High frequency criteria for Boolean functions (with an application to percolation)

Christophe Garban
ENS Lyon and CNRS

Workshop on Discrete Harmonic Analysis
Newton Institute, March 2011

Plan

- Spectrum of Boolean functions
- Motivations (percolation ...)
- Randomized algorithms
- A theorem by Schramm and Steif
- Noise sensitivity of percolation
- Where exactly does the spectrum of percolation localize ? (Work with G. Pete and O. Schramm)

Spectrum of a Boolean function

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These functions form an othornormal basis for $L^2(\{-1,1\}^n)$ endowed with the uniform measure $\mu = (1/2\delta_1 + 1/2\delta_{-1})^{\otimes n}$.

The Fourier coefficients $\widehat{f}(S)$ satisfy

$$\widehat{f}(S) := \langle f, \chi_S \rangle = \mathbb{E}\big[f \chi_S \big]$$

Parseval tells us

$$\|f\|_2^2 = \sum_S \widehat{f}(S)^2$$

Energy spectrum of a Boolean function

For any Boolean or real-valued function $f : \{-1,1\}^n \to \{0,1\}$ or $\mathbb{R}$, we define its **energy spectrum** E_f to be

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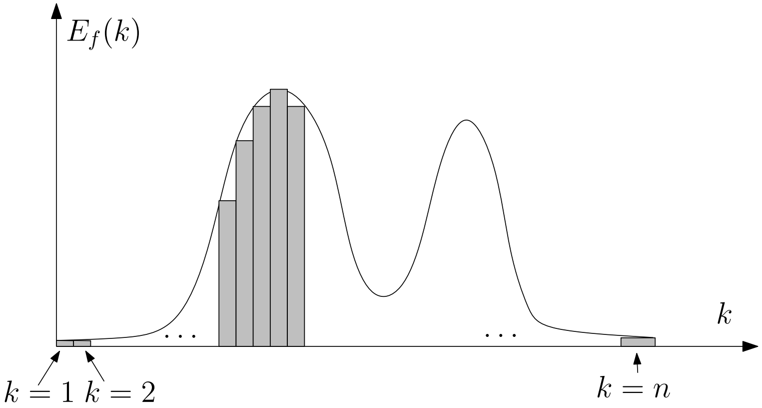

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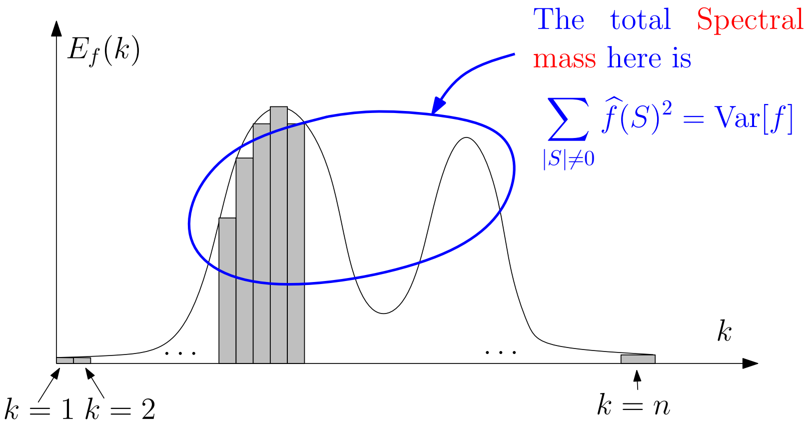

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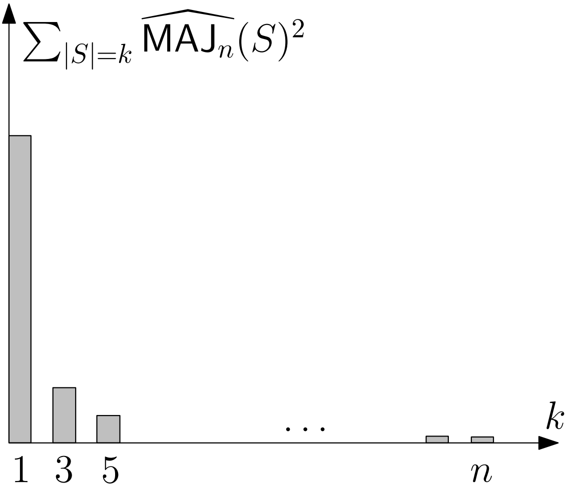

