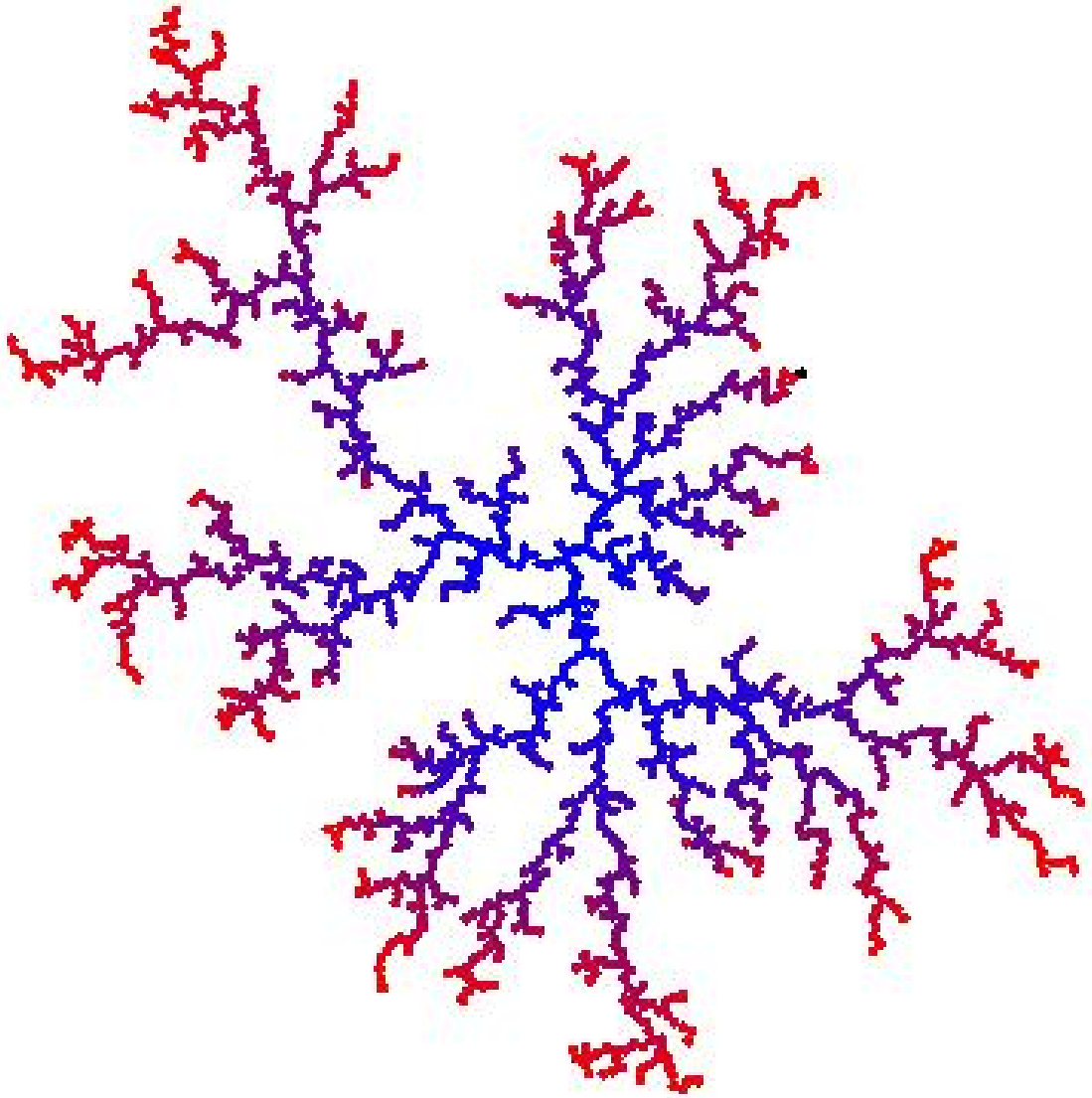
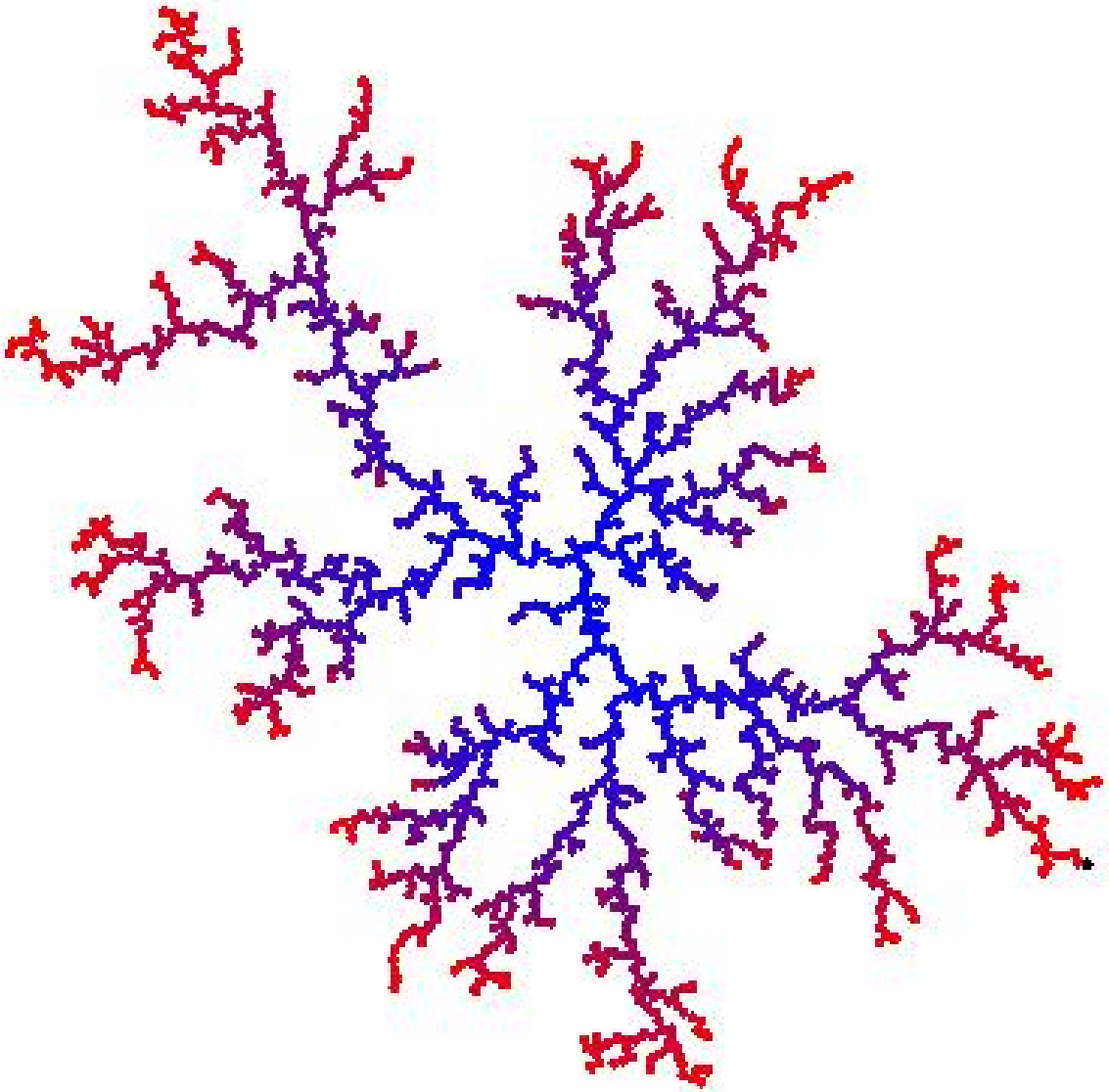
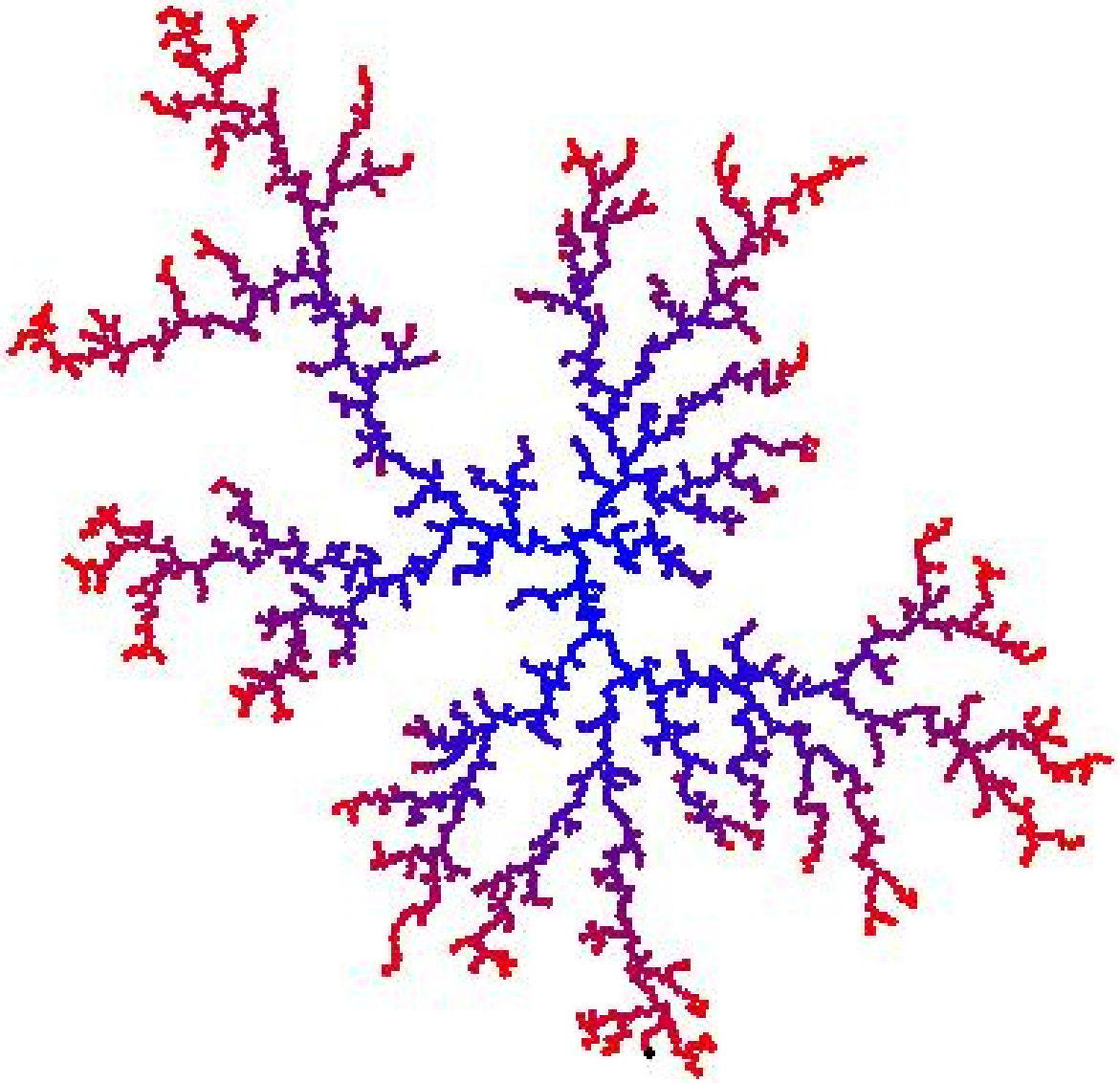
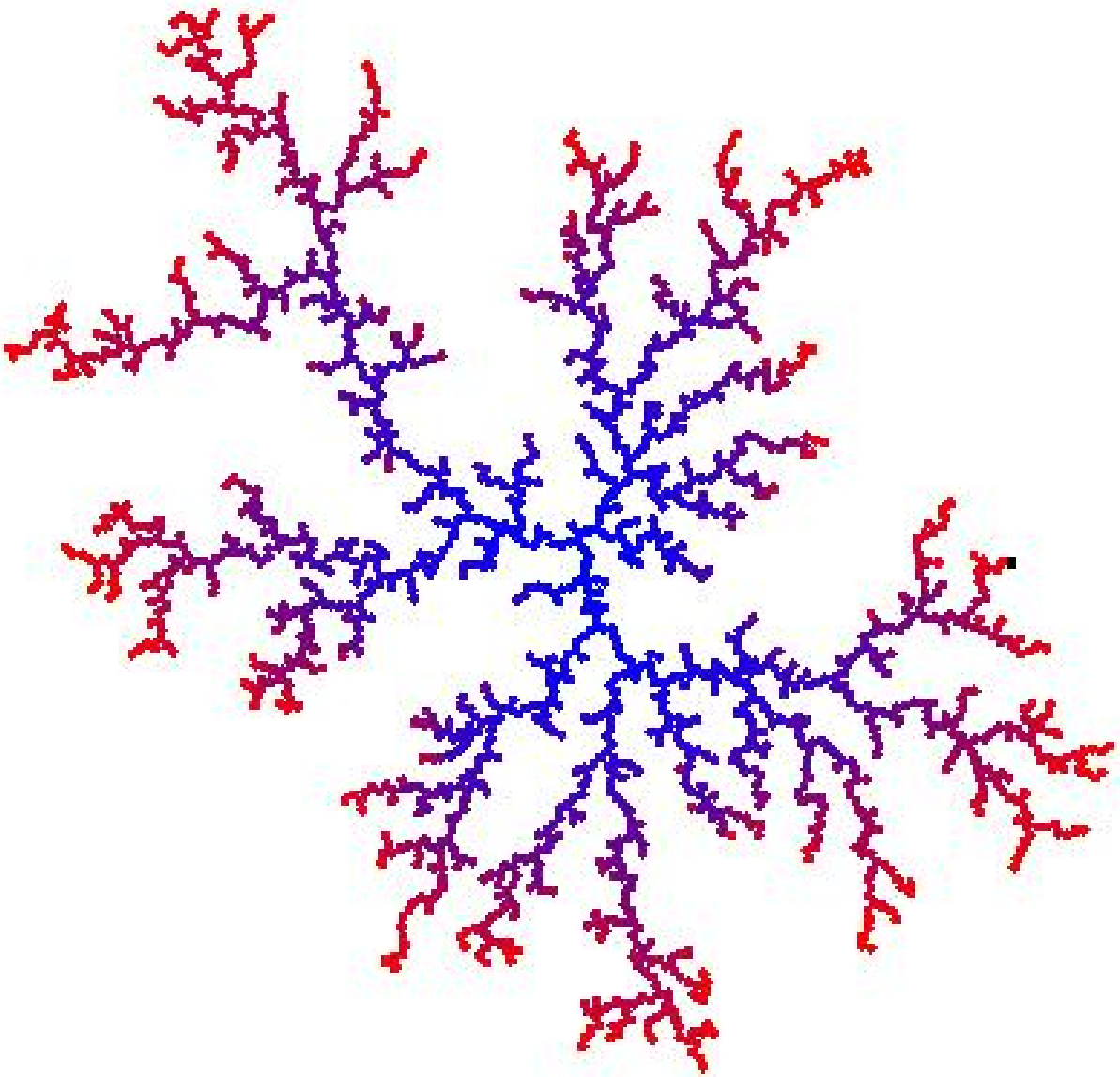
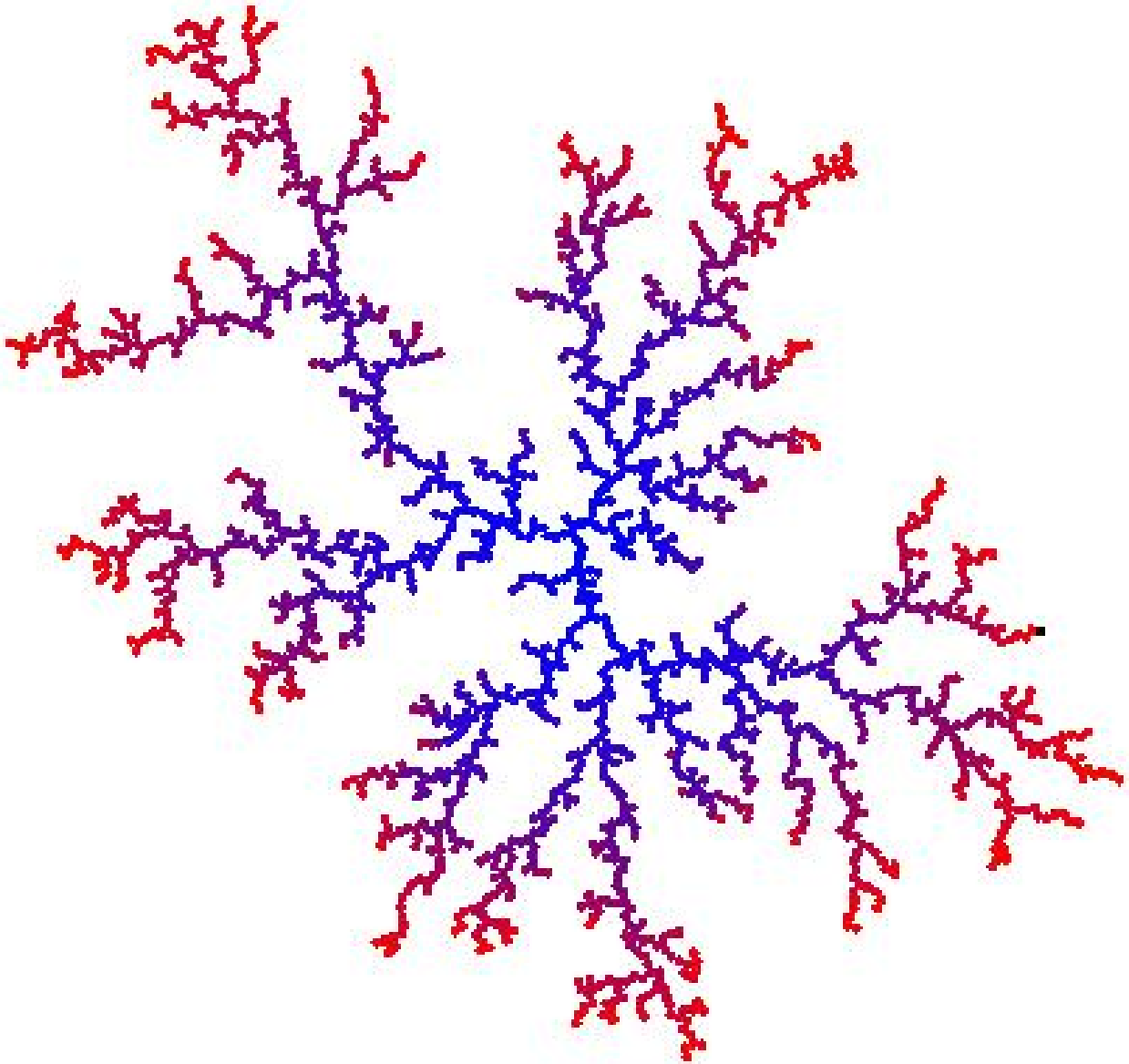
