Critical percolation under conservative dynamics

Christophe Garban
ENS Lyon and CNRS

Joint work with Erik Broman (Uppsala University)
and Jeffrey E. Steif (Chalmers University, Göteborg)

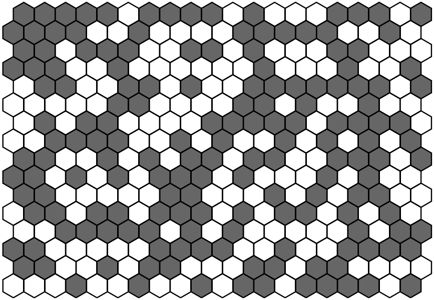
PASI conference, Buenos Aires, January 2012

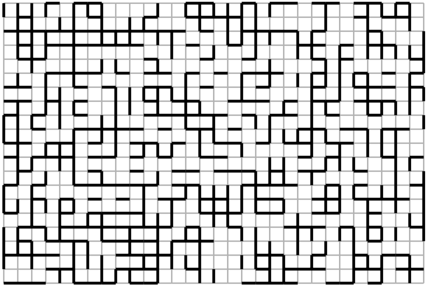
Overview

- Dynamical percolation
- Conservative dynamics on percolation

“standard” dynamical percolation

Start with an initial configuration $\omega_{t=0}$ at $p = p_c(\mathbb{T}) = p_c(\mathbb{Z}^2) = 1/2$.





And let evolve each edge (or site) **independently** at rate 1. This gives a **Markov** process $(\omega_t)_{t\geq 0}$ on critical percolation configurations.

Main results known

Theorem (Schramm, Steif, 2005)

On the **triangular lattice** $\mathbb{T}$, *there exist exceptional times* t *for which* $0 \overset{\omega_t}{\longleftrightarrow} \infty$. *Furthermore, a.s.*

$$\dim_{\mathcal{H}}(\mathrm{Exc}) \in \left[\frac{1}{6}, \frac{31}{36}\right]$$

Main results (continued)

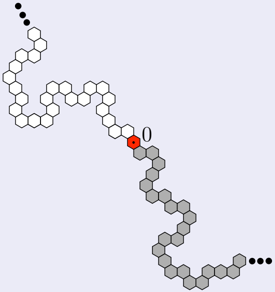
Theorem (G. , Pete, Schramm, 2008)

- *On the* **square lattice** $\mathbb{Z}^2$*, there are exceptional times as well (with* $\dim_{\mathcal{H}}(\mathrm{Exc}) \geq \epsilon > 0$ *a.s.)*

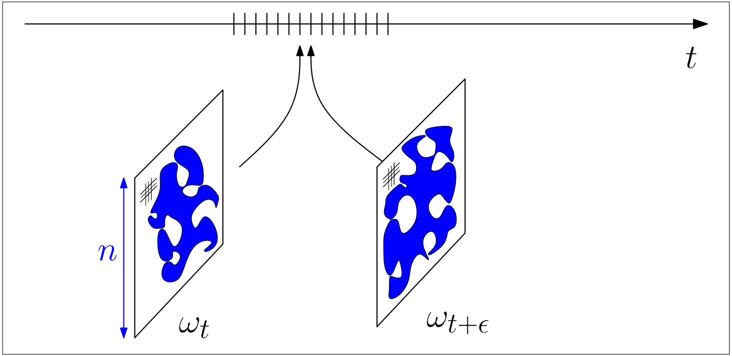
Main results (continued)

Theorem (G. , Pete, Schramm, 2008)

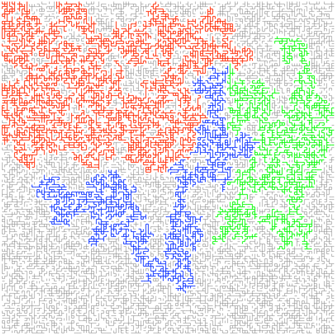
- *On the* **square lattice** $\mathbb{Z}^2$*, there are exceptional times as well (with* $\dim_{\mathcal{H}}(\mathrm{Exc}) \geq \epsilon > 0$ *a.s.)*
- *On the* **triangular lattice** $\mathbb{T}$
  - *a.s.* $\dim_{\mathcal{H}}(\mathrm{Exc}) = \frac{31}{36}$
  - *There exist exceptional times* $\mathrm{Exc}^{(2)}$ *such that*