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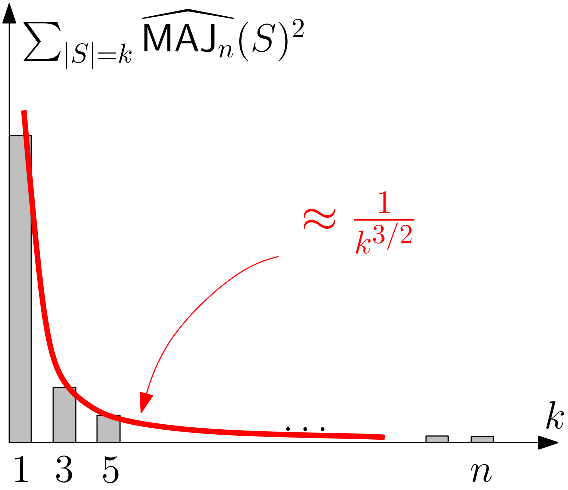

Noise stability

Let $f : \{-1, 1\}^n \to \{-1, 1\}$ be a Boolean function.
For any $\epsilon > 0$, if $\omega_n = (x_1, \ldots, x_n)$ is sampled uniformly in Ω_n, let ω_n^ϵ be the **noised** configuration obtained out of ω_n by resampling each bit with probability ϵ.
The noise stability of f is given by

$$\mathbb{S}^\epsilon(f) := \mathbb{P}\big[f(\omega) = f(\omega^\epsilon)\big]$$

It is easy to check that

$$\mathbb{S}^\epsilon(f) = \sum_S \widehat{f}(S)^2 (1-\epsilon)^{|S|}$$
$$= \mathbb{E}\big[f\big]^2 + \sum_{k \geq 1} E_f(k)(1-\epsilon)^k$$

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Here is a deep result by Bourgain:

Theorem (Bourgain, 2001)

If f is a balanced Boolean function with **low influences**, *then*

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Theorem ("Majority is Stablest" Mossel, O'Donnell and Oleszkiewicz, 2005)

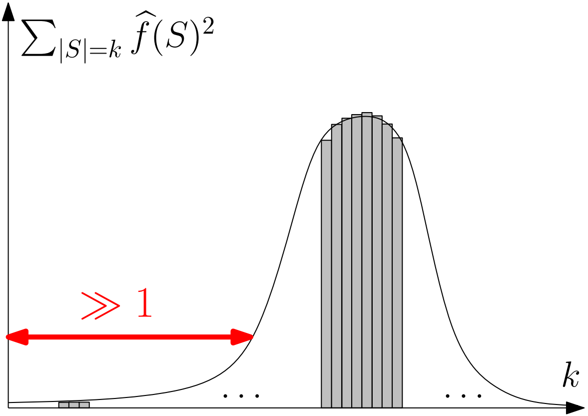
If f is a balanced Boolean function with **low influences**, *then*

$$\mathbb{S}^{\epsilon}(f) \lesssim \frac{2}{\pi}\arcsin(1-\epsilon)$$

Recognizing high frequency behavior

In what follows, we will be interested in Boolean (or real-valued) functions which are highly sensitive to small perturbations (or small noise). Such functions are called noise sensitive.

They are such that most of their Fourier mass is localized on high frequencies. In particular, their energy spectrum should look as follows



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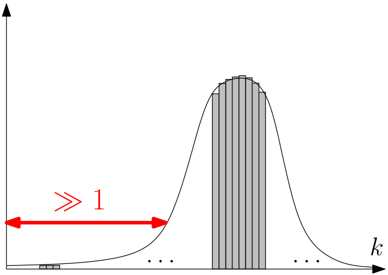

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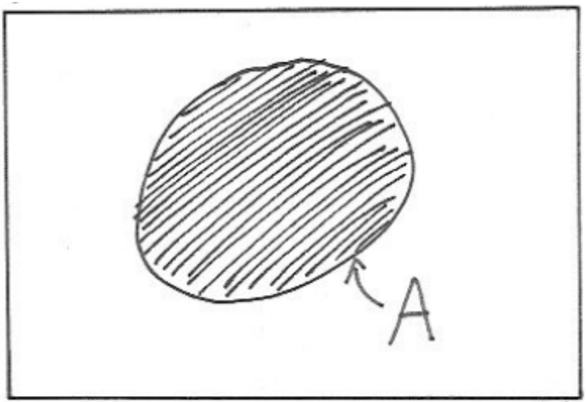

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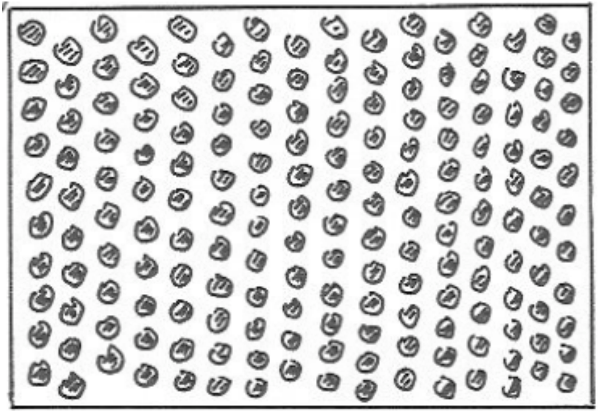