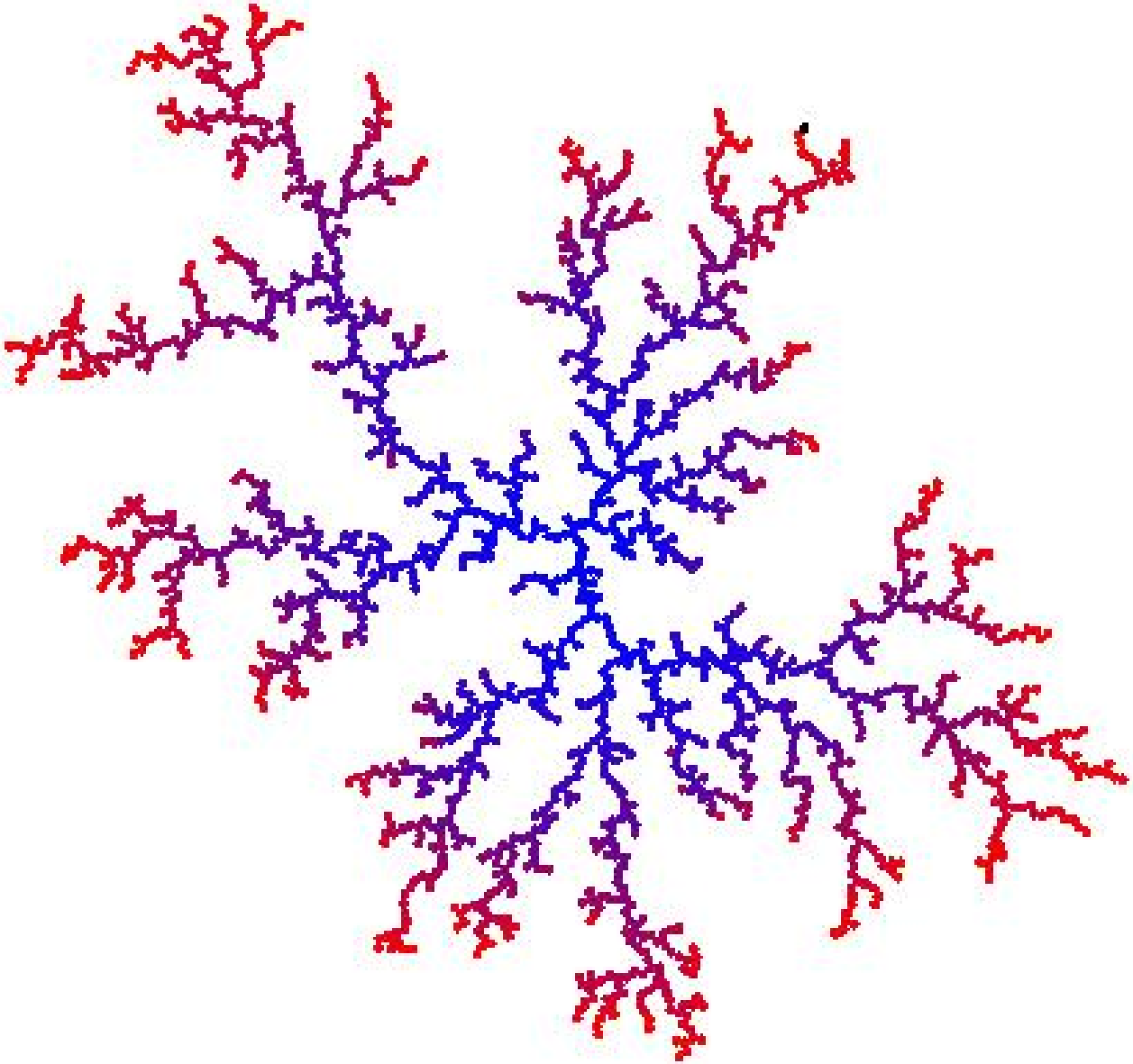
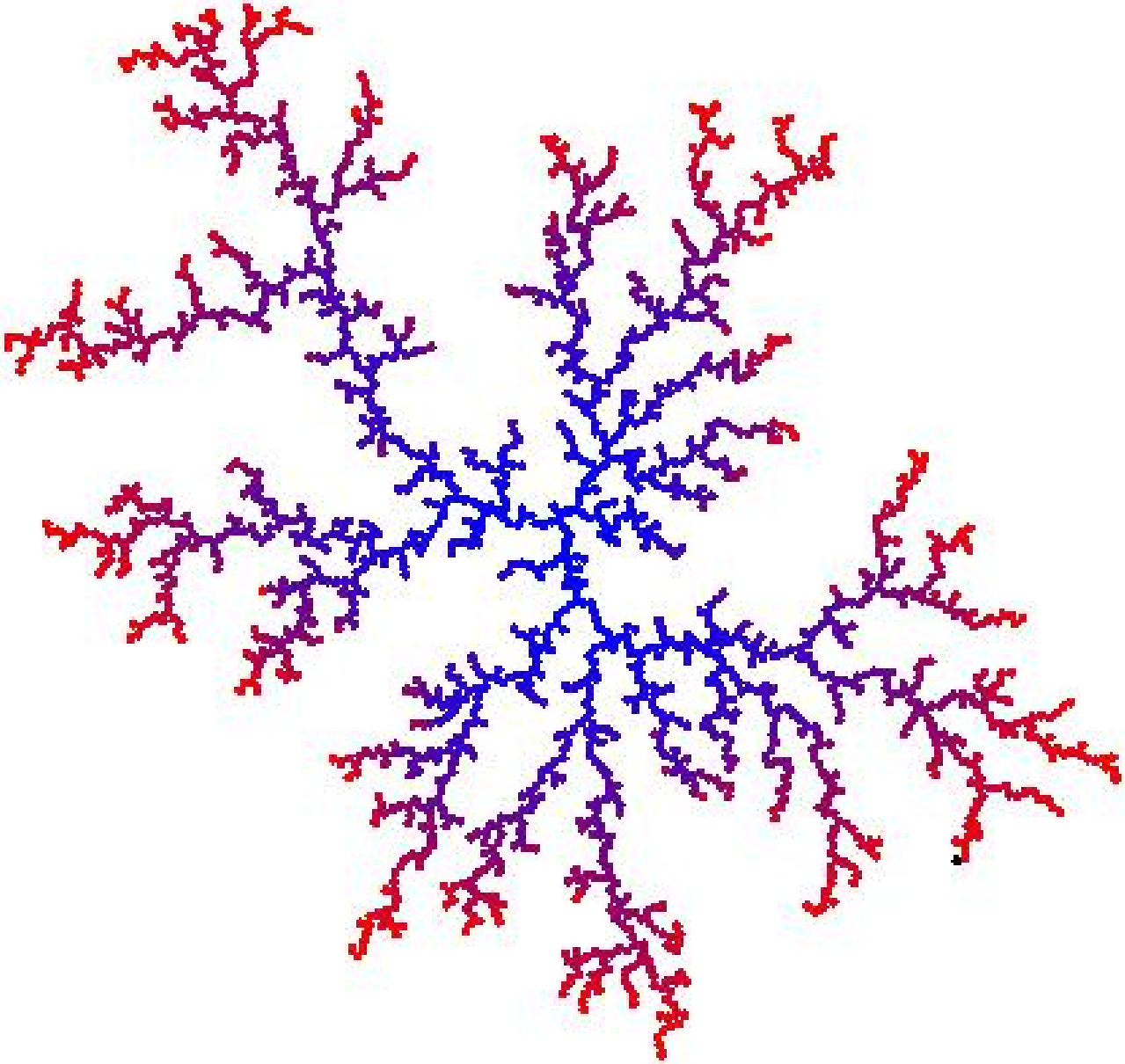
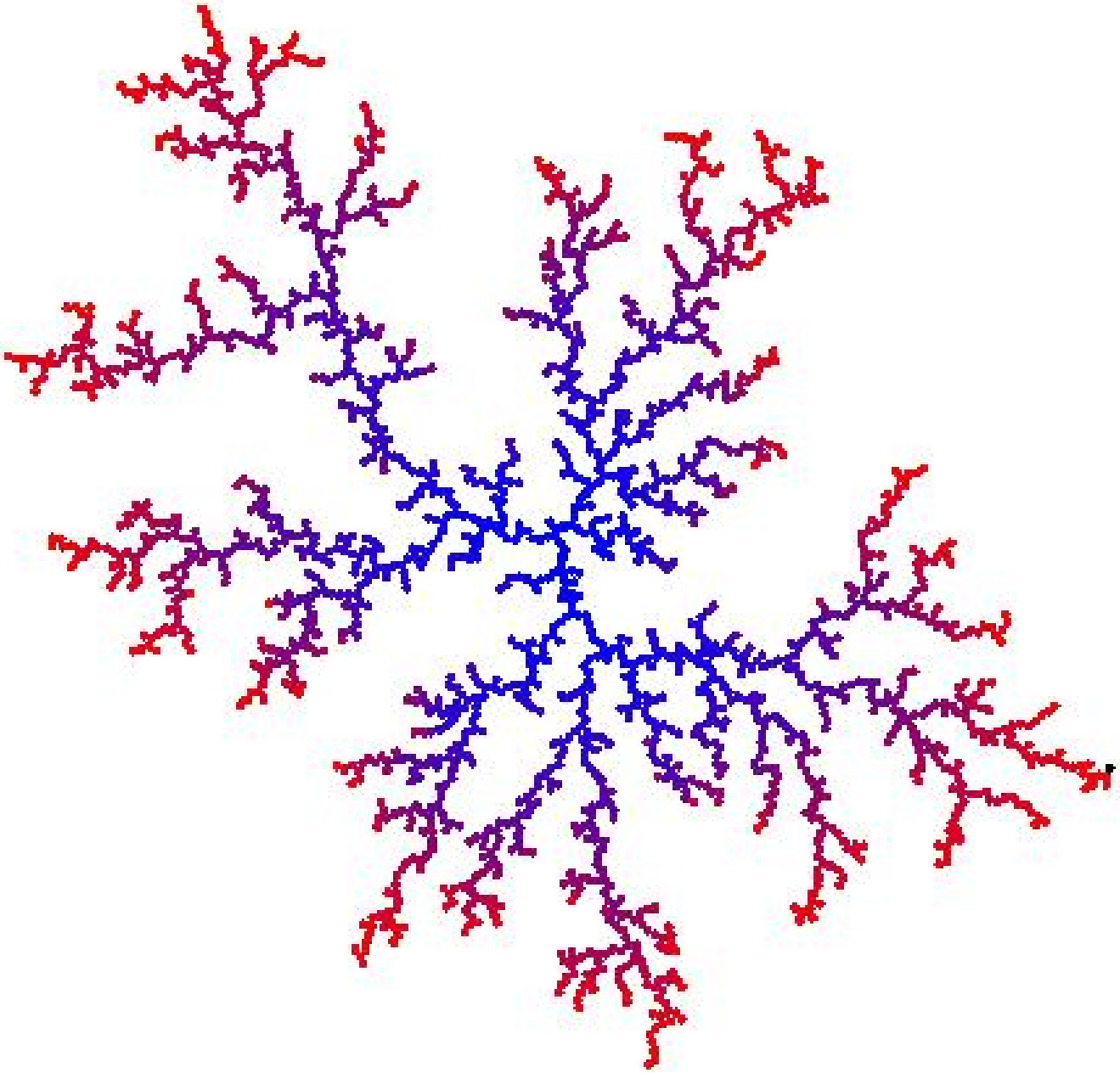
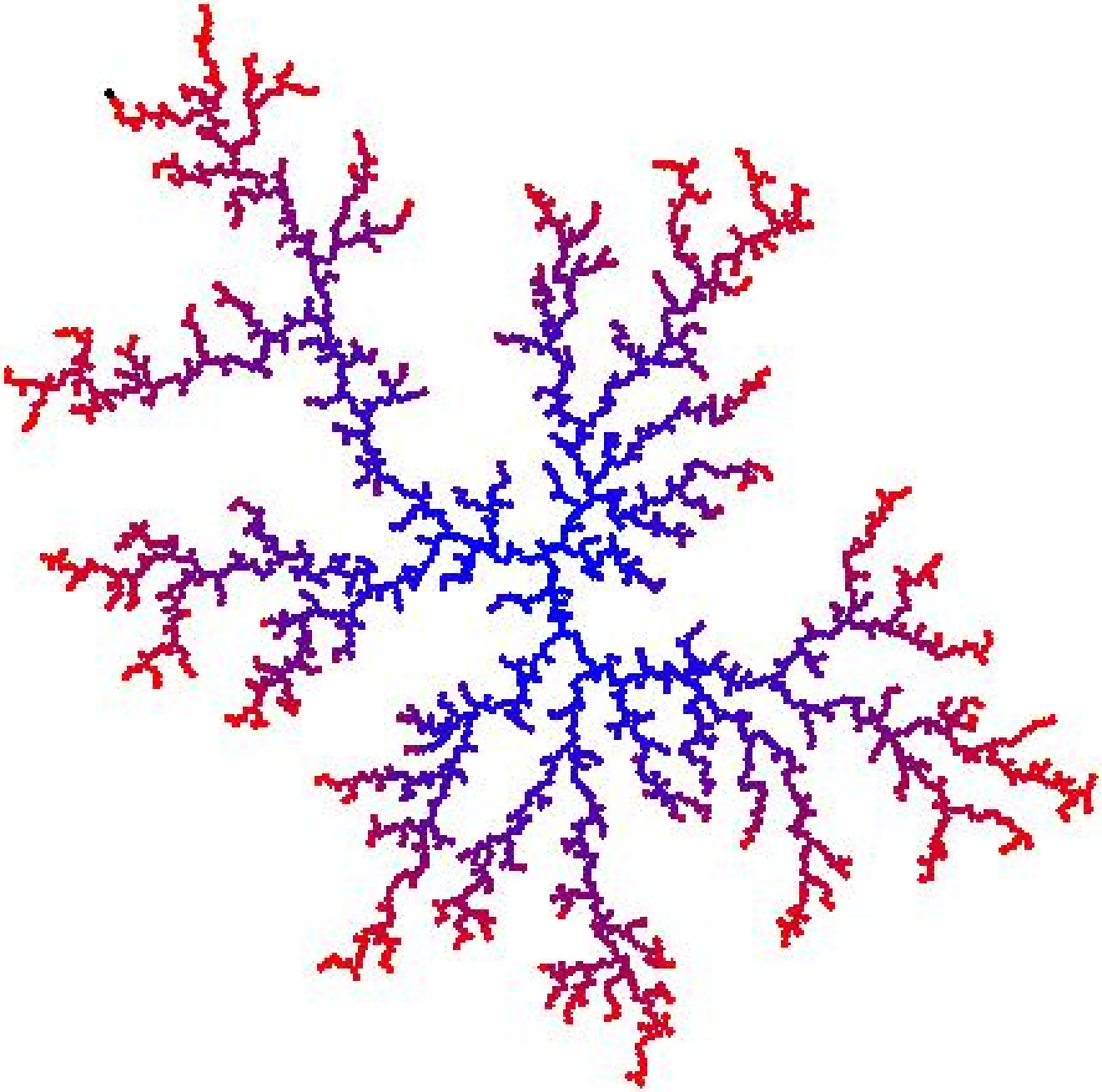
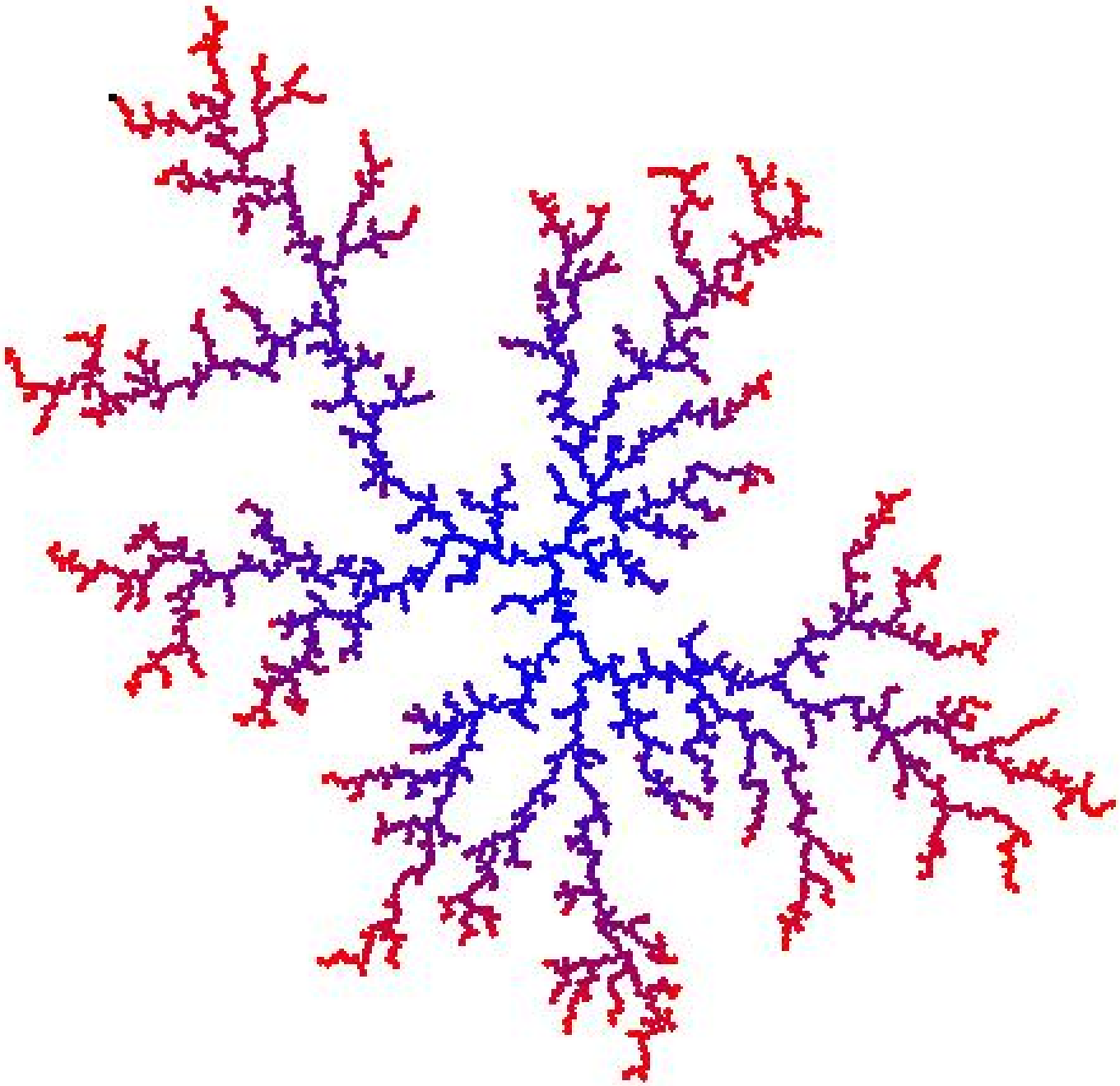
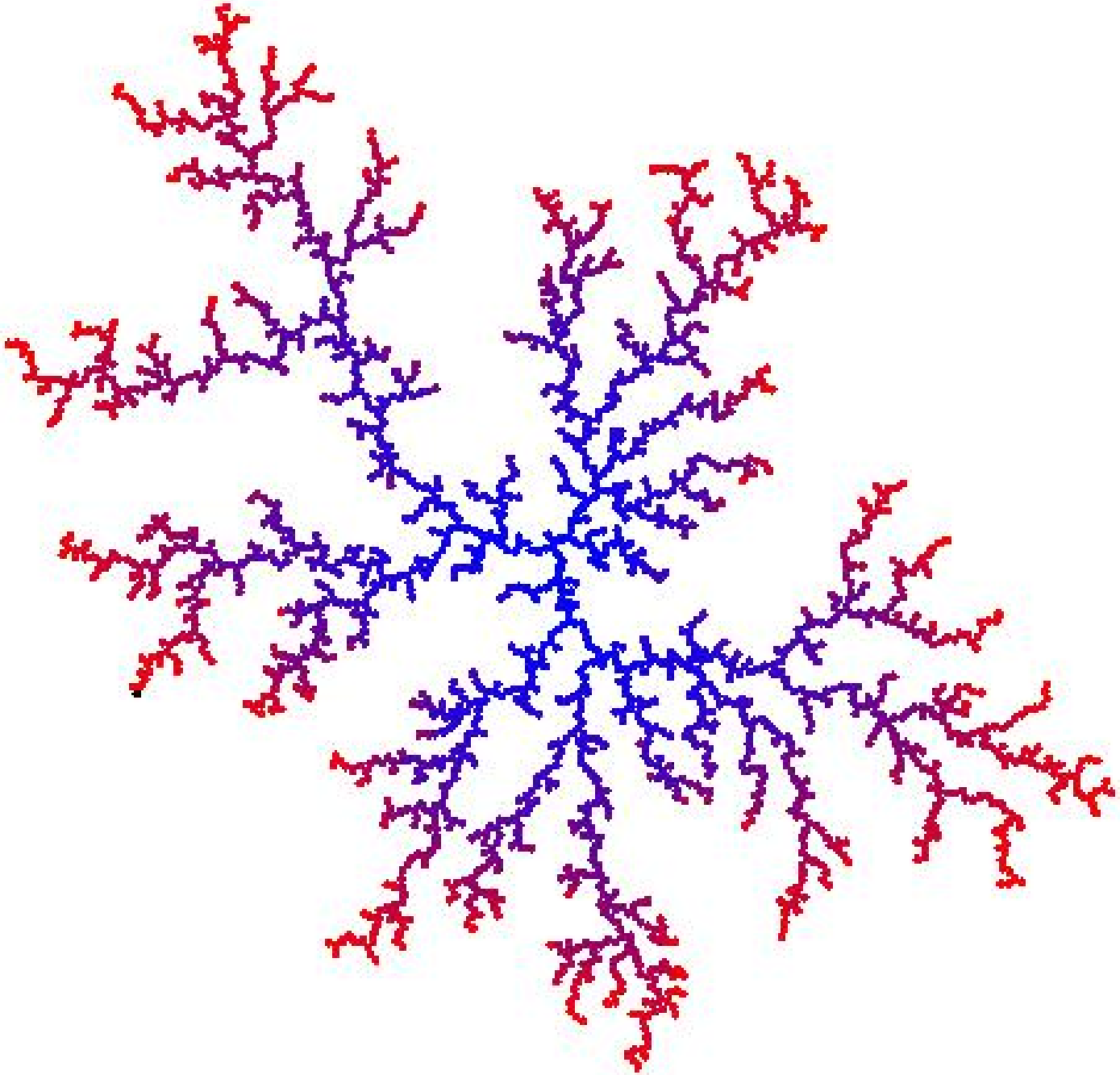
