High frequency criteria for Boolean functions
(with an application to percolation)

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Workshop on Discrete Harmonic Analysis
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Plan

- Spectrum of Boolean functions
- Motivations (percolation ...)
- Randomized algorithms
- A theorem by Schramm and Steif
- Noise sensitivity of percolation
- Where exactly does the spectrum of percolation localize? (Work with G. Pete and O. Schramm)
Spectrum of a Boolean function

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f = \sum_{S \subset \{1, \ldots, n\}} \hat{f}(S) \chi(S),
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with \( \chi(S) := \prod_{i \in S} x_i. \)
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These functions form an orthonormal basis for \( L^2(\{-1, 1\}^n) \) endowed with the uniform measure \( \mu = (1/2\delta_1 + 1/2\delta_{-1})^\otimes n \).

The Fourier coefficients \( \hat{f}(S) \) satisfy

\[
\hat{f}(S) := \langle f, \chi_S \rangle = \mathbb{E}[f \chi_S]
\]

Parseval tells us

\[
\|f\|_2^2 = \sum_S \hat{f}(S)^2
\]
Energy spectrum of a Boolean function

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The total Spectral mass here is

\[
\sum_{|S| \neq 0} \hat{f}(S)^2 = \text{Var}[f]
\]
Energy spectrum of Majority

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Its energy spectrum can be computed explicitly. It has the following shape:

$$\sum |S| = k \overbrace{\text{MAJ}_n(S)}^{2}$$

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Noise stability

Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be a Boolean function. For any $\epsilon > 0$, if $\omega_n = (x_1, \ldots, x_n)$ is sampled uniformly in $\Omega_n$, let $\omega_n^\epsilon$ be the noised configuration obtained out of $\omega_n$ by resampling each bit with probability $\epsilon$.

The noise stability of $f$ is given by

$$S^\epsilon(f) := \mathbb{P}[f(\omega) = f(\omega^\epsilon)]$$

It is easy to check that

$$S^\epsilon(f) = \sum_S \hat{f}(S)^2 (1 - \epsilon)^{|S|}$$

$$= \mathbb{E}[f]^2 + \sum_{k \geq 1} E_f(k)(1 - \epsilon)^k$$
What can be said on the spectrum in general?

Here is a deep result by Bourgain:

**Theorem (Bourgain, 2001)**
If $f$ is a balanced Boolean function with low influences, then
$$\sum |S| \geq k^\hat{f}(S)^2 \geq 1/\sqrt{k},$$
if $k$ is not "too large".

**Theorem ("Majority is Stablest" Mossel, O’Donnell and Oleszkiewicz, 2005)**
If $f$ is a balanced Boolean function with low influences, then
$$S_\epsilon(f) \lesssim 2\pi \arcsin(1-\epsilon) C.$$
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Recognizing high frequency behavior

In what follows, we will be interested in Boolean (or real-valued) functions which are highly sensitive to small perturbations (or small noise). Such functions are called noise sensitive. They are such that most of their Fourier mass is localized on high frequencies. In particular, their energy spectrum should look as follows

$$\sum_{|S|=k} \hat{f}(S)^2 \gg 1$$
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Let $A$ be a subset of the square in the plane.

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C. Garban (ENS Lyon)  High frequency criteria for Boolean functions  10 / 23
Motivations for high spectra

• Influences of Boolean functions, Sharp thresholds...

• Dynamical percolation

• Fluctuations for natural random metrics on $\mathbb{Z}^d$ (First passage percolation).
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ω₀:
Dynamical percolation

\[ \omega_0 \rightarrow \omega_t: \]
Key step: percolation is noise sensitive
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$$f_n(\omega) := \begin{cases} 1 & \text{if there is a left-right crossing} \end{cases}$$
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$$f_n(\omega) := \begin{cases} 1 & \text{if there is a left-right crossing} \\ 0 & \text{else} \end{cases}$$
The energy spectrum of macroscopic events

Question: how does the energy spectrum of the above Boolean functions $f_n, n \geq 1$ look?
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\[ \sum_{|S|=k} |\hat{f}_n(S)|^2 \]

At which speed does the Spectral mass “spread” as the scale $n$ goes to infinity?
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At which speed does the Spectral mass “spread” as the scale $n$ goes to infinity?
Informal definition (First Passage Percolation)

Let $0 < a < b$. Define the random metric on the graph $\mathbb{Z}^d$ as follows: for each edge $e \in \mathbb{E}^d$, fix its length $\tau_e$ to be $a$ with probability $1/2$ and $b$ with probability $1/2$. 

It is well-known that the random ball $B_\omega(R) := \{x \in \mathbb{Z}^d, \text{dist}_\omega(0, x) \leq R\}$ has an asymptotic shape. Question: What are the fluctuations around this asymptotic shape?
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**Question**

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Three different approaches to localize the Spectrum

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- **Hypercontractivity**, 1998
  Benjamini, Kalai, Schramm

- **Randomized Algorithms**, 2005
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- **Geometric study of the ’frequencies’**, 2008
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  ↔ Gil’s talk

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- **Geometric study of the ‘frequencies’, 2008**
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If $\mathcal{A}$ is such a randomized algorithm, let $J = J_\mathcal{A} \subset [n]$ be the random set of bits that are examined along the algorithm.
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We are looking for algorithms which examine the least possible number of bits.
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We are looking for algorithms which examine the least possible number of bits. This can be quantified by the revealment:

$$\delta = \delta_\mathcal{A} := \sup_{i \in [n]} \mathbb{P}[i \in J].$$
Examples

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- Recursive Majority:
Percolation is very suitable to randomized algorithms
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\[ f_n = 0 \]
Percolation is very suitable to randomized algorithms
Proposition (Schramm, Steif, 2005)

On the triangular lattice, a slight modification of the above randomized algorithm gives a small revealment for the left-right Boolean functions $f_n$ of order

$$\delta_n \approx n^{-1/4}$$
How is it related with the Fourier expansion of $f$?
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**Theorem (Schramm, Steif, 2005)**

Let $f : \{-1, 1\}^n \to \mathbb{R}$ be a *real-valued* function.
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**Theorem (Schramm, Steif, 2005)**

Let $f : \{-1, 1\}^n \to \mathbb{R}$ be a real-valued function. Let $A$ be a randomized algorithm computing $f$ whose revealment is $\delta = \delta_A$.
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**Theorem (Schramm, Steif, 2005)**

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Then, for any $k = 1, 2, \ldots$ the Fourier coefficients of $f$ satisfy

$$\sum_{|S|=k} \hat{f}(S)^2 \leq k \delta \|f\|^2$$