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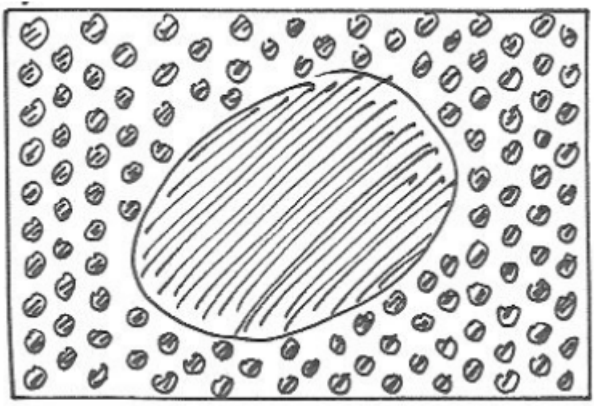

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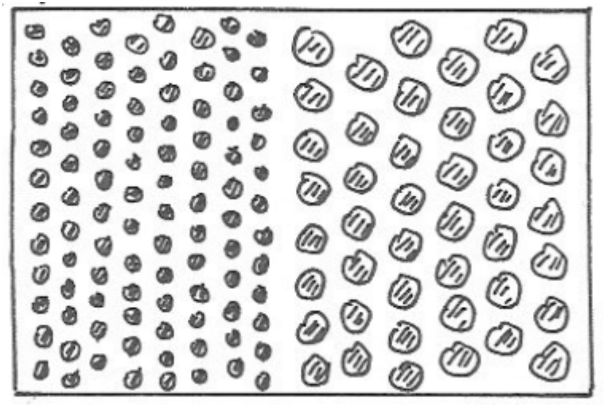

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- Fluctuations for natural random metrics on $\mathbb{Z}^d$ (First passage percolation).

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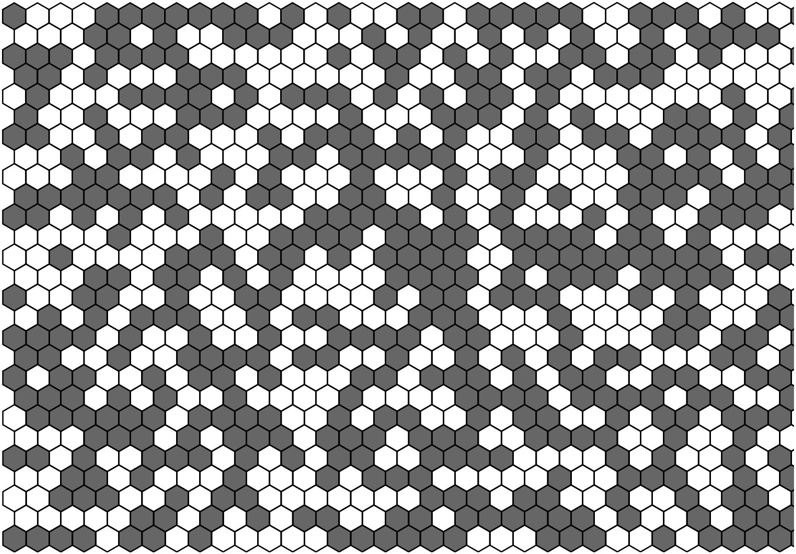
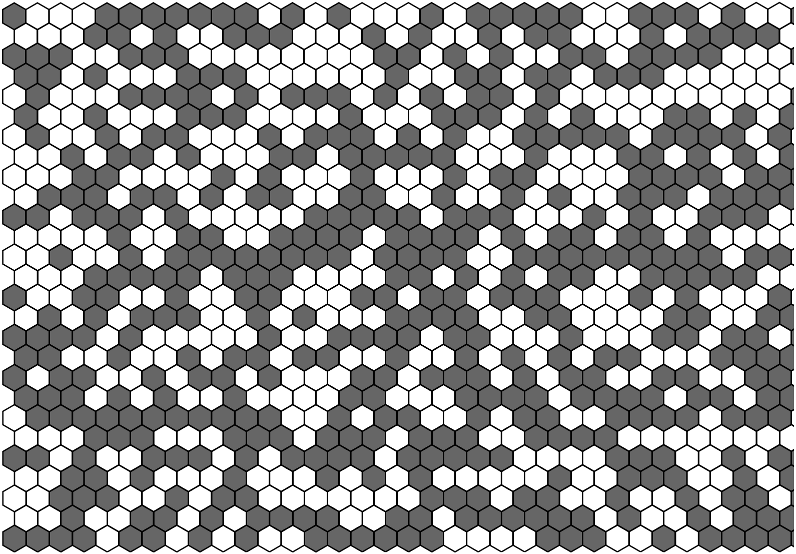
ω_0:



Dynamical percolation

$\omega_0 \rightarrow \omega_t$:



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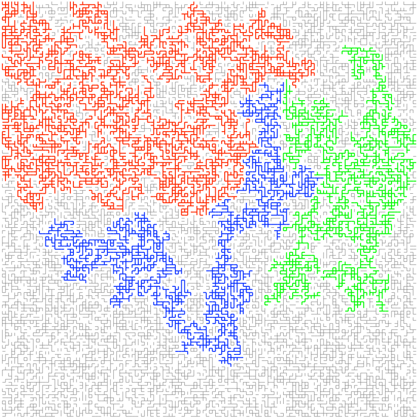

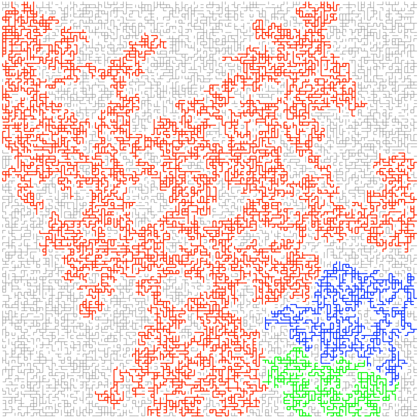

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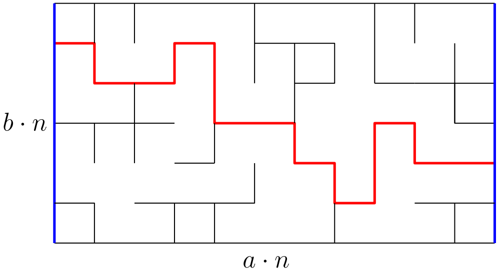

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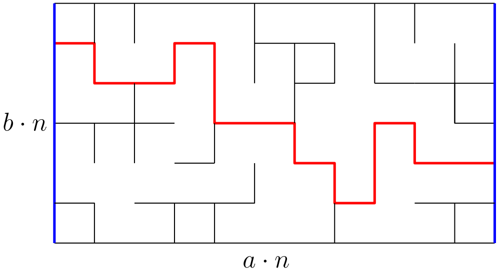

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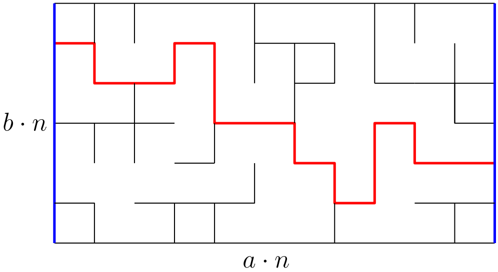

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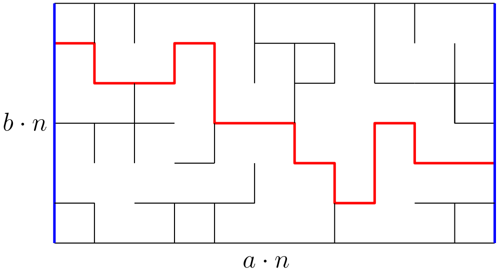

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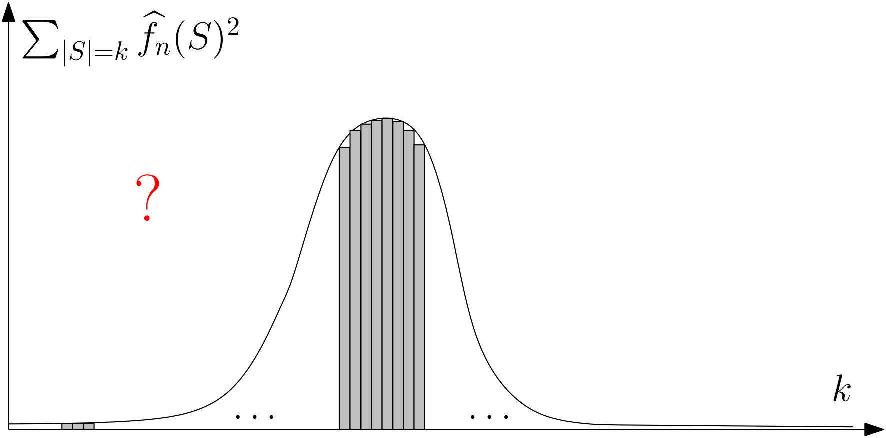