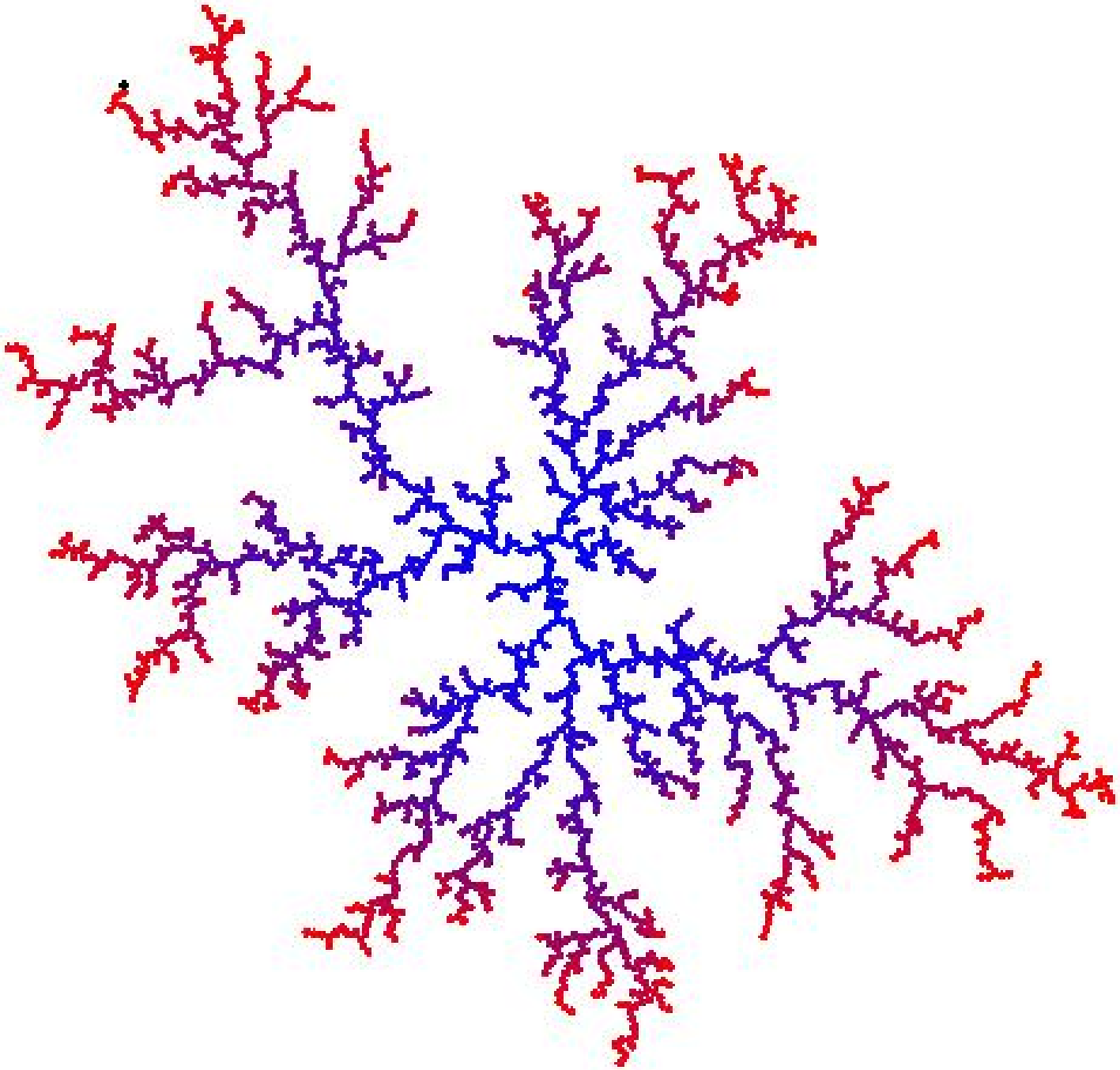
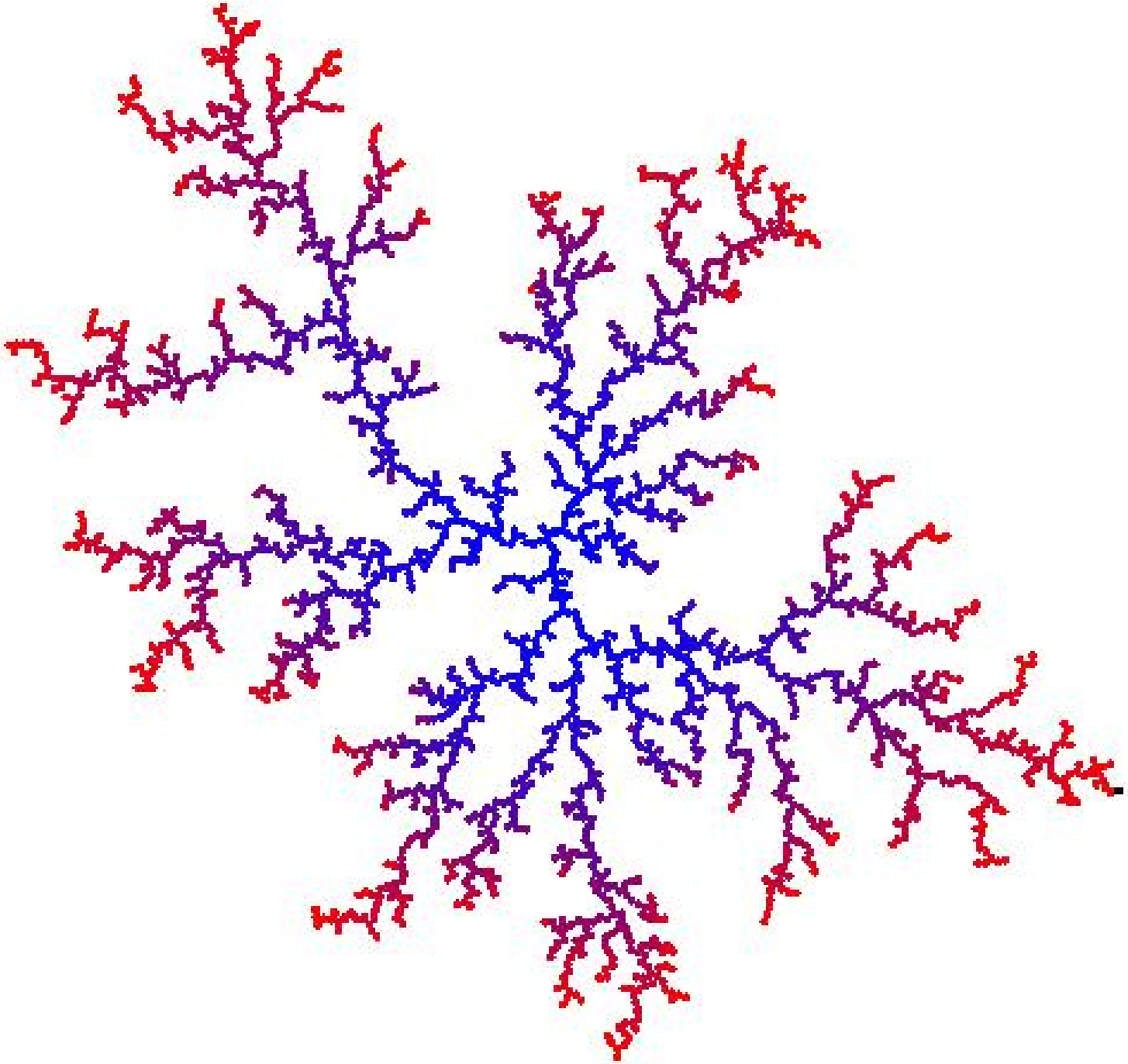
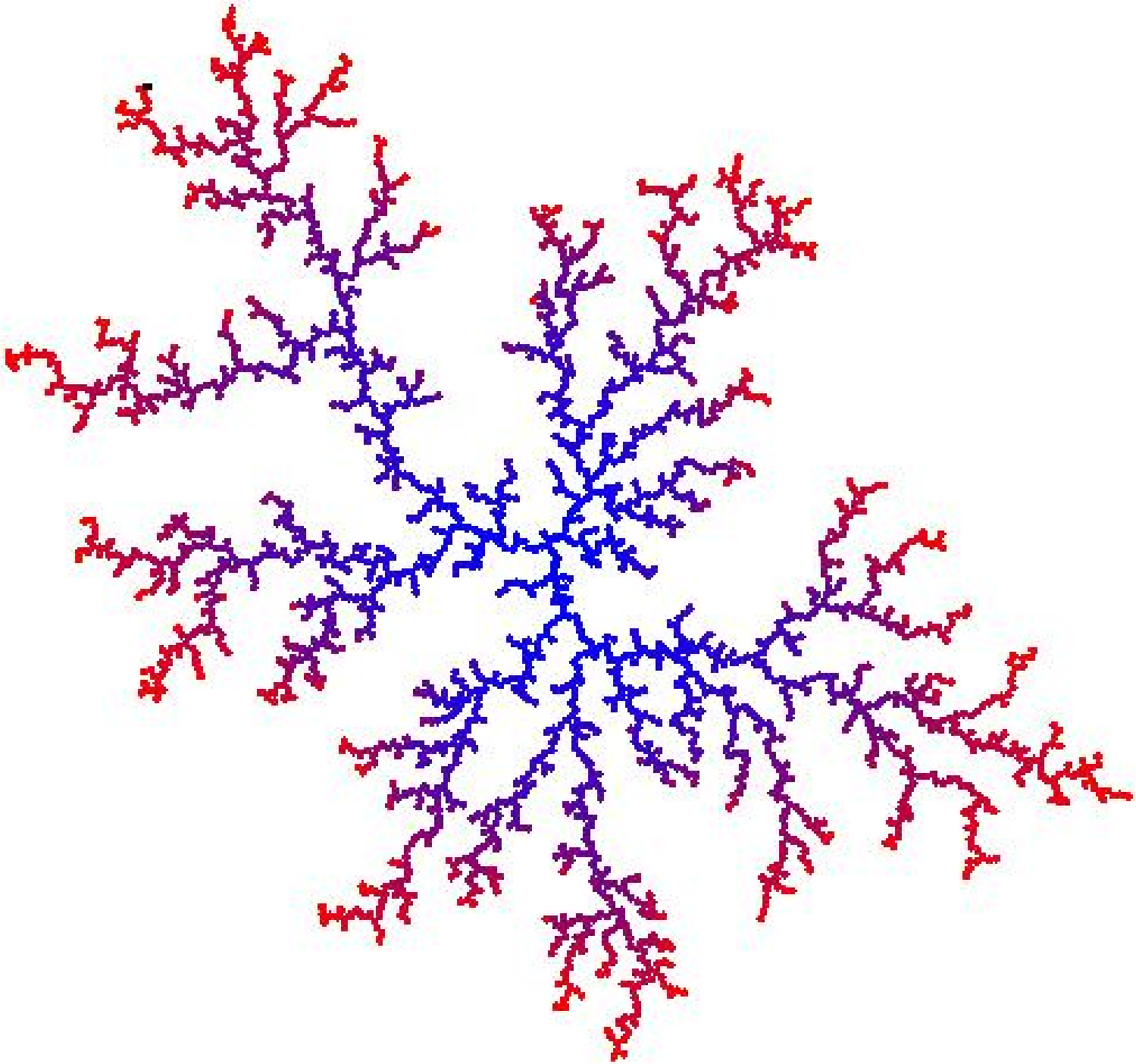
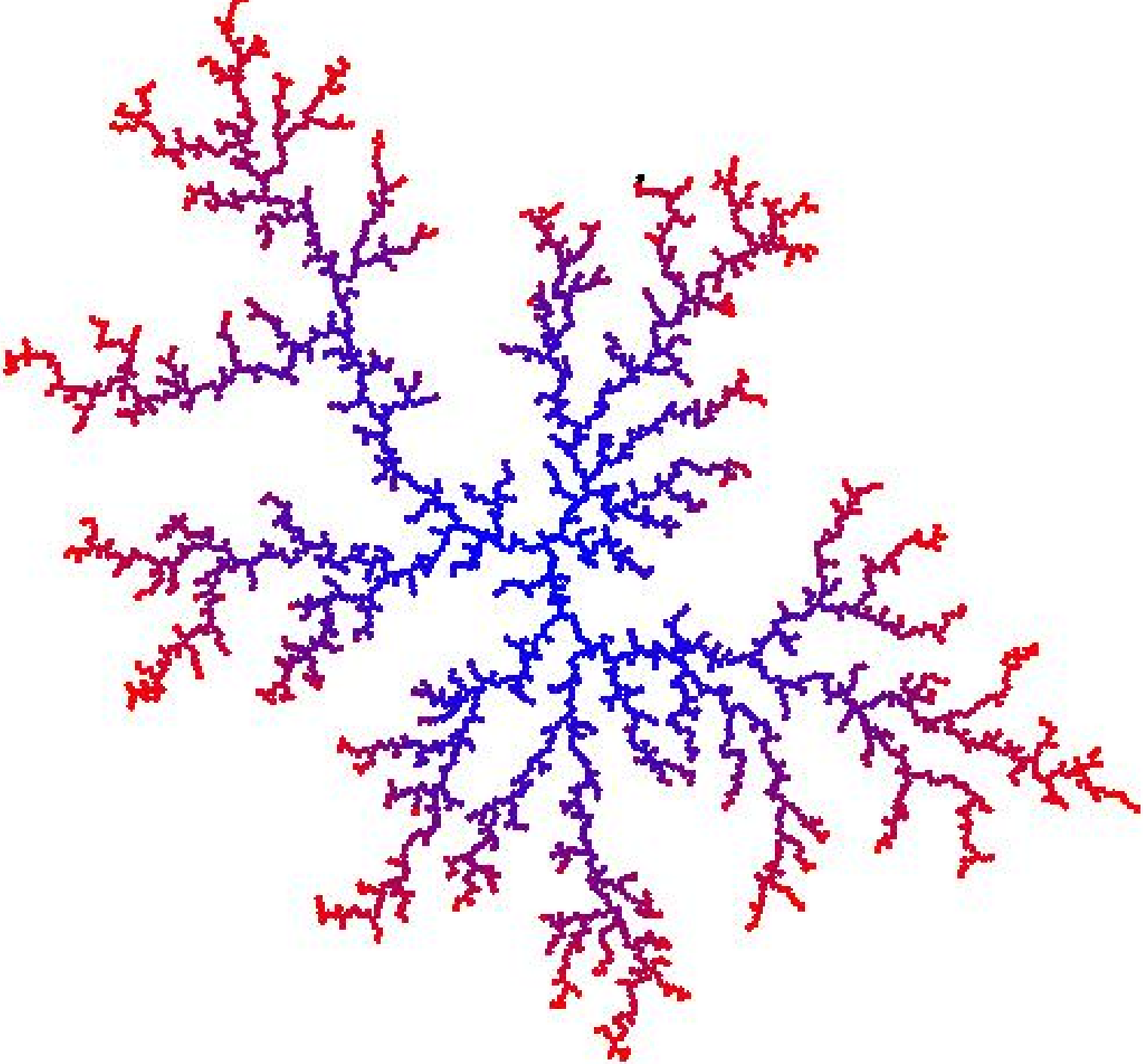
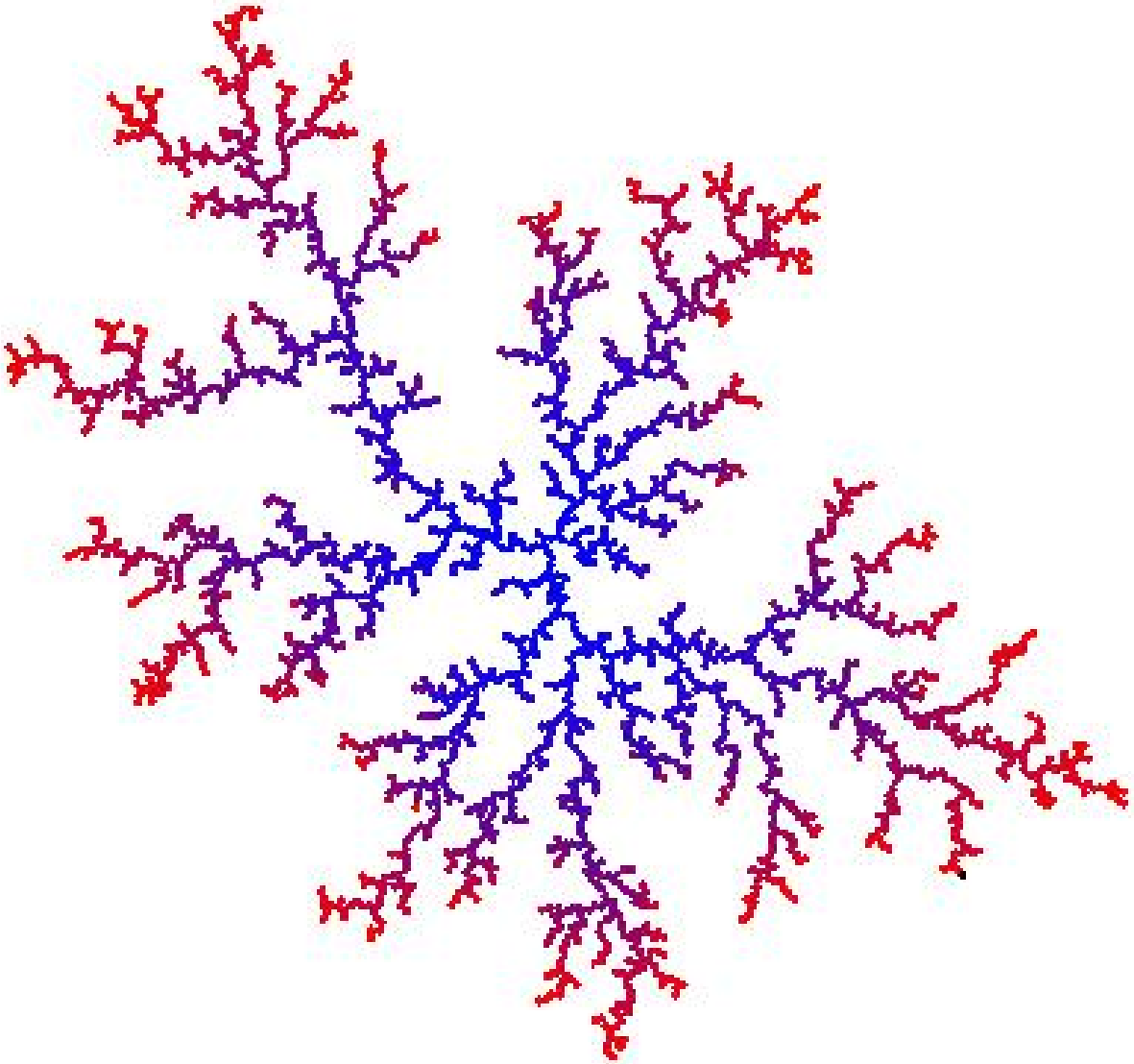