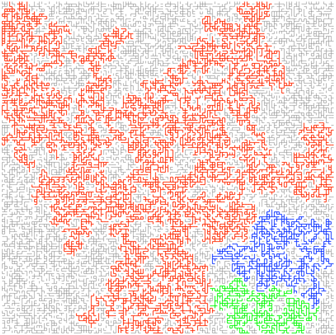


Strategy: **noise sensitivity** of percolation

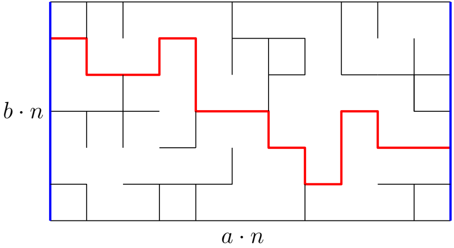


Strategy: **noise sensitivity** of percolation





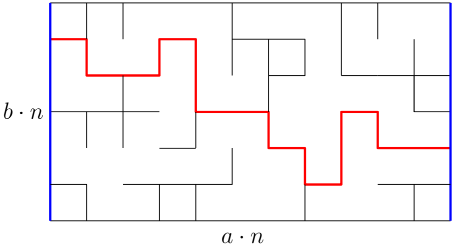
Large scale properties are encoded by Boolean functions of the 'inputs'



Let $f_n : \{-1,1\}^{O(1)n^2} \to \{0,1\}$ be the Boolean function defined as follows

$$f_n(\omega) := \begin{cases} 1 & \text{if left-right crossing} \\ 0 & \text{else} \end{cases}$$

Large scale properties are encoded by Boolean functions of the ‘inputs’



Let $f_n : \{-1,1\}^{O(1)n^2} \to \{0,1\}$ be the Boolean function defined as follows

$$f_n(\omega) := \begin{cases} 1 & \text{if left-right crossing} \\ 0 & \text{else} \end{cases}$$

Theorem (Benjamini, Kalai, Schramm, 1998)

For any fixed $t > 0$:

$$\mathrm{Cov}\Big[f_n(\omega_0)\,,\, f_n(\omega_t)\Big] \underset{n\to\infty}{\longrightarrow} 0$$

We say in such a case that $(f_n)_{n\geq 1}$ is **noise sensitive**.

Main tool to study noise sensitivity: Fourier analysis

Decompose $f : \{-1, 1\}^m \to \{0, 1\}$ into "Fourier" series

$$f(\omega) = \sum_S \hat{f}(S)\, \chi_S(\omega)\,,$$

where $\chi_S(x_1, \ldots, x_m) := \prod_{i \in S} x_i$.

Main tool to study noise sensitivity: Fourier analysis

Decompose $f : \{-1,1\}^m \to \{0,1\}$ into "Fourier" series

$$f(\omega) = \sum_S \hat{f}(S)\, \chi_S(\omega)\,,$$

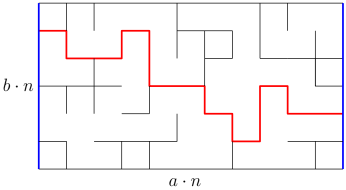
where $\chi_S(x_1, \ldots, x_m) := \prod_{i\in S} x_i$.

$$\begin{aligned}
\mathbb{E}\big[f(\omega_0)\, f(\omega_t)\big] &= \mathbb{E}\Big[\Big(\sum_{S_1} \widehat{f}(S_1)\chi_{S_1}(\omega_0)\Big)\Big(\sum_{S_2} \widehat{f}(S_2)\chi_{S_2}(\omega_t)\Big)\Big] \\
&= \sum_S \widehat{f}(S)^2\, \mathbb{E}\big[\chi_S(\omega_0)\chi_S(\omega_t)\big] \\
&= \sum_S \widehat{f}(S)^2\, e^{-t\,|S|}
\end{aligned}$$

Thus the covariance can be written:

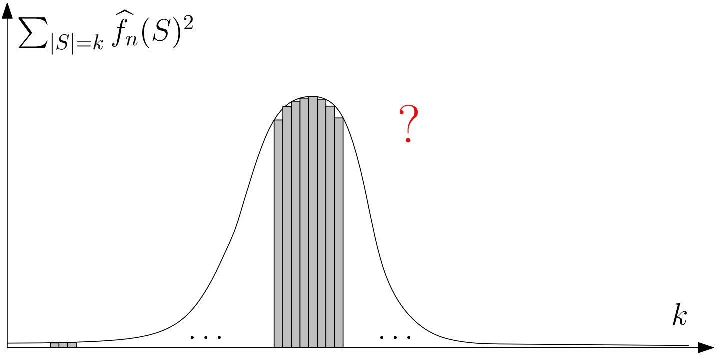
$$\mathbb{E}\big[f(\omega_0)\, f(\omega_t)\big] - \mathbb{E}\big[f(\omega)\big]^2 = \sum_{S\neq\emptyset} \widehat{f}(S)^2\, e^{-t\,|S|}$$

Fourier spectrum of critical percolation

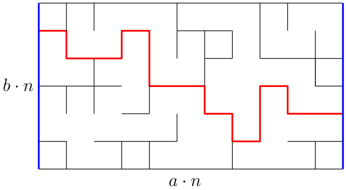


Let $f_n, n \geq 1$ be Boolean functions defined above.

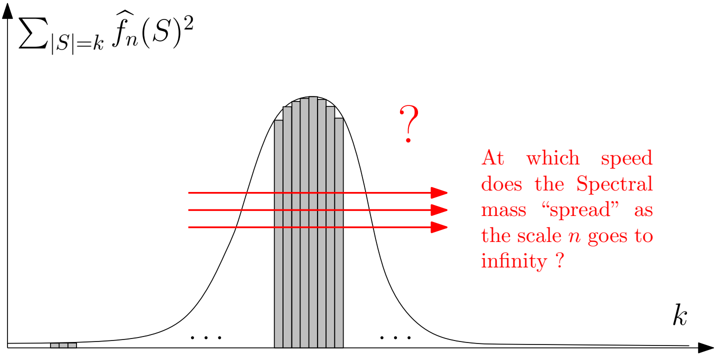
One is interested in the shape of their Fourier spectrum.



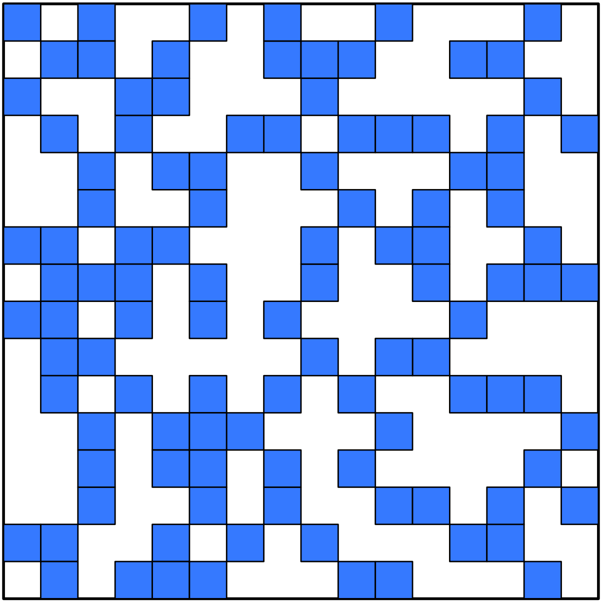
Fourier spectrum of critical percolation



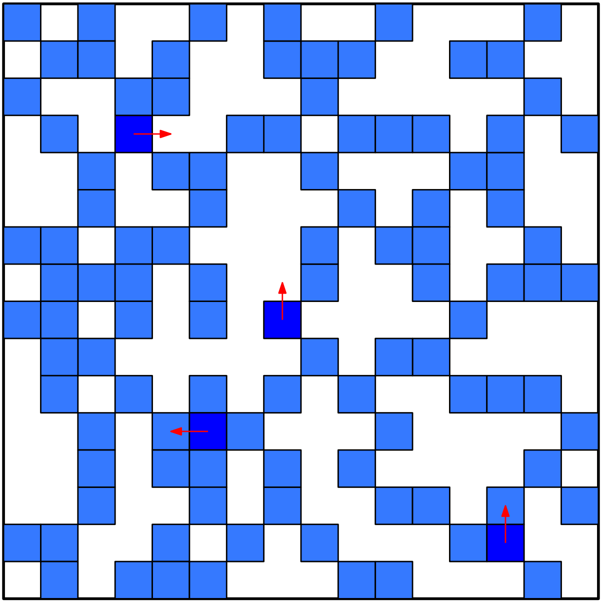
Let $f_n, n \geq 1$ be Boolean functions defined above.
One is interested in the shape of their Fourier spectrum.