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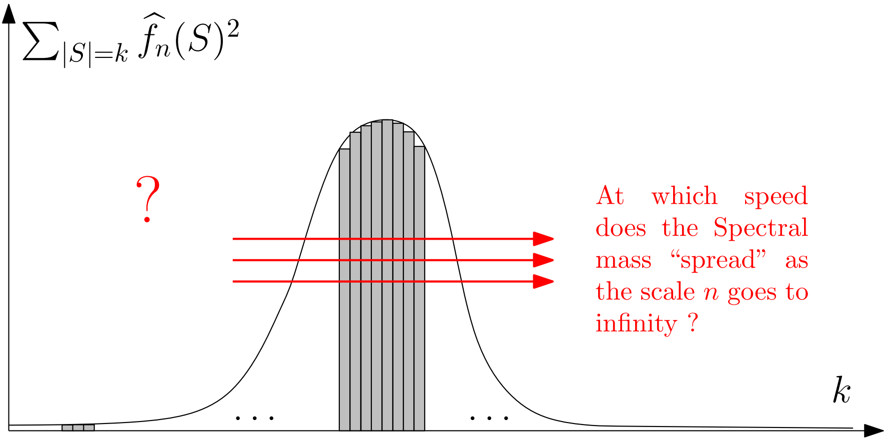

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Informal definition (First Passage Percolation)

Let $0 < a < b$*. Define the* **random metric** *on the graph* $\mathbb{Z}^d$ *as follows: for each edge* $e \in \mathbb{E}^d$*, fix its length* τ_e *to be* a *with probability 1/2 and* b *with probability* $1/2$*.*

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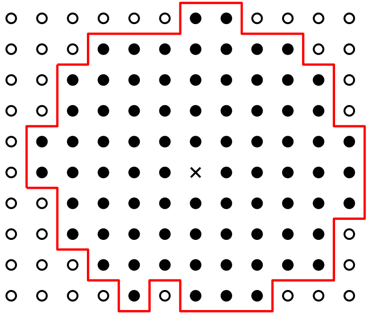

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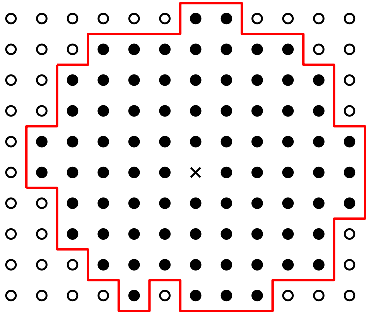

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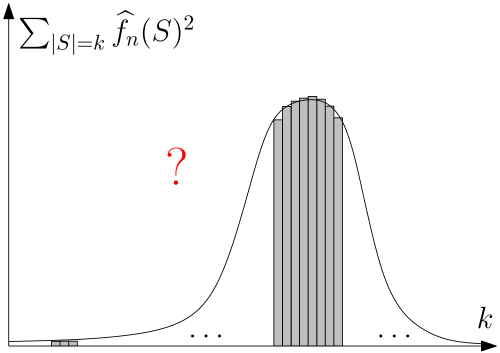
What are the fluctuations around this asymptotic shape ?

Three different approaches to localize the Spectrum



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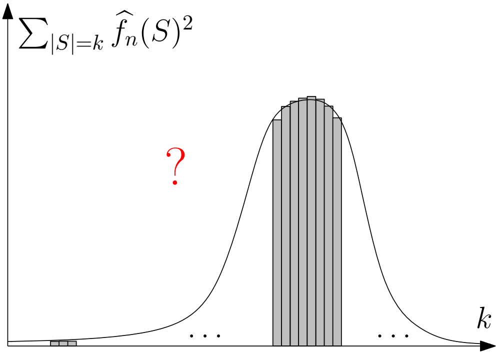

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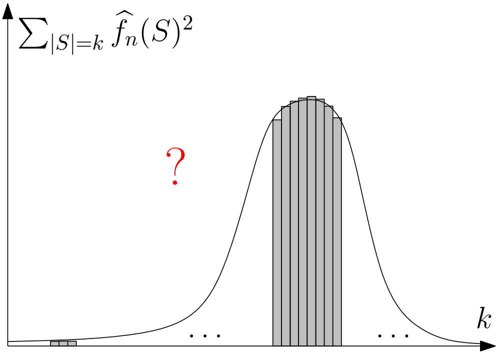

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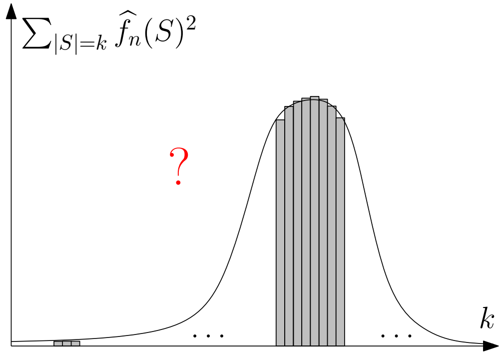

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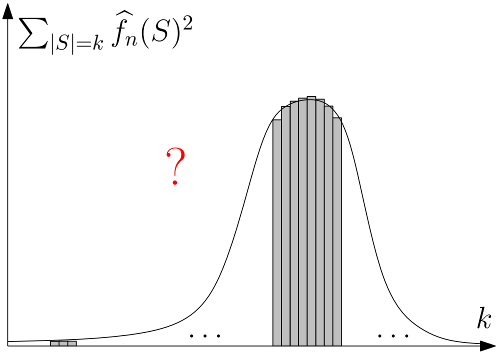