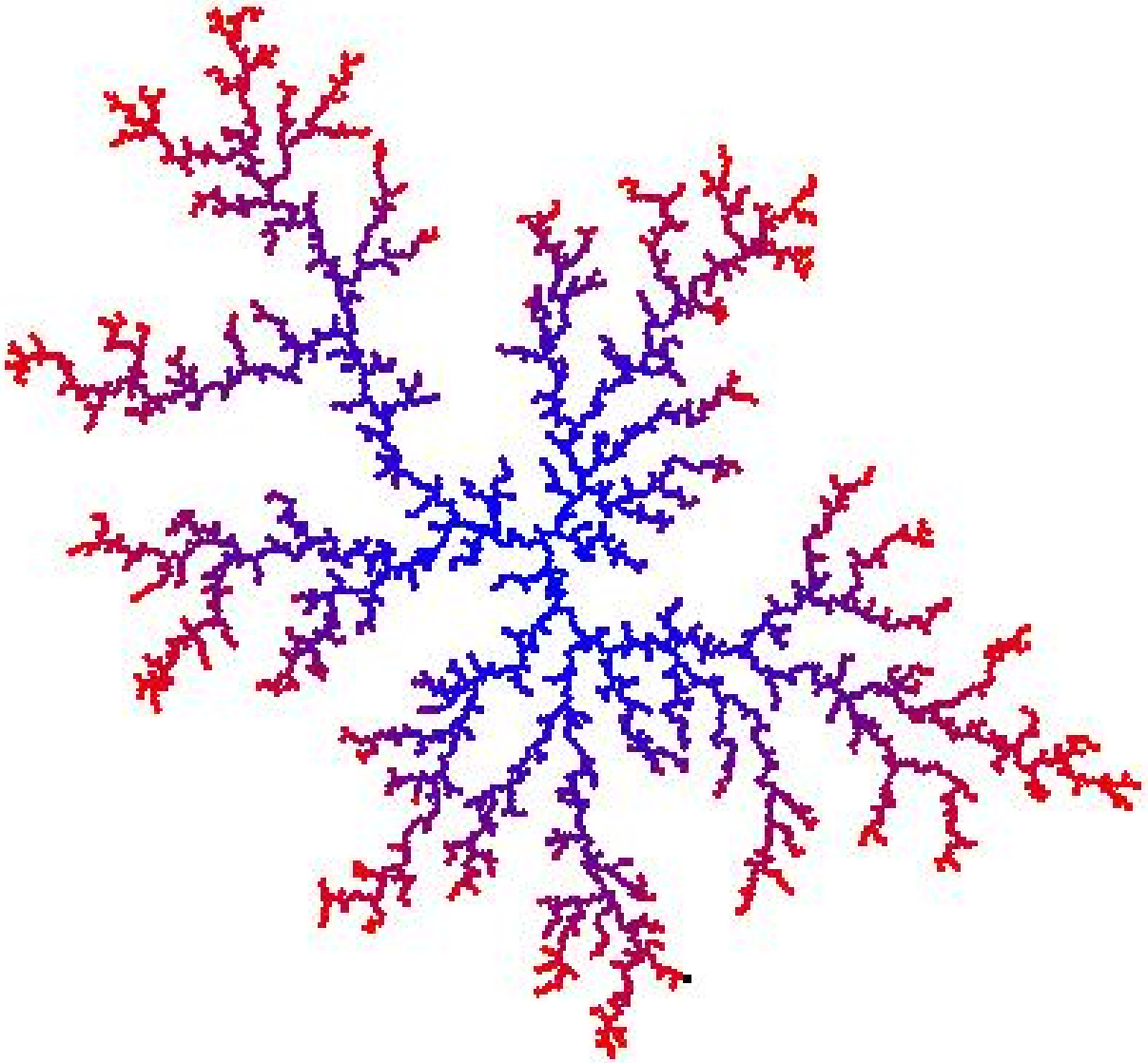
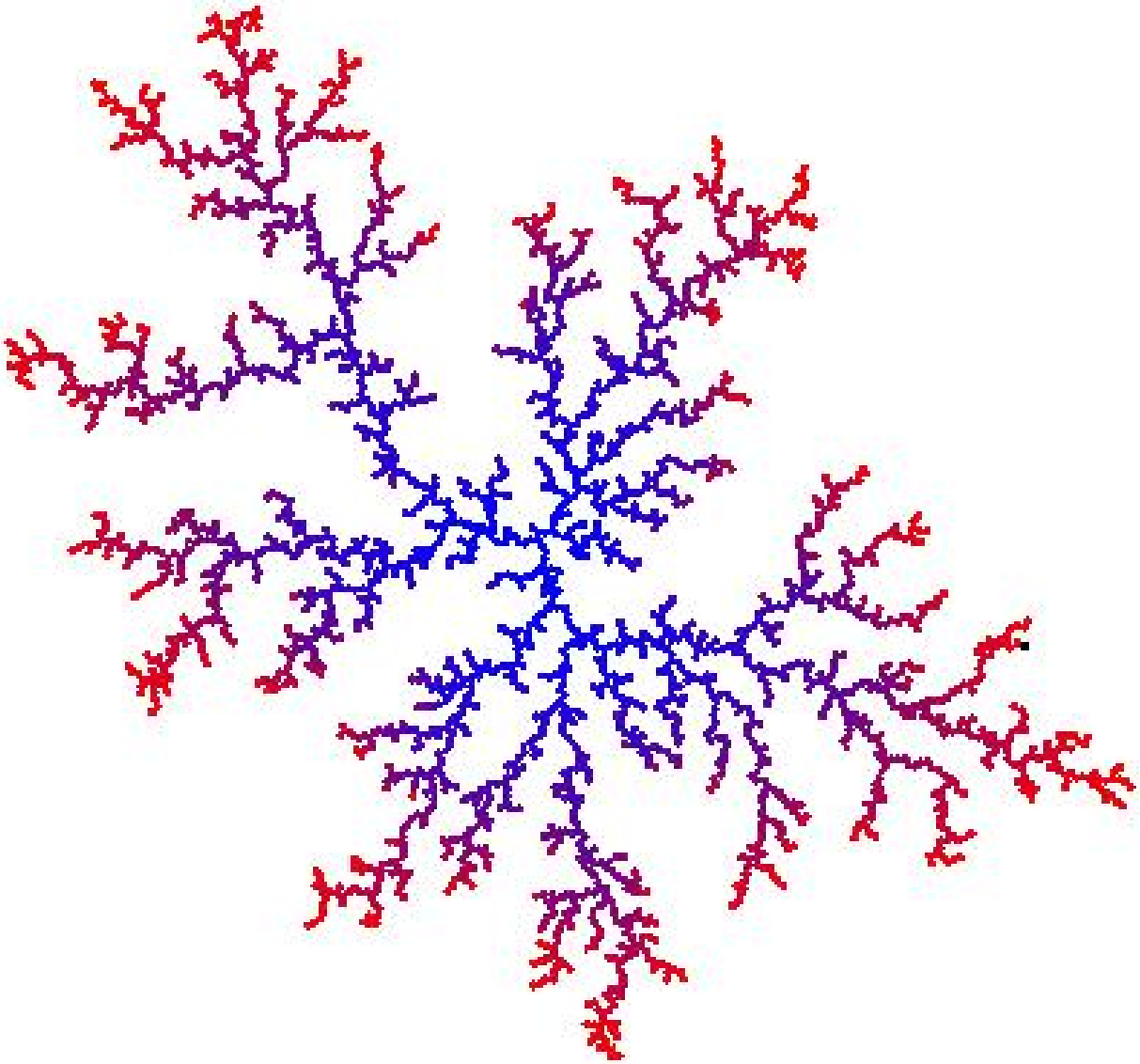
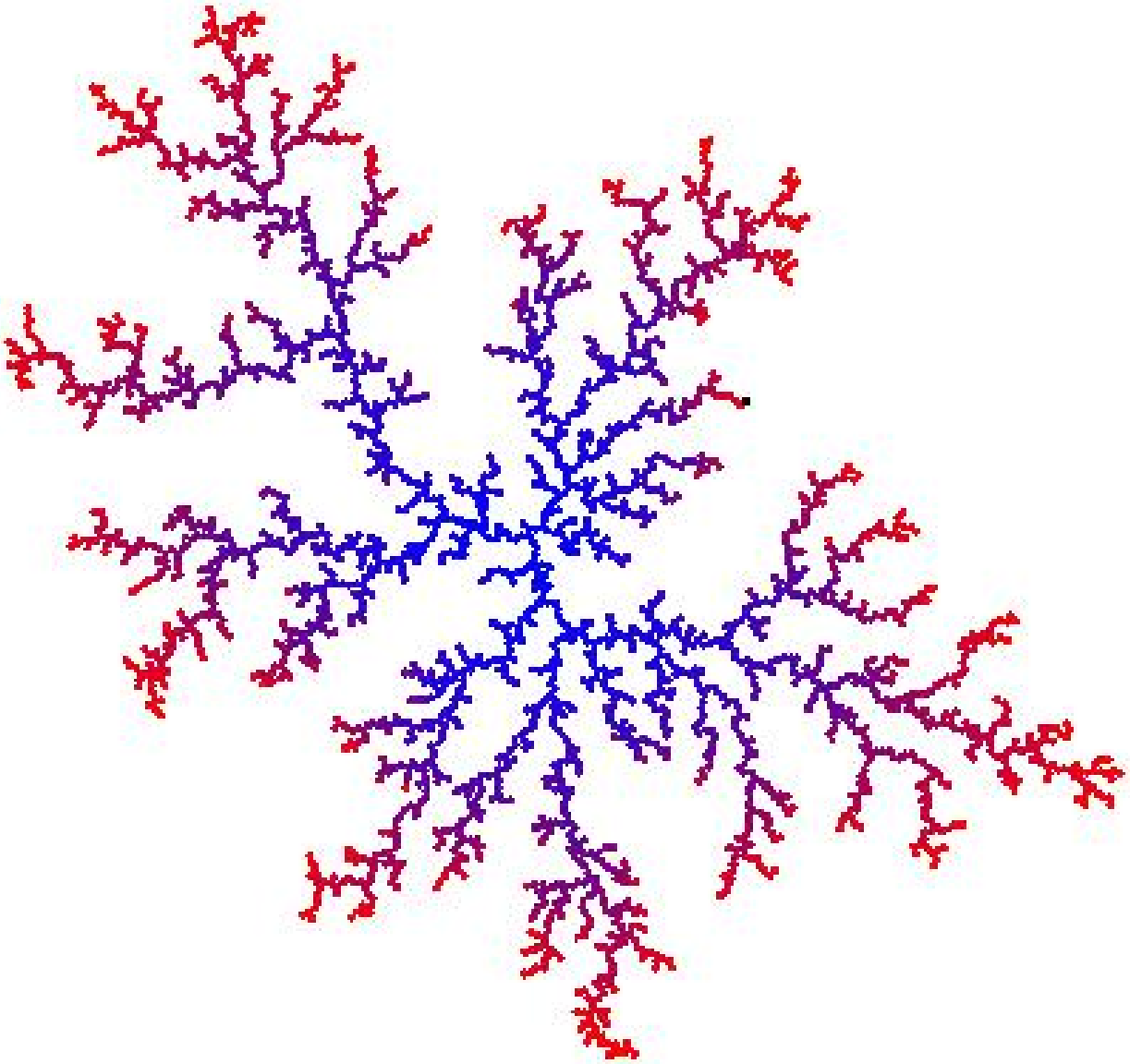
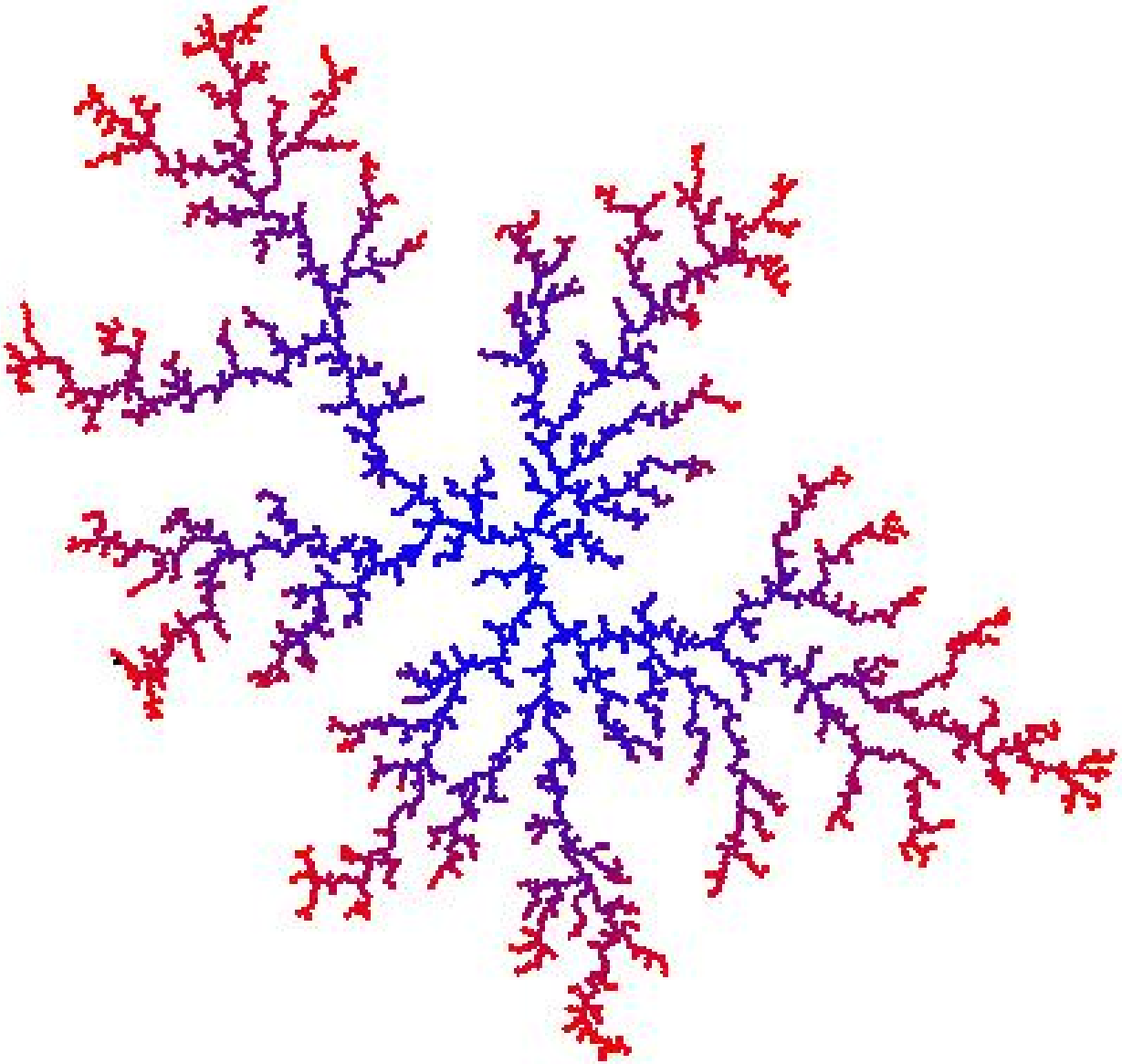
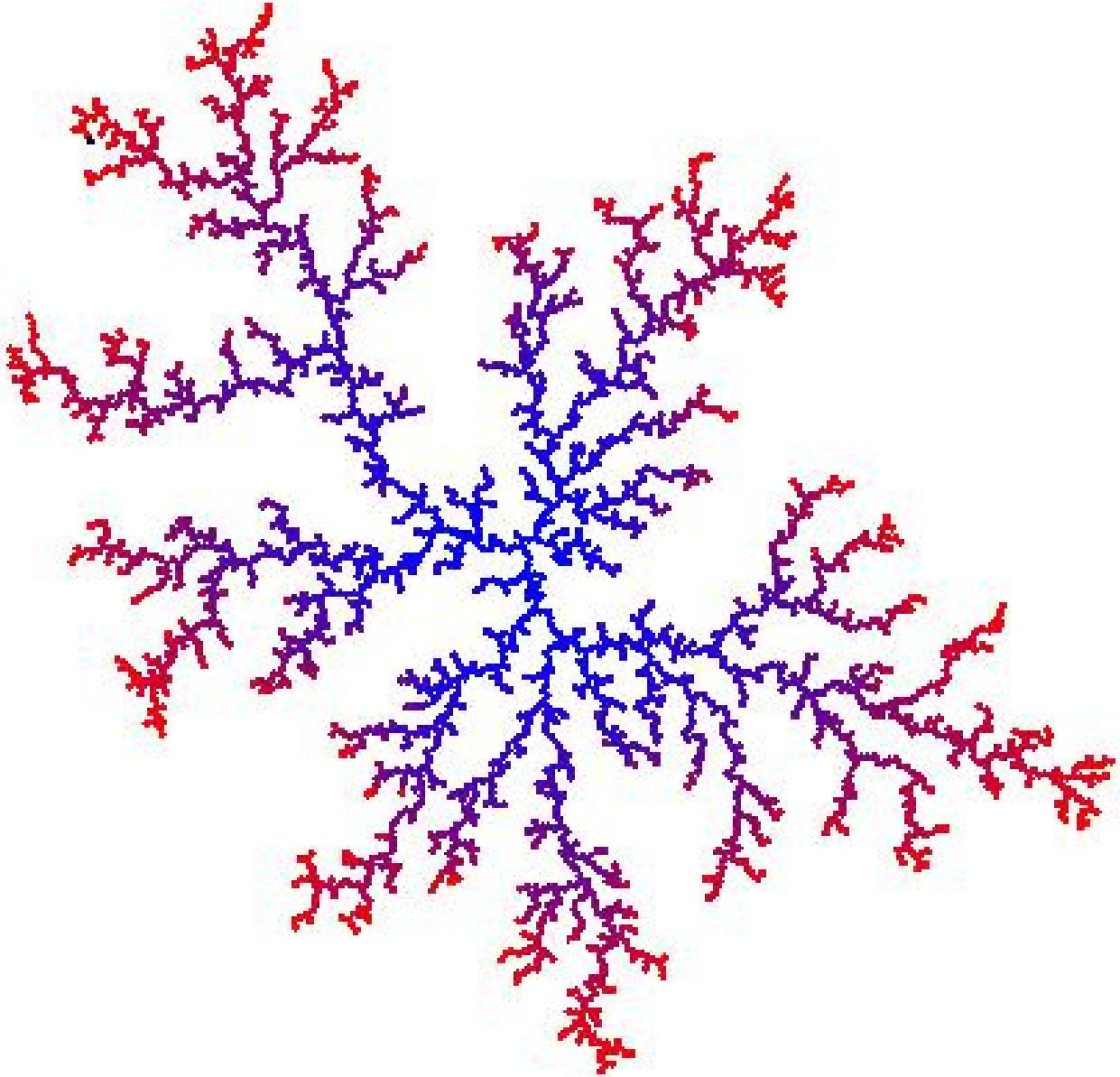
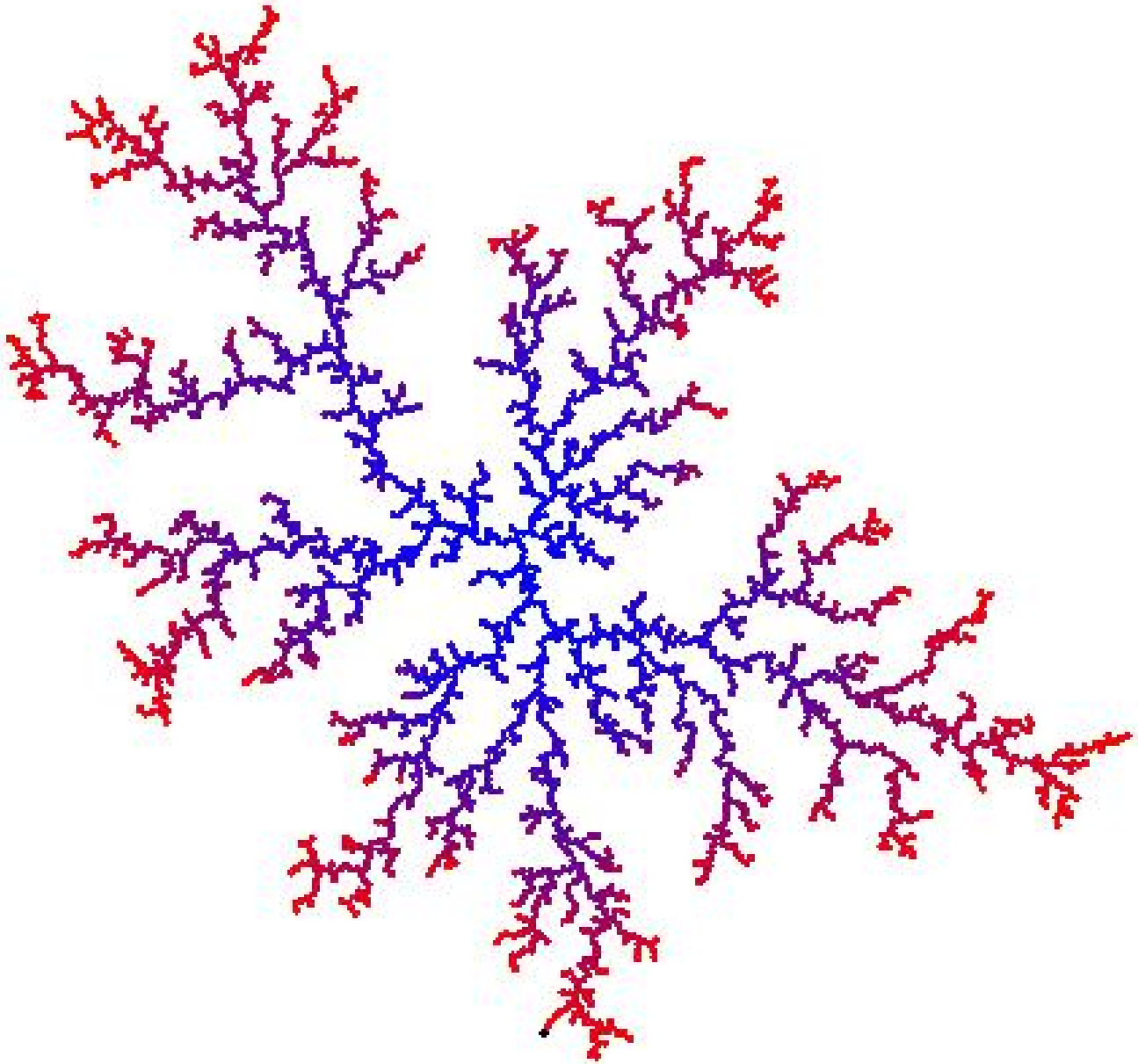