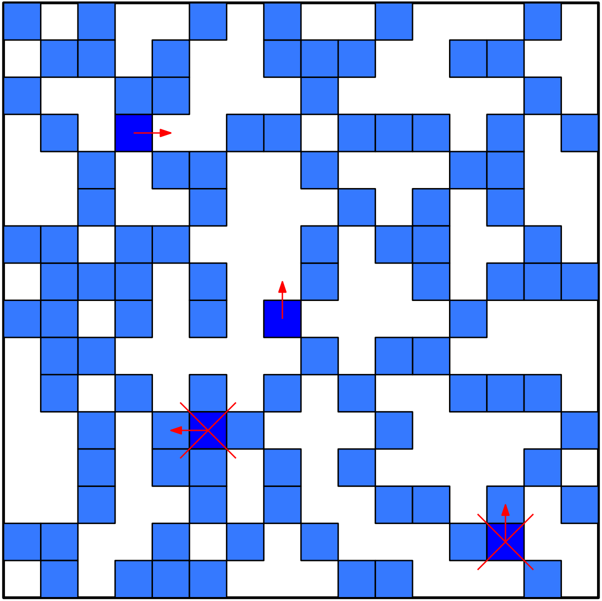
Percolation undergoing conservative dynamics



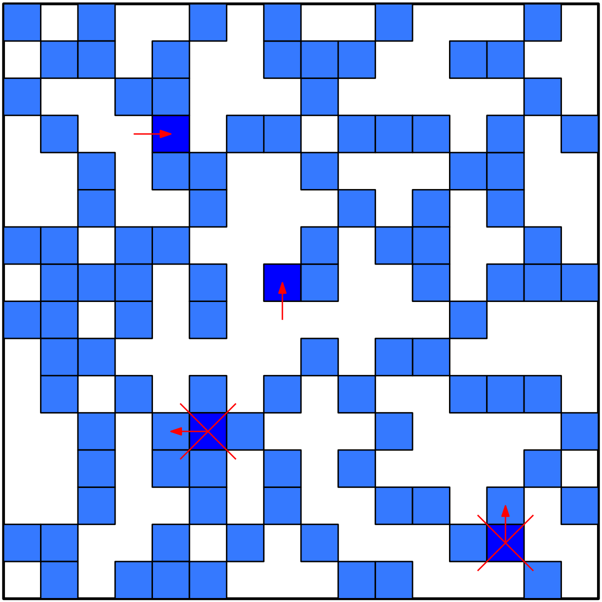
Percolation undergoing conservative dynamics



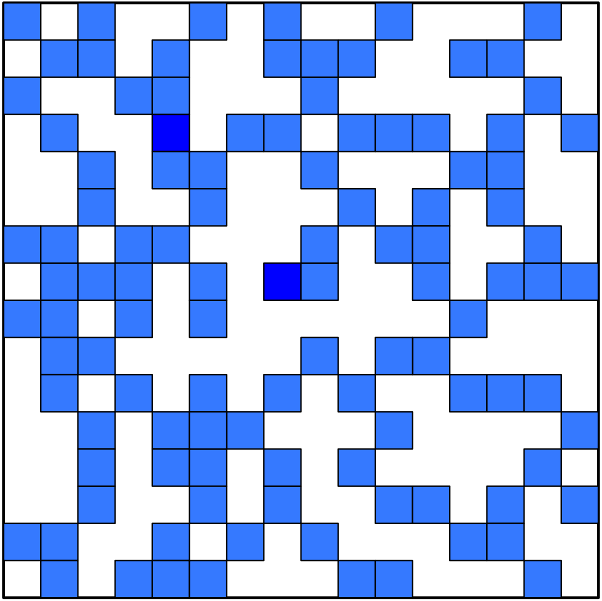
Percolation undergoing conservative dynamics



Percolation undergoing conservative dynamics



Percolation undergoing conservative dynamics



The system evolves according to the symmetric exclusion process

Let $(\omega_t^P)_{t\geq 0}$ be a sample of a symmetric exclusion process with symmetric kernel $P(x,y)$, $(x,y)\in\mathbb{Z}^2\times\mathbb{Z}^2$ or $(x,y)\in\mathbb{T}\times\mathbb{T}$

We distinguish 2 cases:

(a) **Nearest neighbor** dynamics:

$$P(x,y)=\frac{1}{\mathrm{degree}}\,1_{x\sim y}$$

(b) **Medium-range** dynamics:

$$P(x,y)\asymp\frac{1}{\|x-y\|^{2+\alpha}}\quad\text{for some exponent } \alpha>0$$

What we can and cannot :-(prove about these dynamics

1. Let's start with the bad news: we don't know if there are exceptional times for ω_t^P.

What we can and cannot :-(prove about these dynamics

1. Let's start with the bad news: we don't know if there are exceptional times for ω_t^P.
2. If the dynamics is **medium-range** with exponent $\alpha > 0$ (recall $P(x,y) \asymp \|x-y\|^{-2-\alpha}$), then we get quantitative bounds on the noise sensitivity of the crossing events f_n under ω_t^P. More precisely:

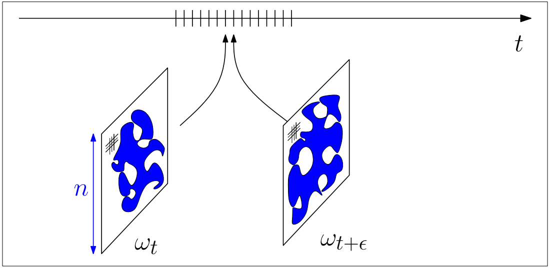
Theorem (Broman, G., Steif, 2011)

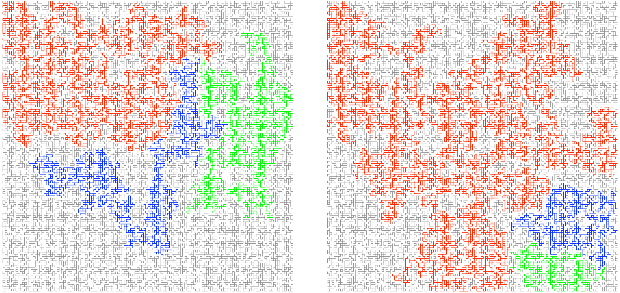
If P is any transition kernel with exponent $\alpha > 0$, then on $\mathbb{Z}^{2,\mathrm{site}}$, $\mathbb{Z}^{2,\mathrm{bond}}$ or $\mathbb{T}$, at the critical point, one has

$$\mathrm{Cov}(f_n(\omega_0^P),\, f_n(\omega_t^P)) \underset{n\to\infty}{\longrightarrow} 0$$

Furthermore, one can choose $t = t_n \geq n^{-\beta(\alpha)}$.

In other words, for medium-range exclusion dynamics ($\alpha > 0$), we also obtain this "picture"





Which approach for this problem ?

Two strategies:

1. Either the noise sensitivity results for the iid case transfer to these conservative dynamics ?
2. Or an "appropriate" spectral approach ?

Which approach for this problem ?

Two strategies:

1. Either the noise sensitivity results for the iid case transfer to these conservative dynamics ?
2. Or an "appropriate" spectral approach ?

strategy 1. is "hopeless" since

Fact

There exist Boolean functions $(f_n)_n$ which are highly noise sensitive to i.i.d. noise but which are **stable** *to symmetric exclusion P- dynamics.*

What about the spectral approach ?

Natural attempt: decompose our Boolean function f on a basis of eigenvectors which diagonalize the **generator** $\mathcal{L} = \mathcal{L}_P$ of our P-exclusion process.

What about the spectral approach ?

Natural attempt: decompose our Boolean function f on a basis of eigenvectors which diagonalize the **generator** $\mathcal{L} = \mathcal{L}_P$ of our P-exclusion process.

But there are difficulties:

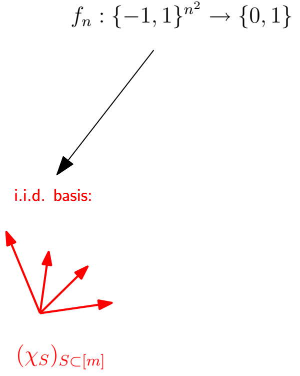
1. In the finite volume case, such a basis obviously exists, but it highly depends on P and it is not very “explicit”.
2. In the infinite volume case, $\mathcal{L}_P$ is of course non-compact and it seems that it does not have **pure-point** spectrum.

What about the spectral approach ?

Natural attempt: decompose our Boolean function f on a basis of eigenvectors which diagonalize the **generator** $\mathcal{L} = \mathcal{L}_P$ of our P-exclusion process.