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$$\delta = \delta_{\mathcal{A}} := \sup_{i \in [n]} \mathbb{P}\big[i \in J \big] .$$

Examples

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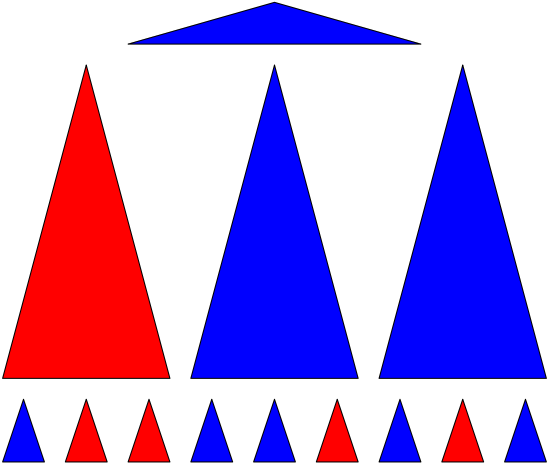

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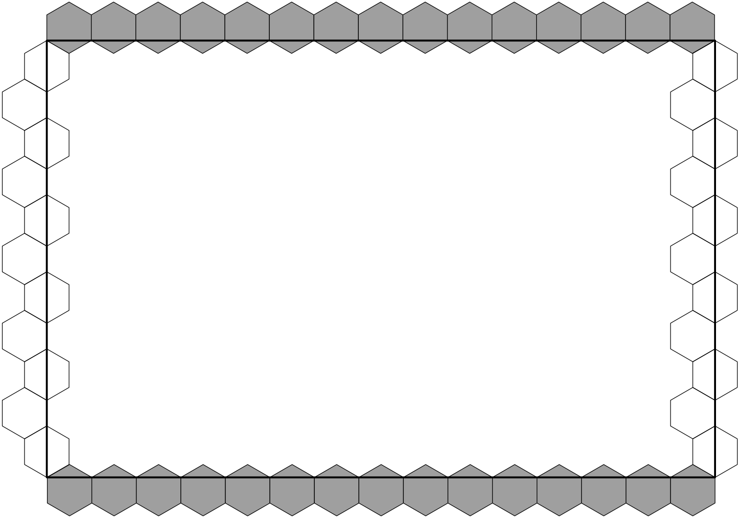

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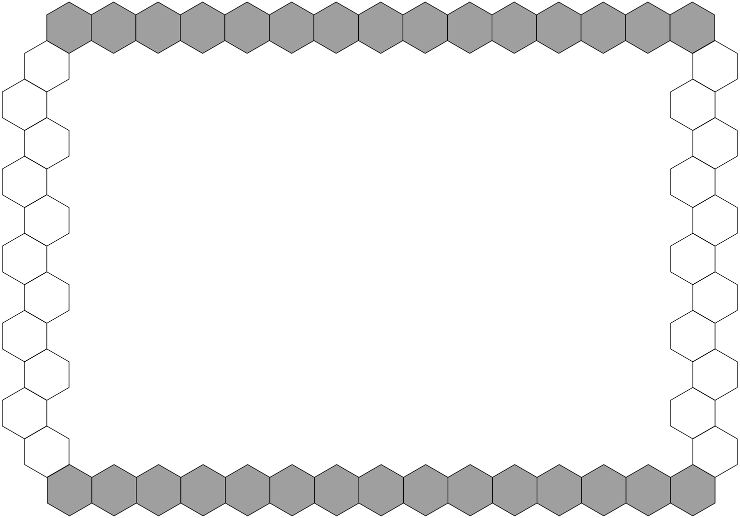

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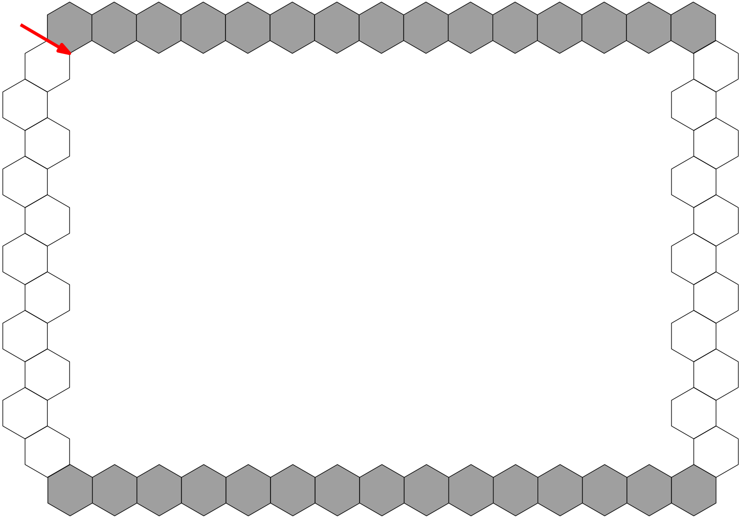