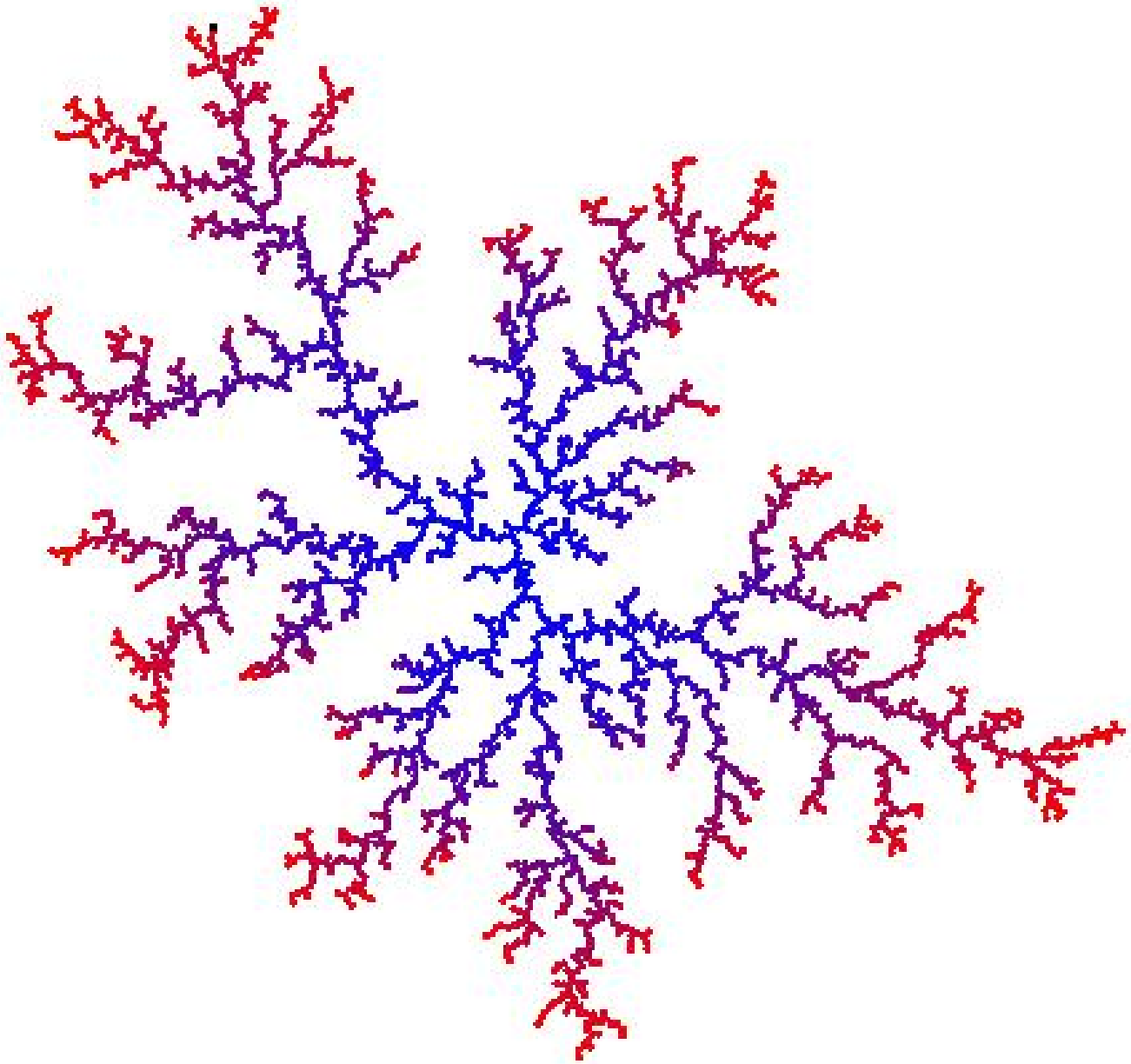
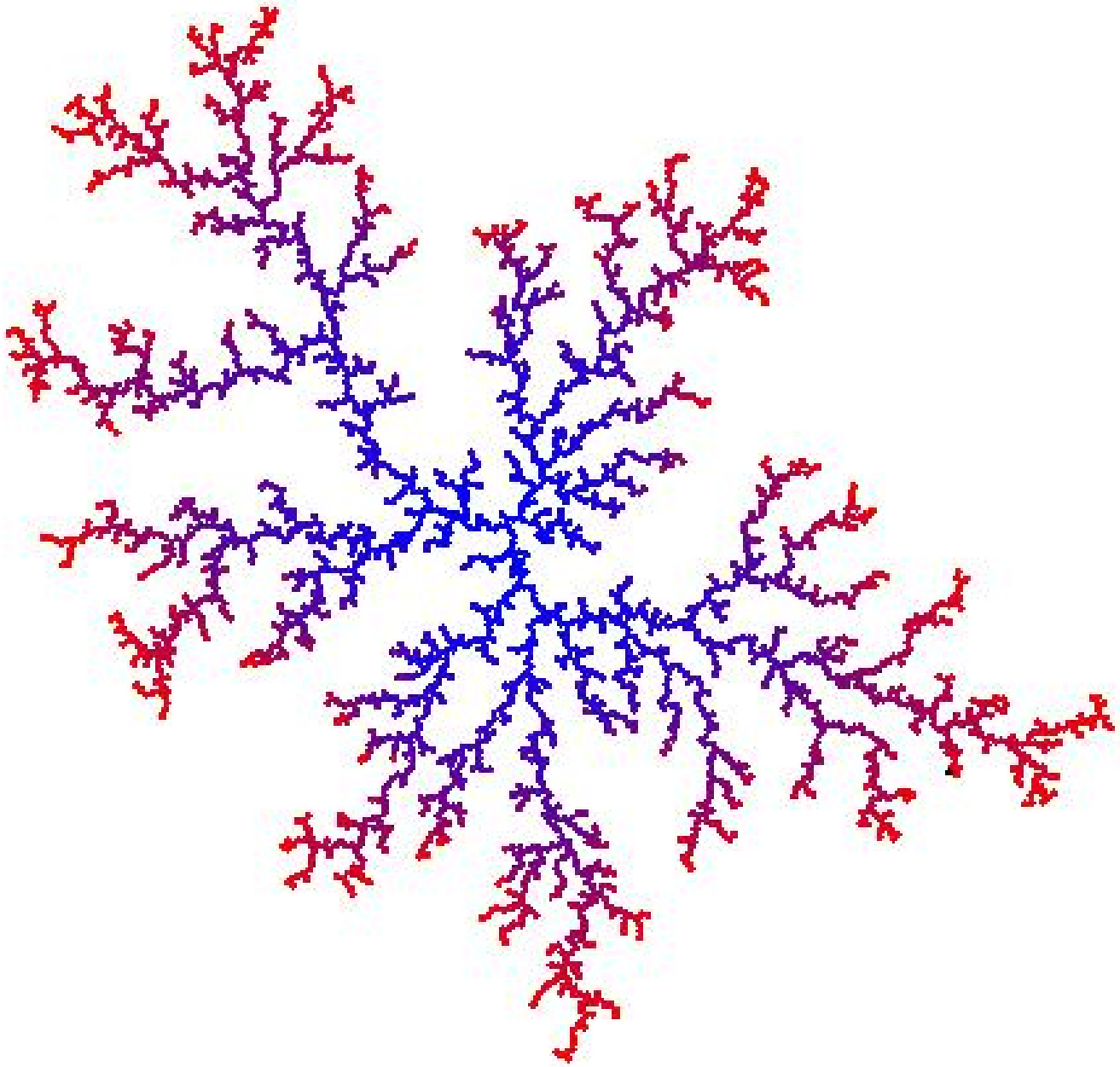
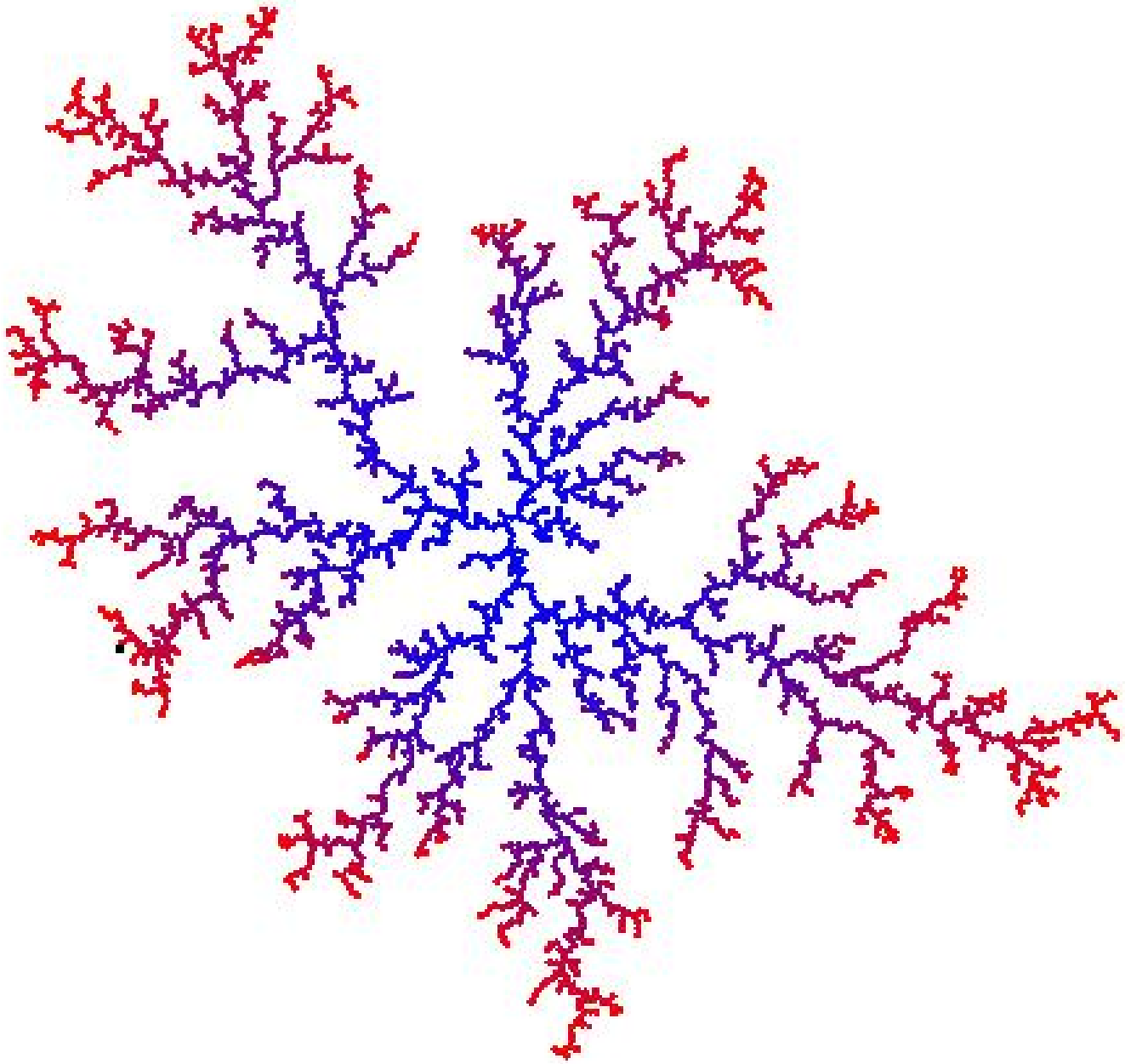
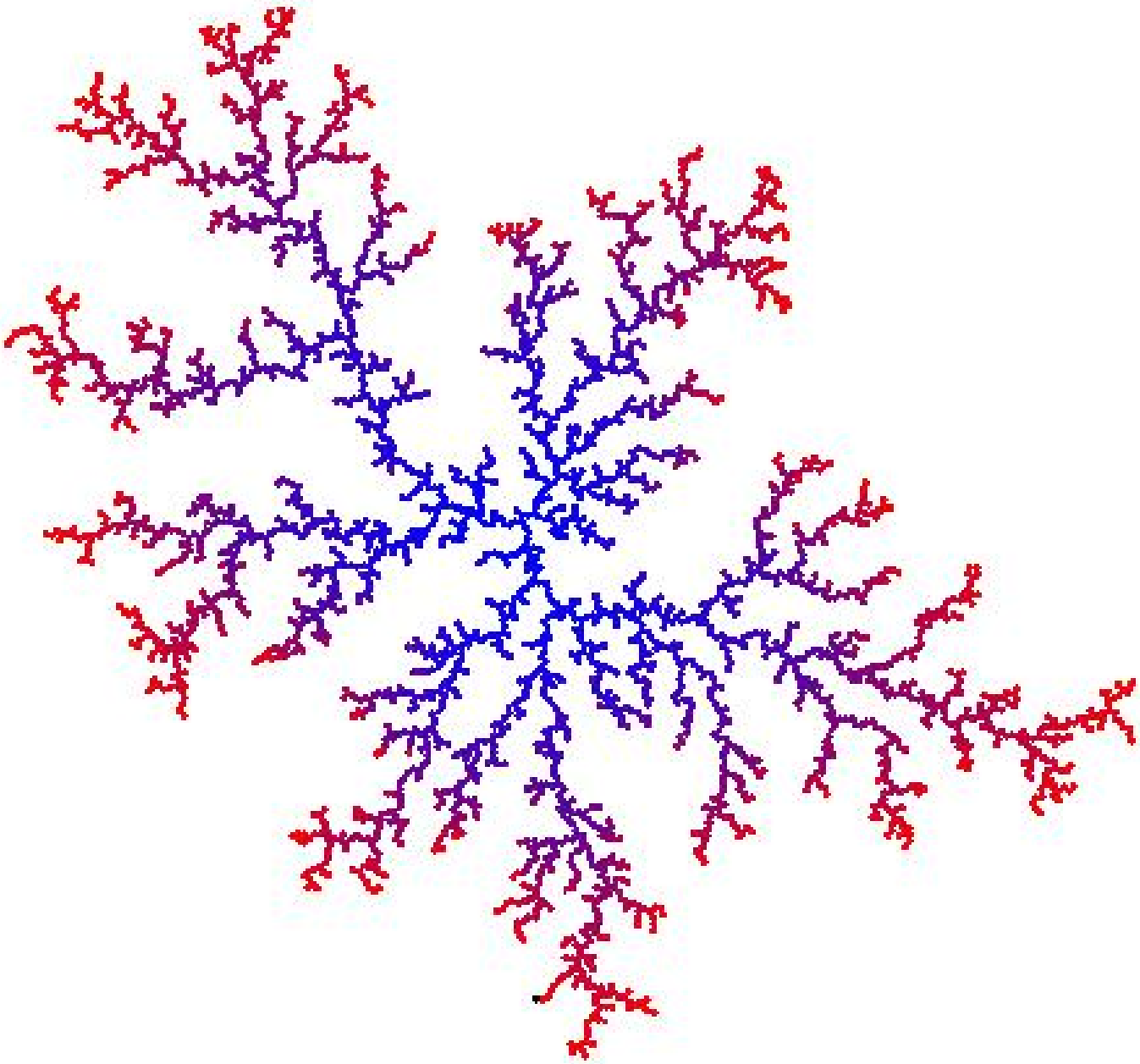
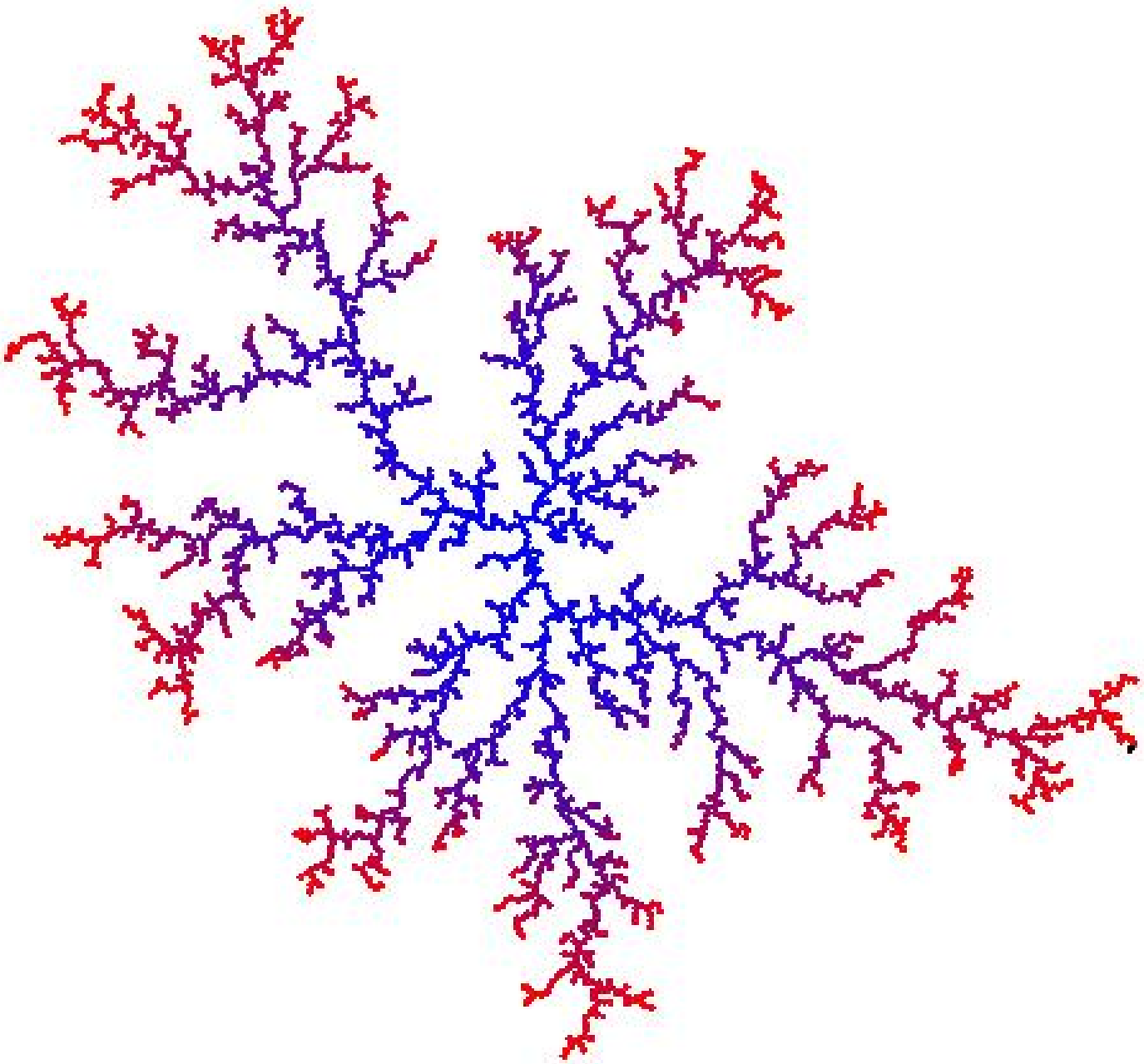
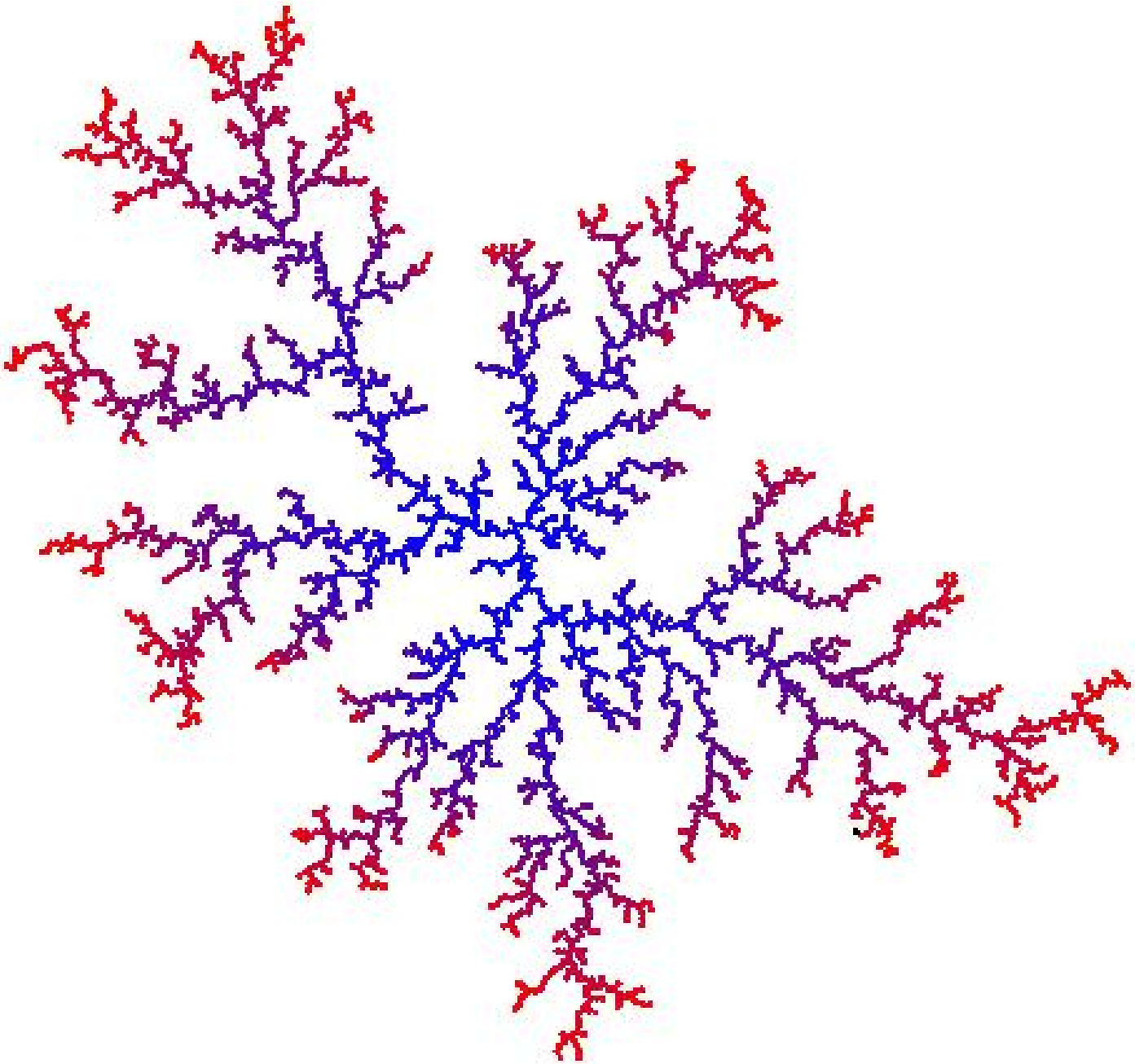
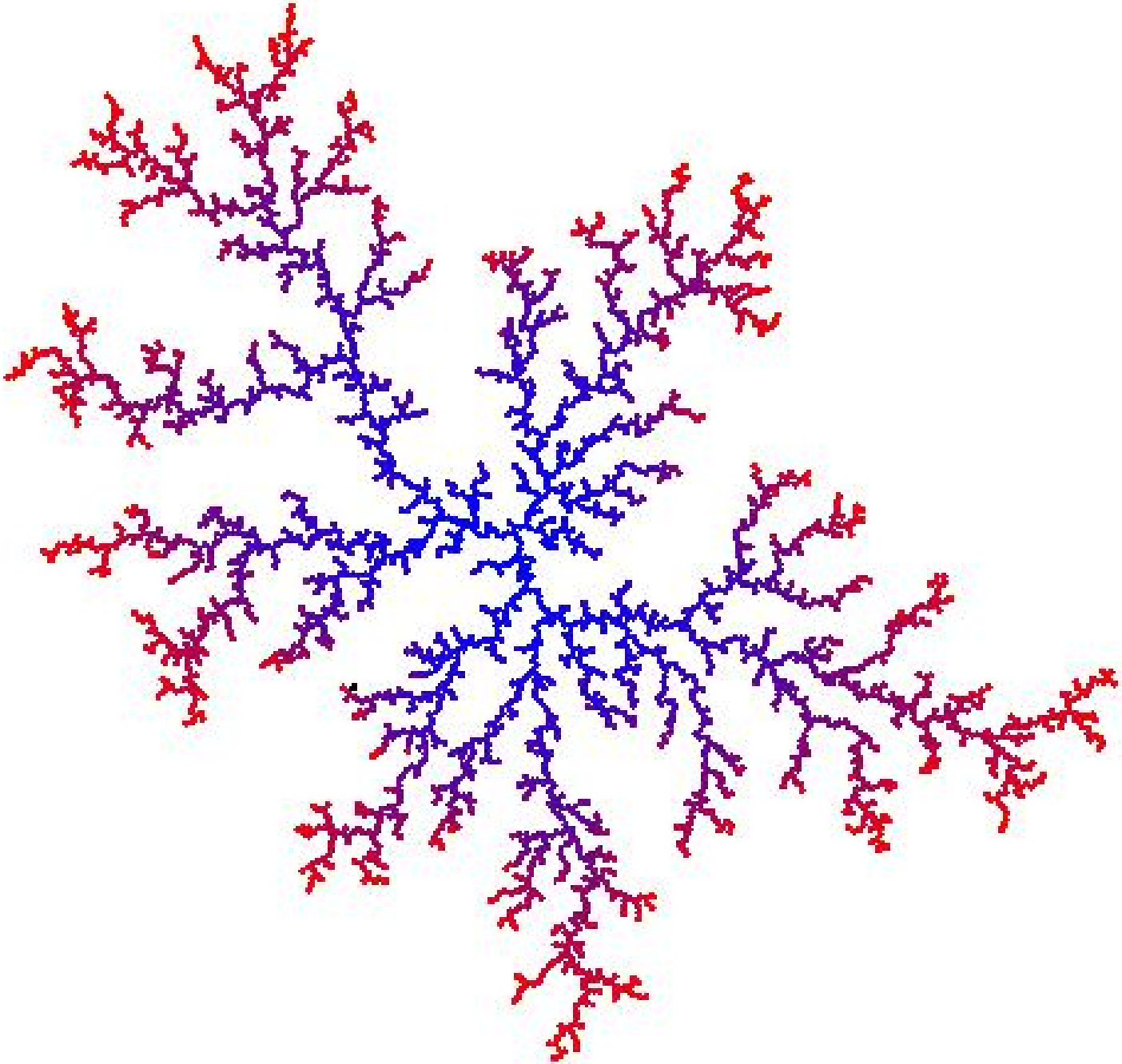
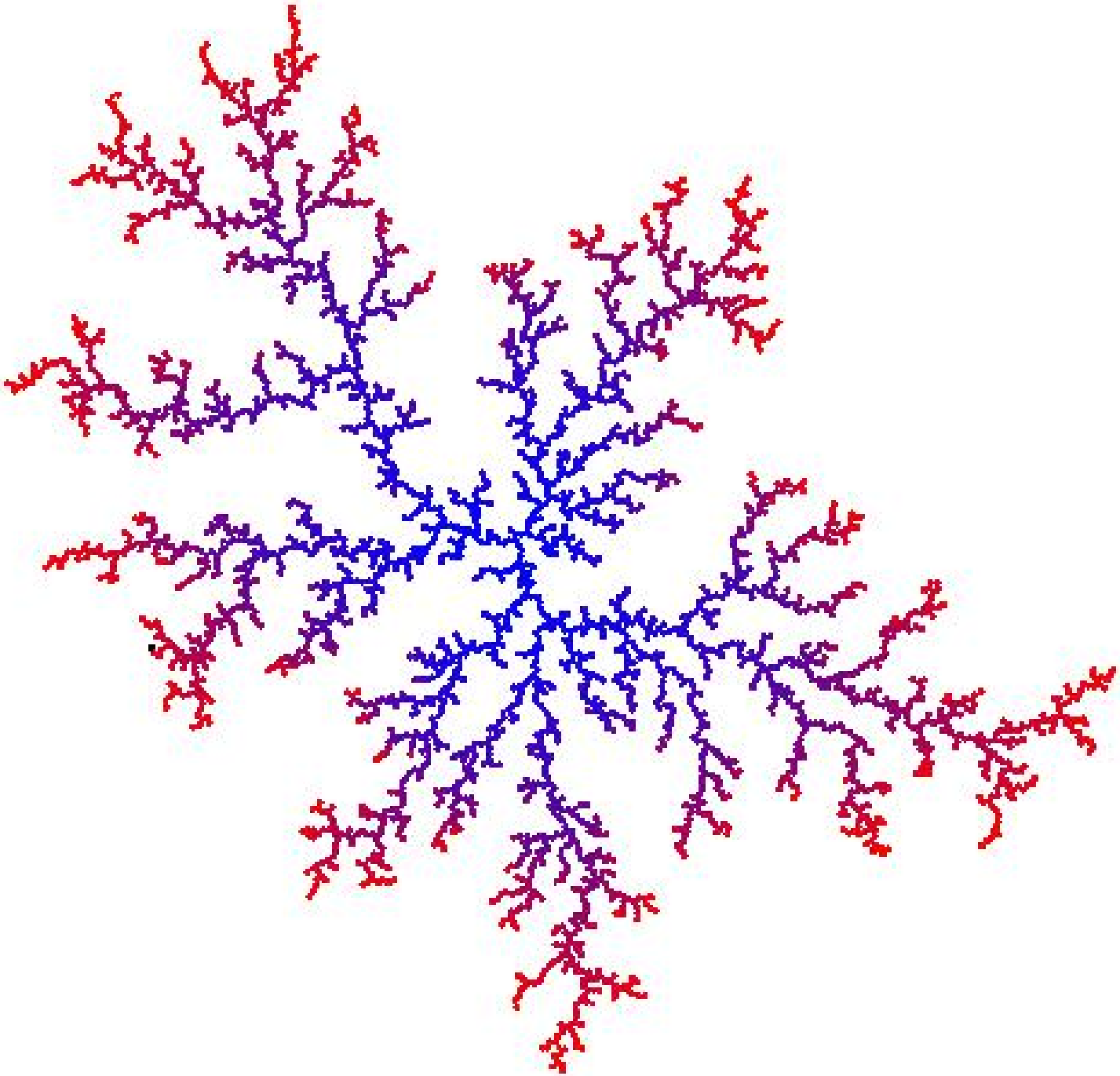
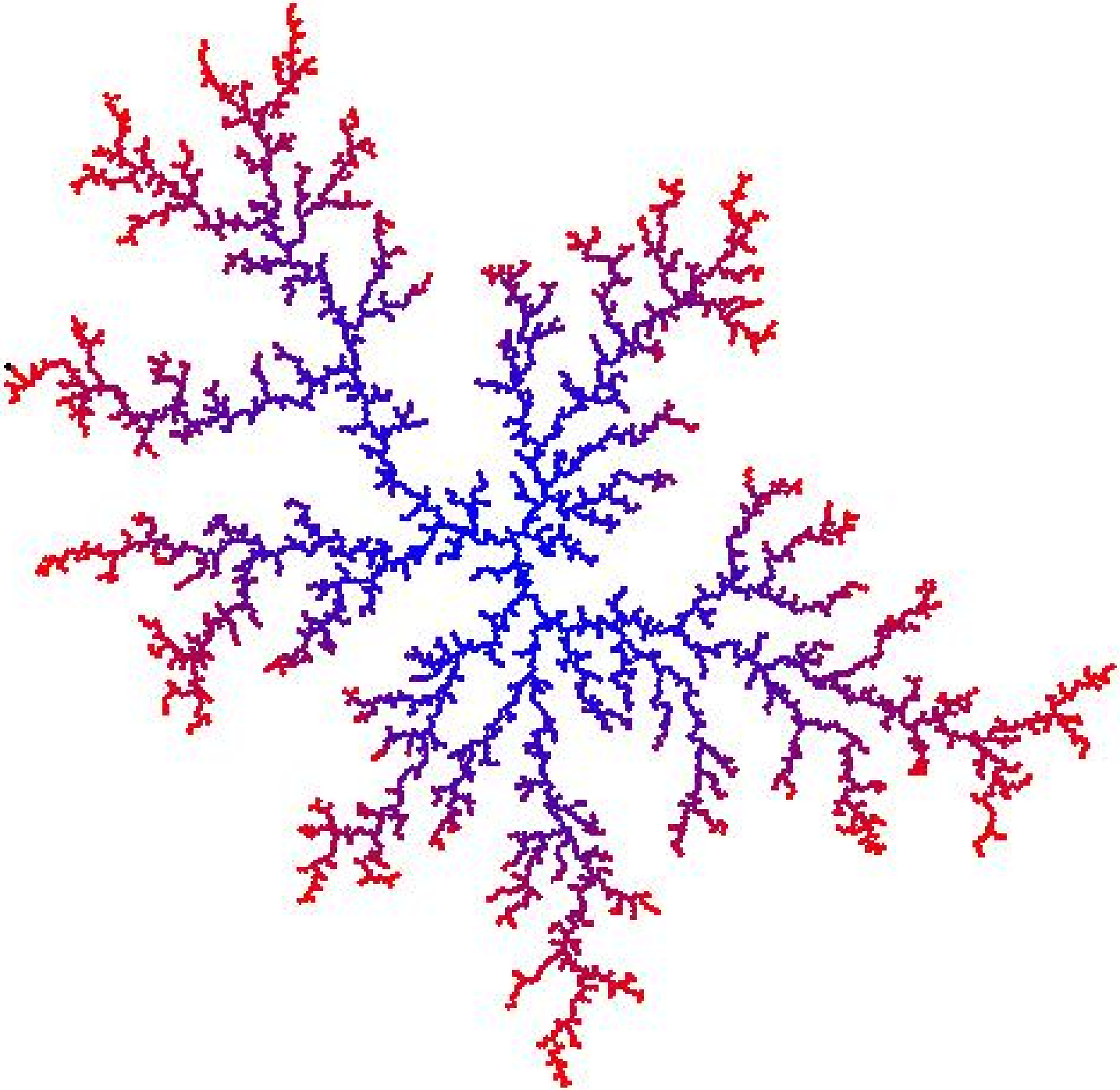
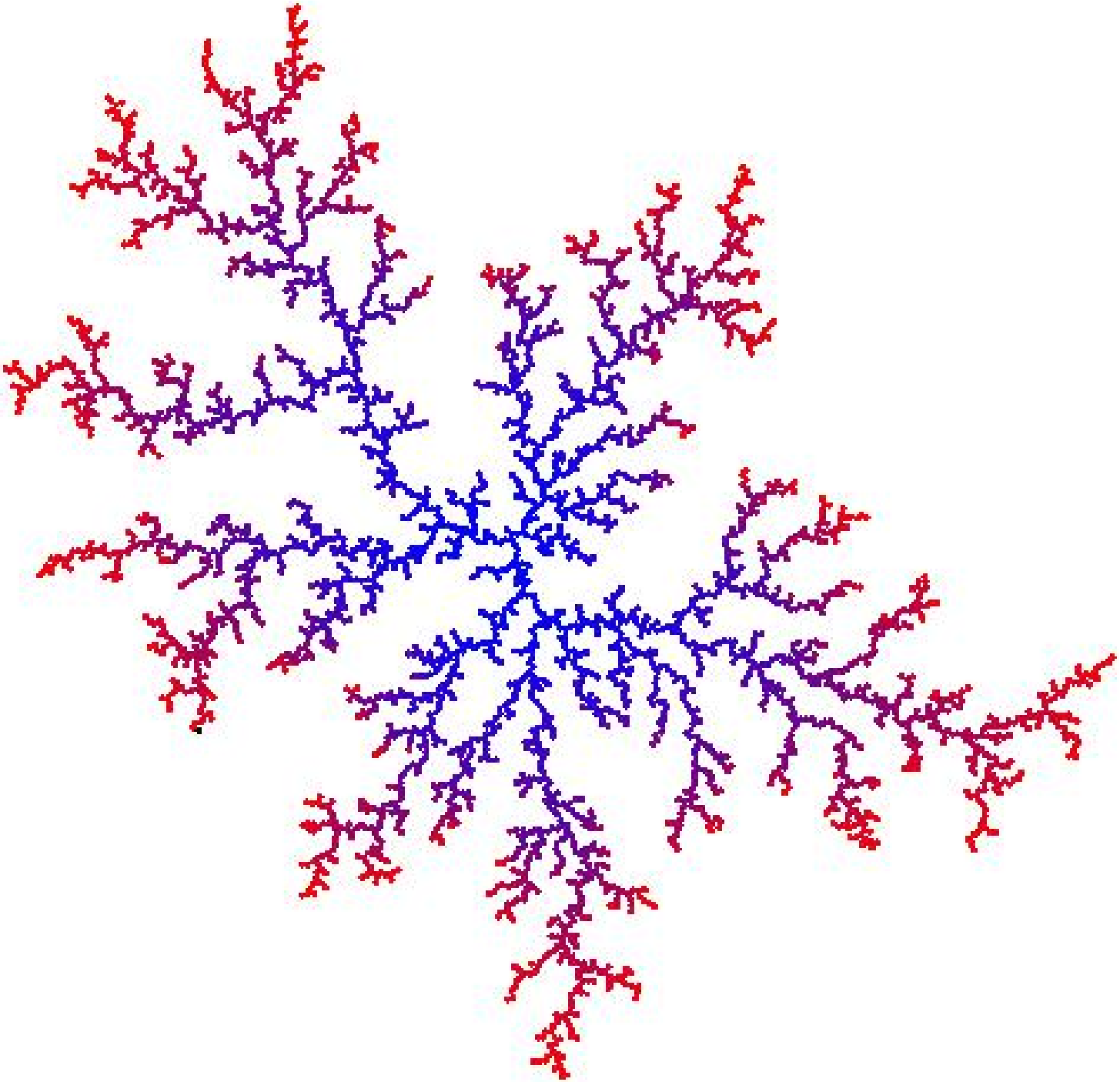
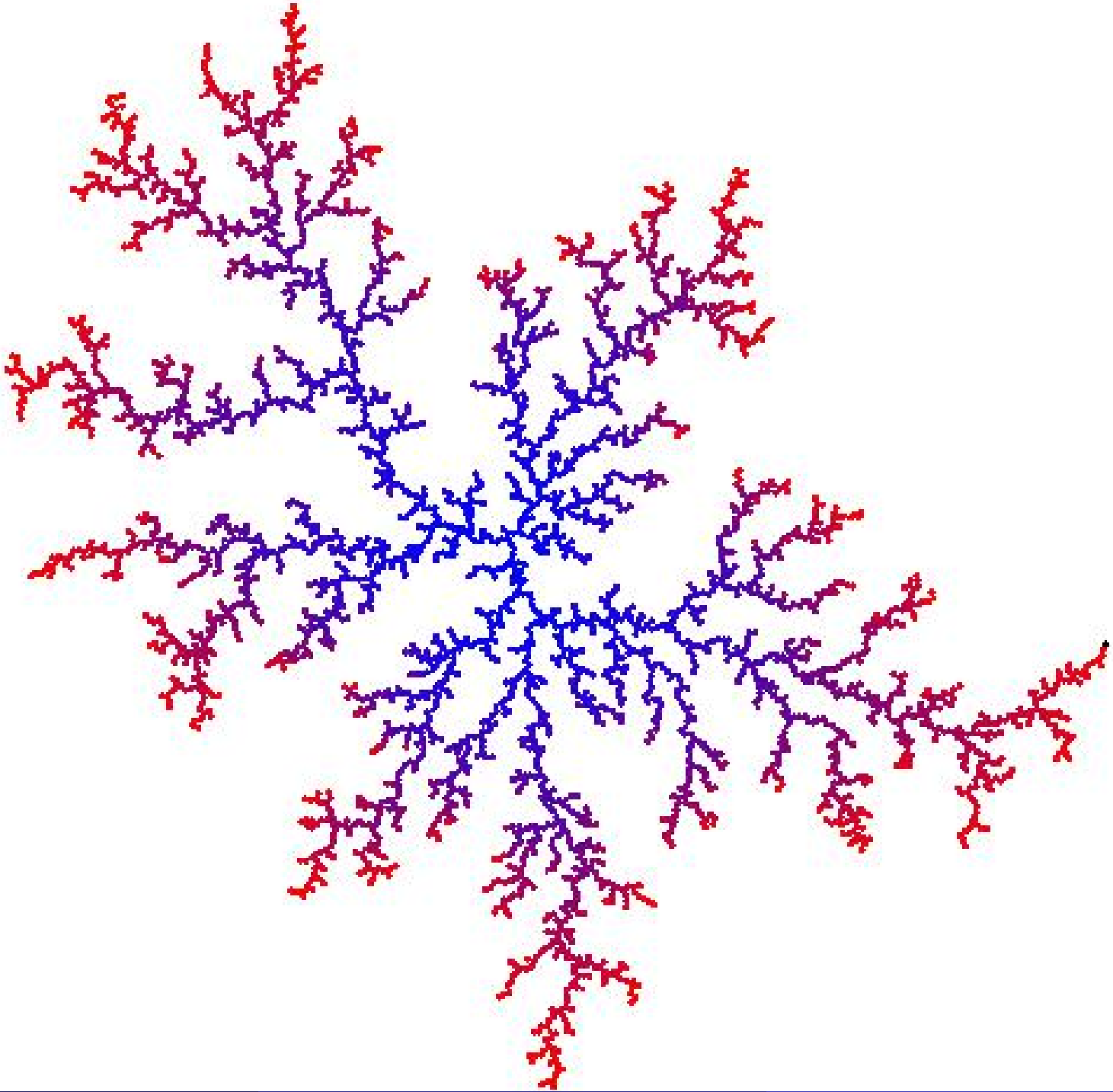