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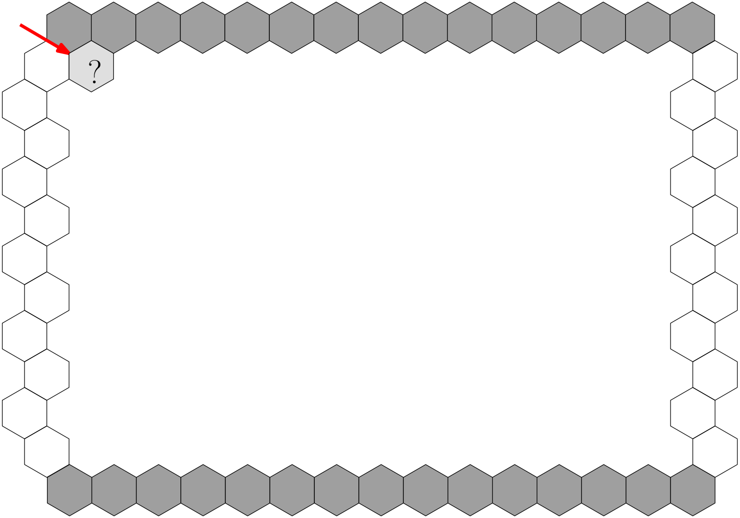

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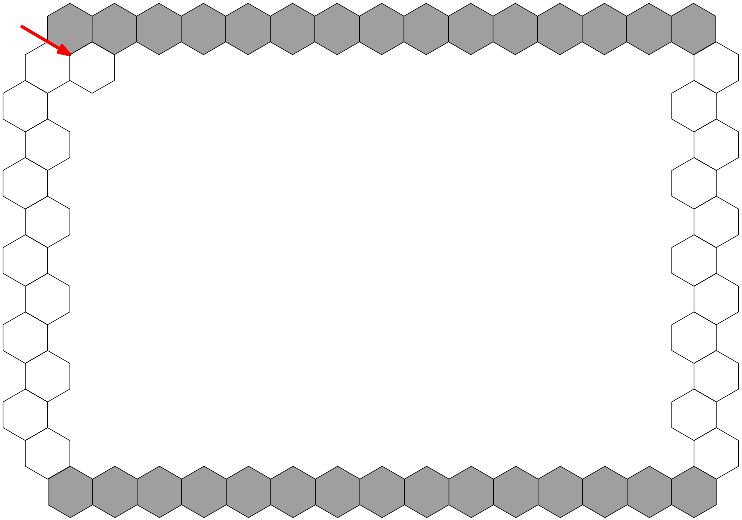

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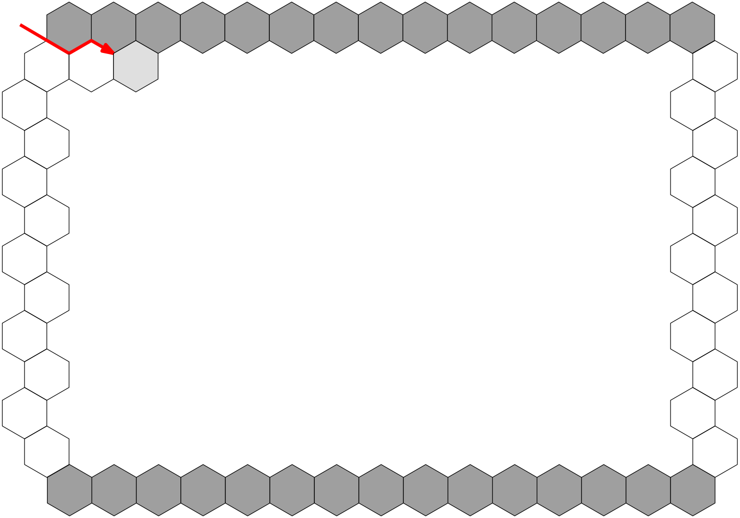

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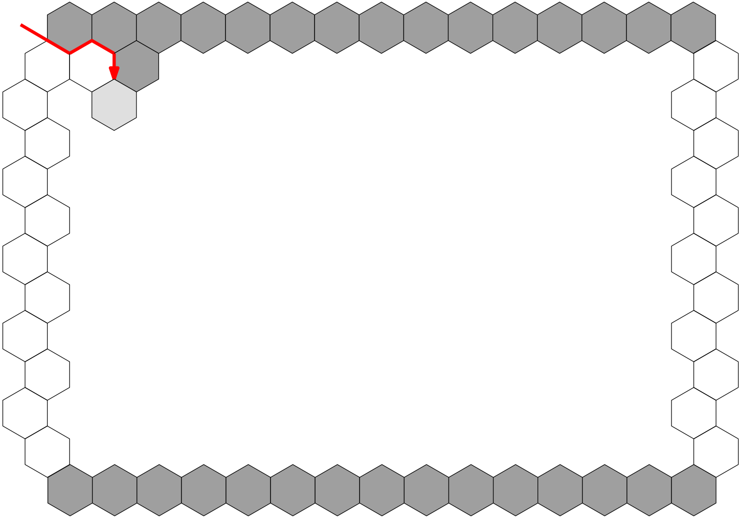