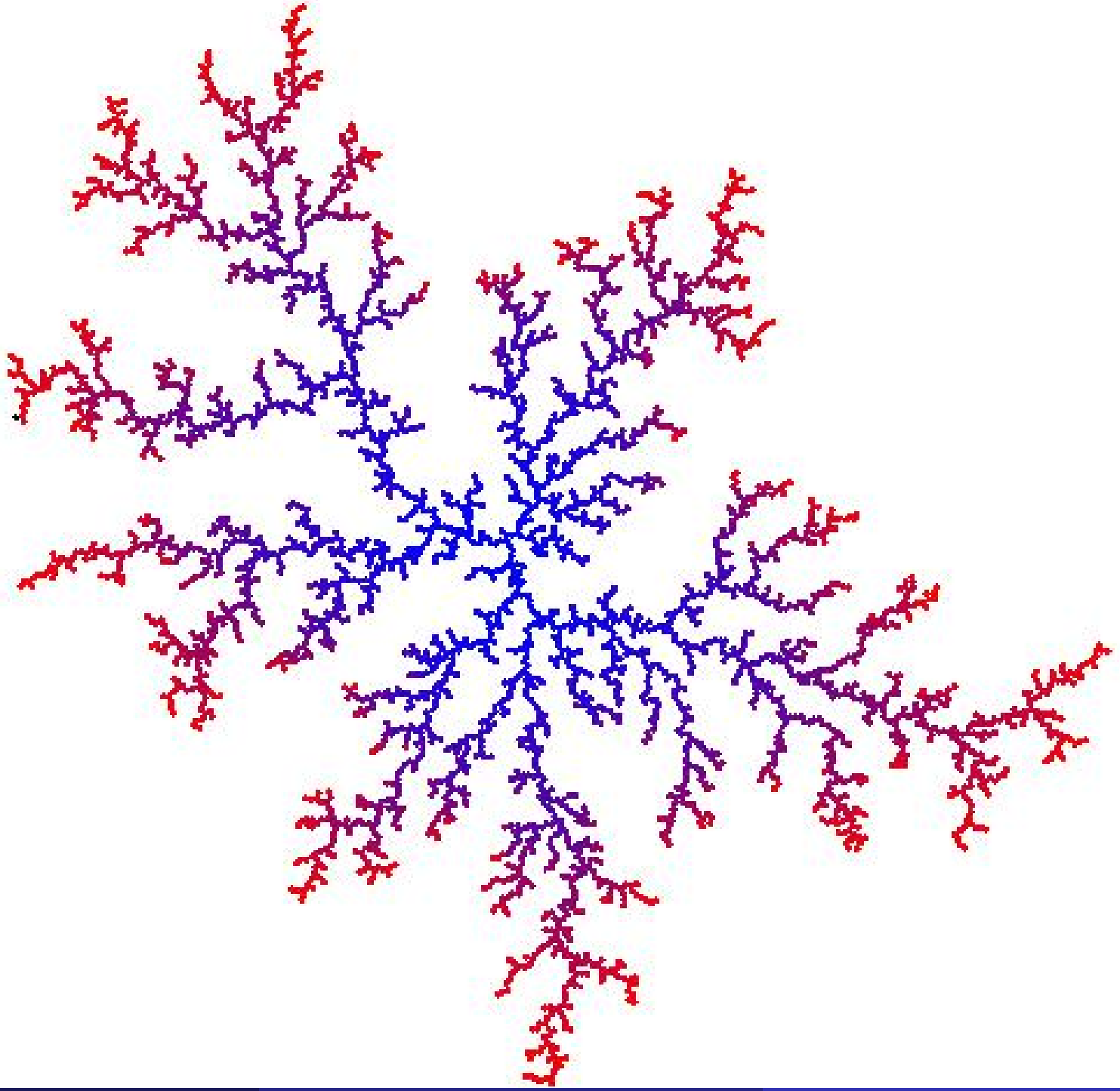
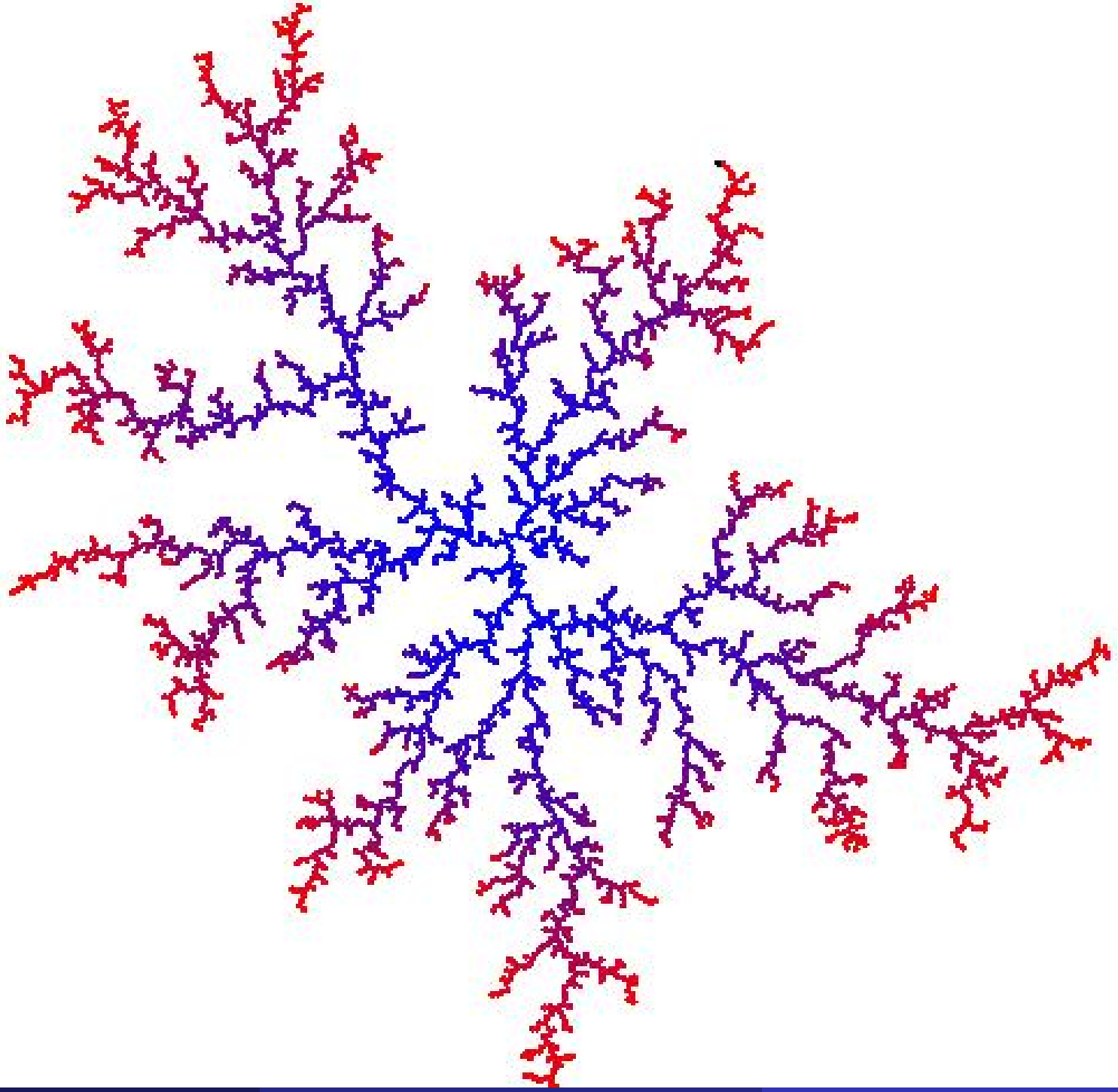
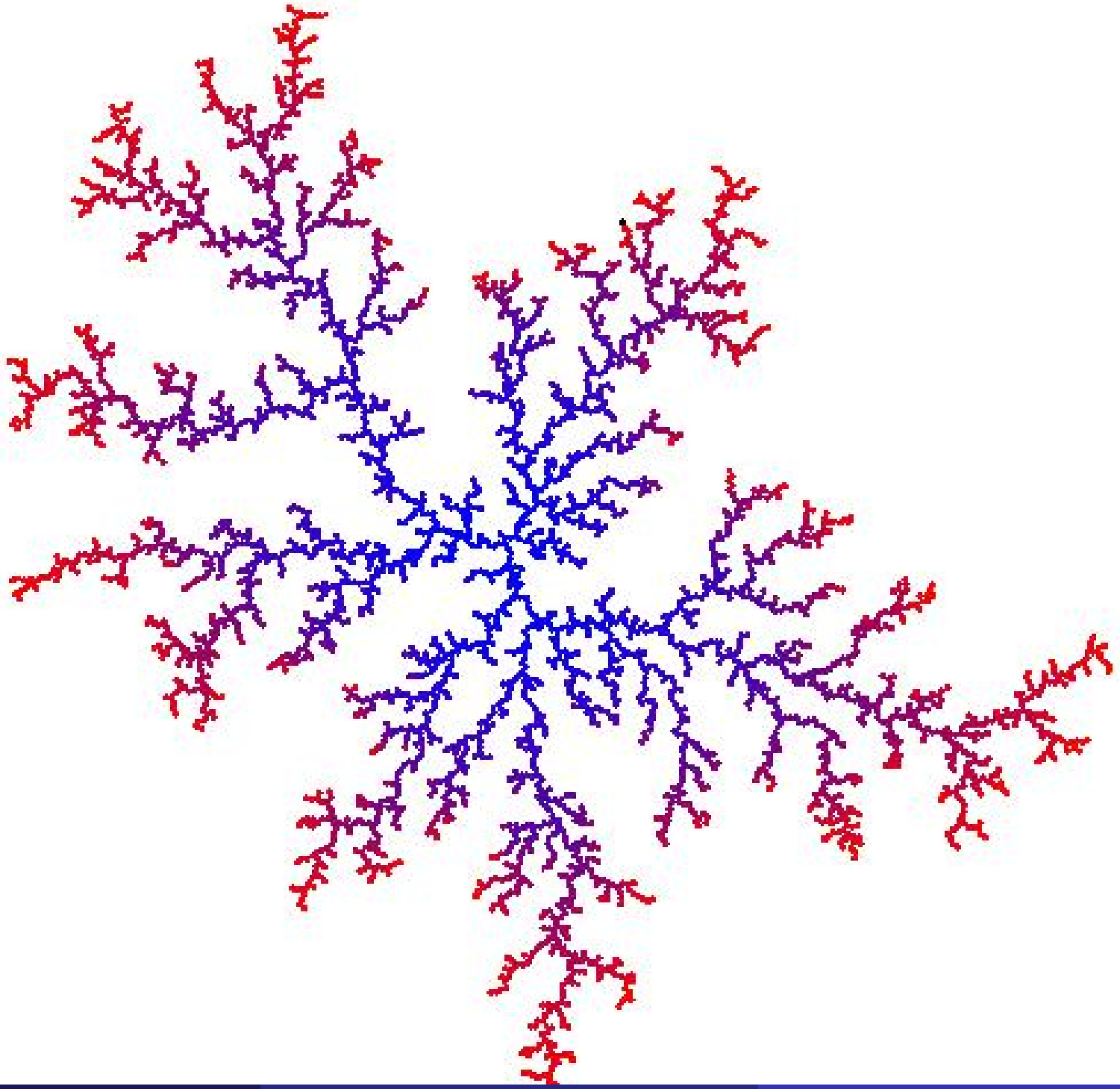
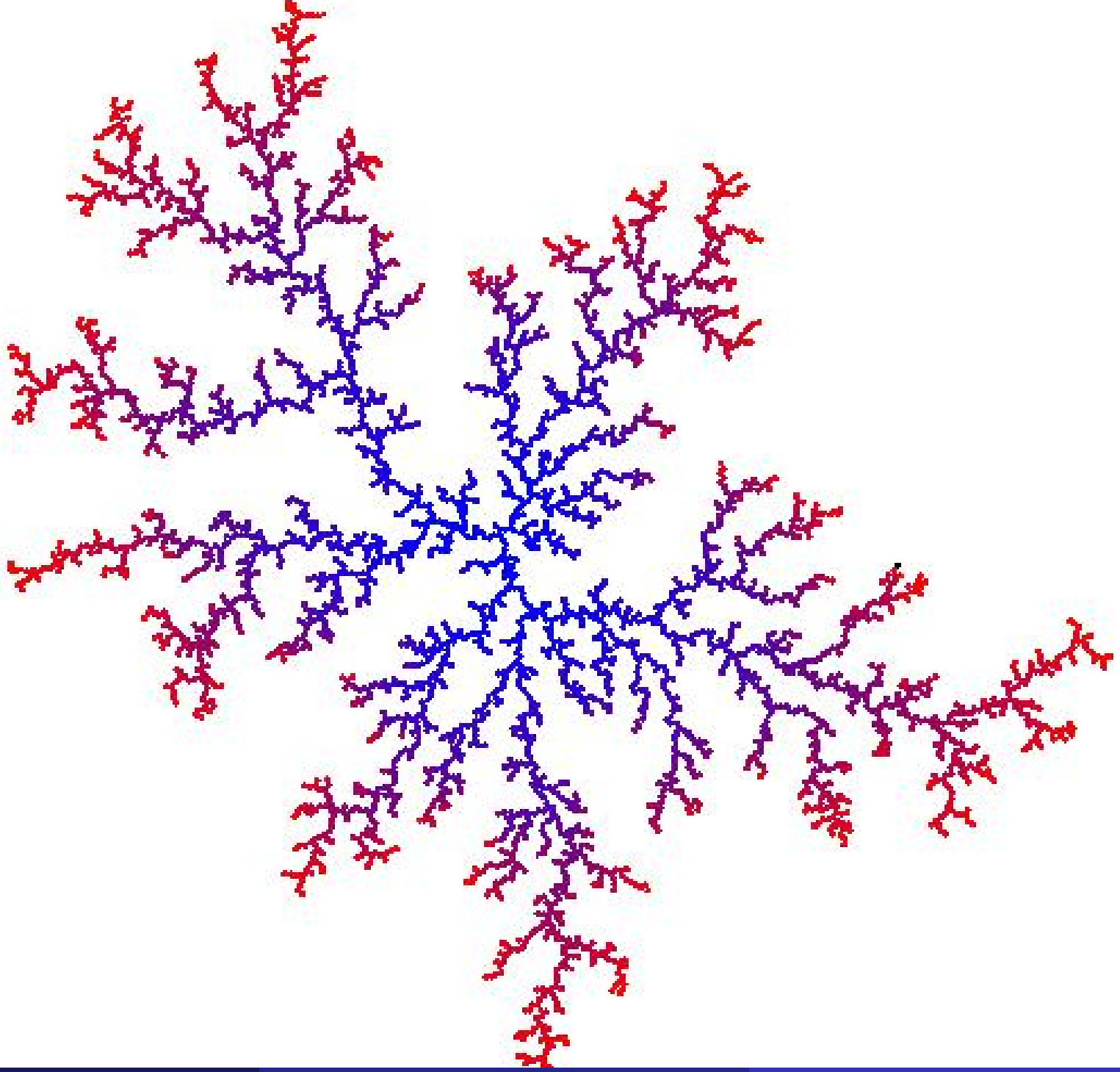
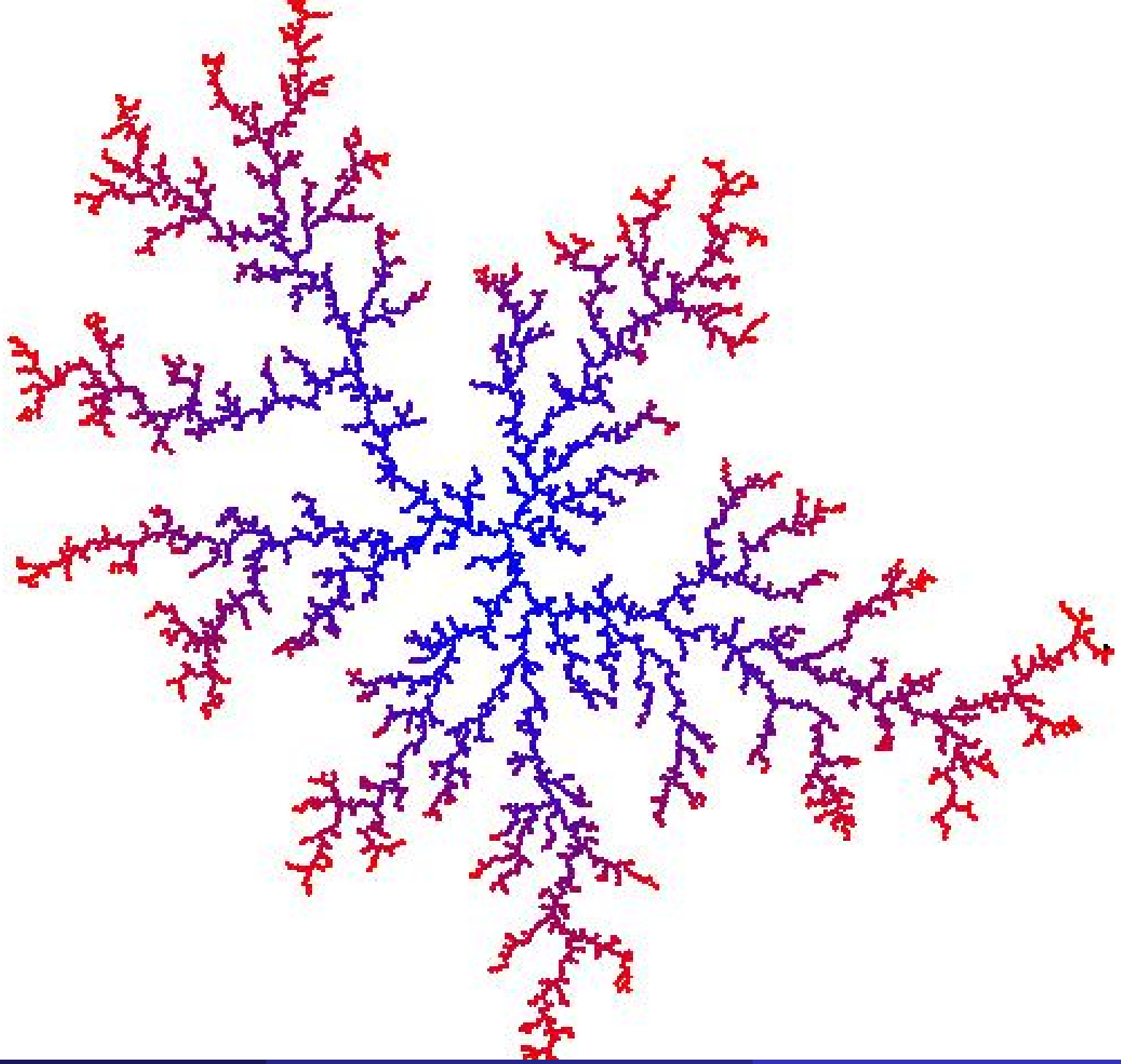
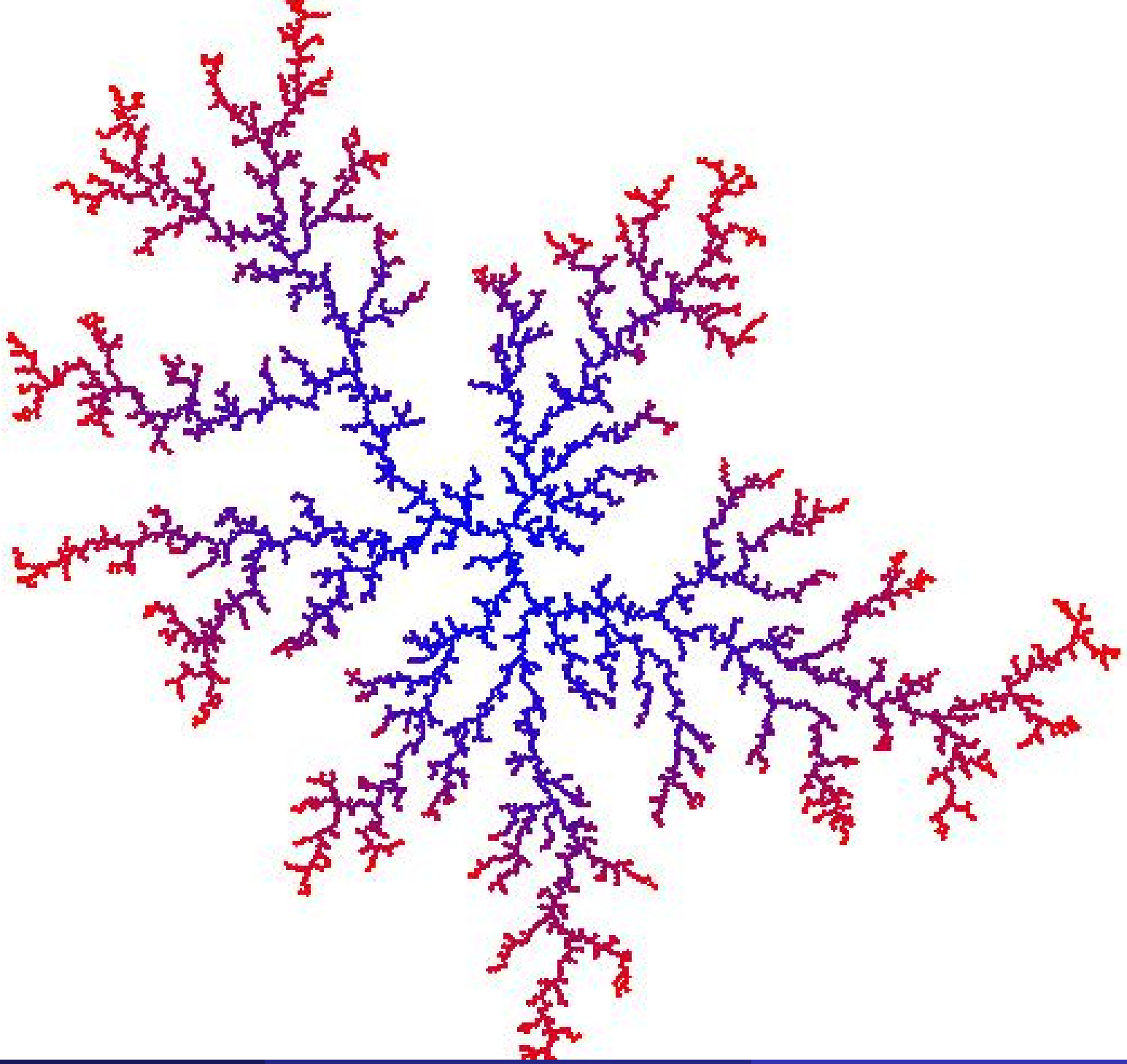
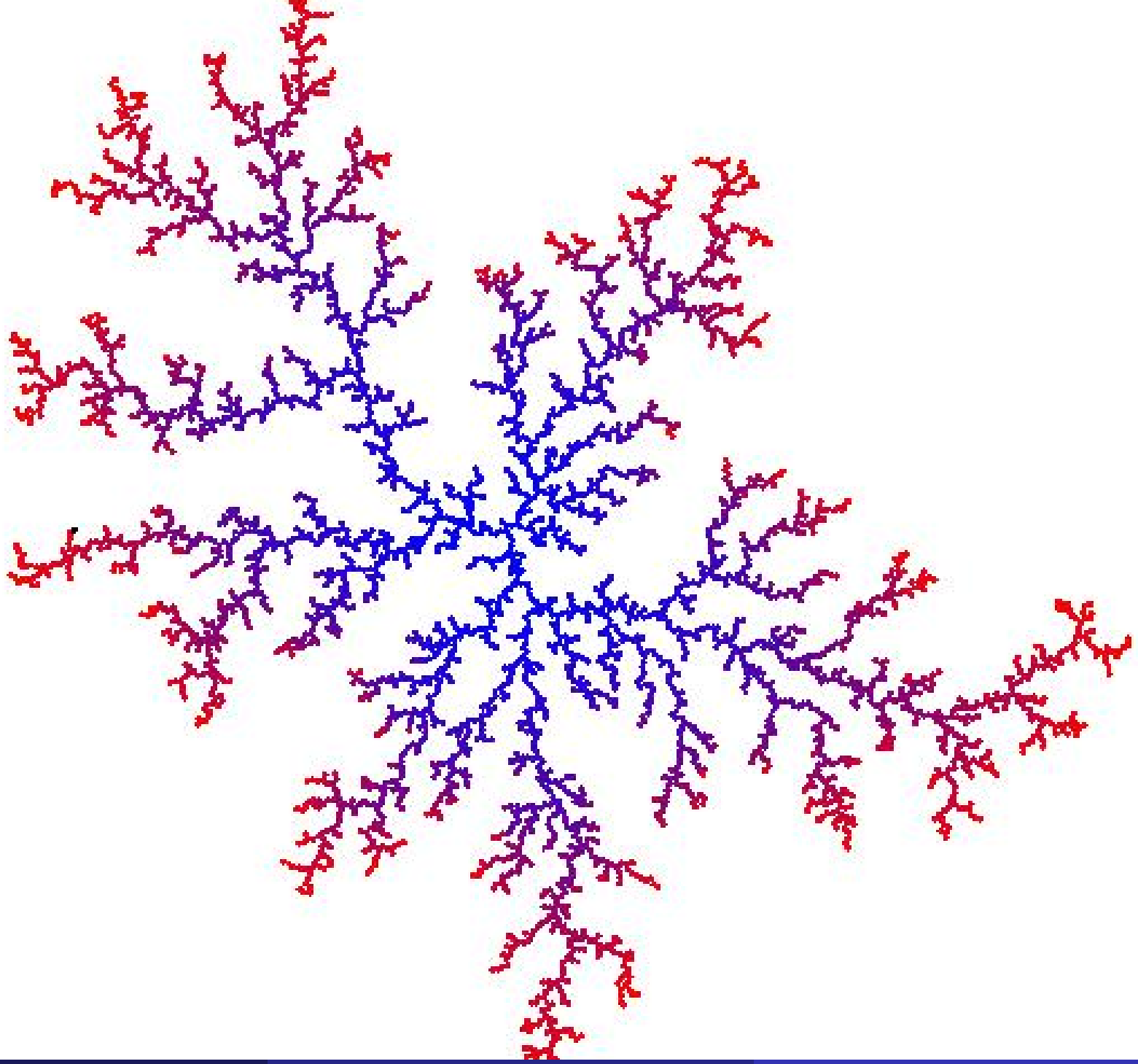
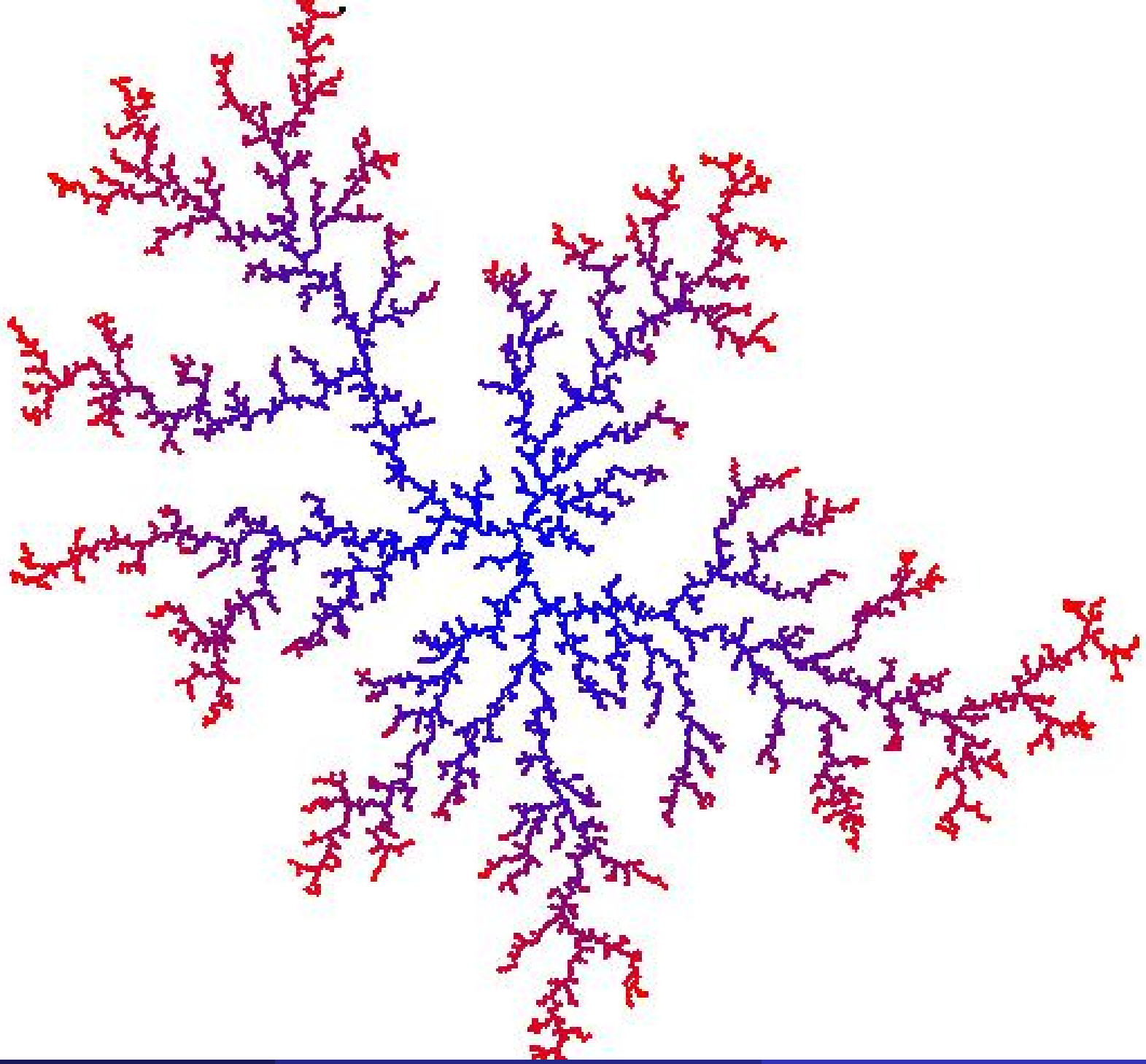
Théorème (M. Aizenman, H. Duminil-Copin (2019))

Pas de blancs en neige pour le modèle $\phi^4_{d=4}$ *!*