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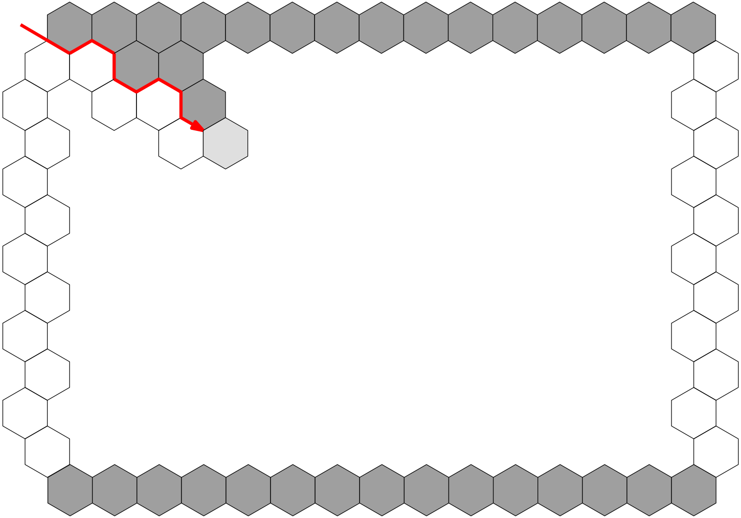

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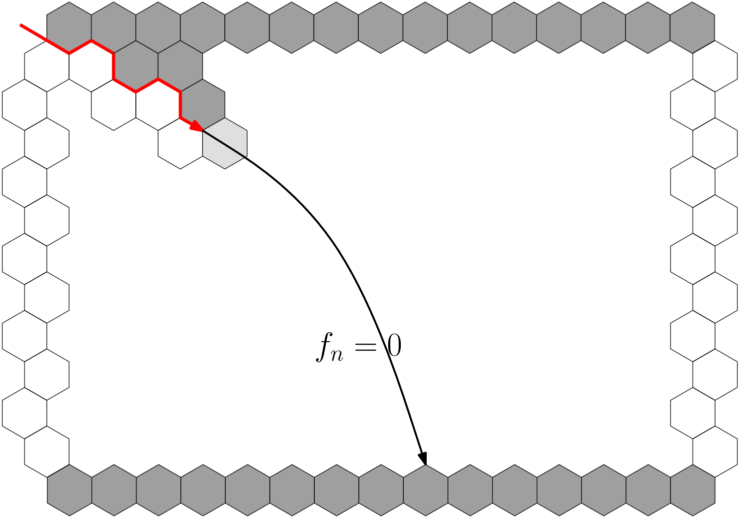

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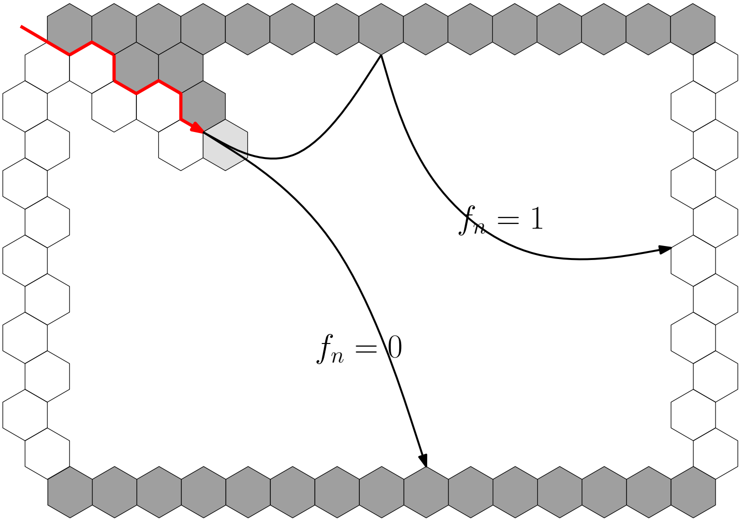

Revealment for percolation

Proposition (Schramm, Steif, 2005)

On the triangular lattice, a slight modification of the above randomized algorithm gives a small revealment for the left-right Boolean functions f_n of order

$$\delta_n \approx n^{-1/4}$$

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Then, for any $k = 1, 2, \ldots$ the Fourier coefficients of f satisfy

$$\sum_{|S|=k} \widehat{f}(S)^2 \leq k\,\delta\,\|f\|^2$$