But there are difficulties:

1. In the finite volume case, such a basis obviously exists, but it highly depends on P and it is not very "explicit".
2. In the infinite volume case, $\mathcal{L}_P$ is of course non-compact and it seems that it does not have **pure-point** spectrum.

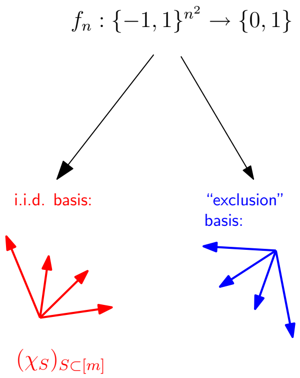


What about the spectral approach ?

Natural attempt: decompose our Boolean function f on a basis of eigenvectors which diagonalize the **generator** $\mathcal{L} = \mathcal{L}_P$ of our P-exclusion process.

But there are difficulties:

1. In the finite volume case, such a basis obviously exists, but it highly depends on P and it is not very "explicit".
2. In the infinite volume case, $\mathcal{L}_P$ is of course non-compact and it seems that it does not have **pure-point** spectrum.



The key identity

We decompose f on the classical "i.i.d." basis even though it does not diagonalize our exclusion process:

$$\mathbb{E}\big[f(\omega_0^P)\, f(\omega_t^P)\big] \;=\; \mathbb{E}\Big[\Big(\sum_S \widehat{f}(S)\chi_S(\omega_0^P)\Big)\Big(\sum_{S'} \widehat{f}(S')\chi_{S'}(\omega_t^P)\Big)\Big]$$

The key identity

We decompose f on the classical "i.i.d." basis even though it does not diagonalize our exclusion process:

$$\begin{aligned}\mathbb{E}\big[f(\omega_0^P)\,f(\omega_t^P)\big] &= \mathbb{E}\Big[\Big(\sum_S \widehat{f}(S)\chi_S(\omega_0^P)\Big)\Big(\sum_{S'} \widehat{f}(S')\chi_{S'}(\omega_t^P)\Big)\Big] \\ &= \sum_{S,S'} \widehat{f}(S)\widehat{f}(S')\,\mathbb{E}\big[\chi_S(\omega_0^P)\chi_{S'}(\omega_t^P)\big]\end{aligned}$$

The key identity

We decompose f on the classical "i.i.d." basis even though it does not diagonalize our exclusion process:

$$
\begin{aligned}
\mathbb{E}\big[f(\omega_0^P)\, f(\omega_t^P)\big] &= \mathbb{E}\Big[\Big(\sum_S \widehat{f}(S)\chi_S(\omega_0^P)\Big)\Big(\sum_{S'} \widehat{f}(S')\chi_{S'}(\omega_t^P)\Big)\Big] \\
&= \sum_{S,S'} \widehat{f}(S)\widehat{f}(S')\,\mathbb{E}\big[\chi_S(\omega_0^P)\chi_{S'}(\omega_t^P)\big] \\
&= \sum_{|S|=|S'|} \widehat{f}(S)\widehat{f}(S')\,\mathbb{E}\big[\chi_S(\omega_0^P)\chi_{S'}(\omega_t^P)\big]
\end{aligned}
$$

The key identity

We decompose f on the classical "i.i.d." basis even though it does not diagonalize our exclusion process:

$$
\begin{aligned}
\mathbb{E}\big[f(\omega_0^P)\, f(\omega_t^P)\big] &= \mathbb{E}\Big[\Big(\sum_S \widehat{f}(S)\chi_S(\omega_0^P)\Big)\Big(\sum_{S'} \widehat{f}(S')\chi_{S'}(\omega_t^P)\Big)\Big] \\
&= \sum_{S,S'} \widehat{f}(S)\widehat{f}(S')\,\mathbb{E}\big[\chi_S(\omega_0^P)\chi_{S'}(\omega_t^P)\big] \\
&= \sum_{|S|=|S'|} \widehat{f}(S)\widehat{f}(S')\,\mathbb{E}\big[\chi_S(\omega_0^P)\chi_{S'}(\omega_t^P)\big] \\
&= \sum_{|S|=|S'|} \widehat{f}(S)\widehat{f}(S')\,\mathbf{P_t(S,S')}
\end{aligned}
$$

where $\mathbf{P_t(S,S')}$ is the probability that the set S travels in time t towards the set S' under the exclusion process.

$$\mathbb{E}\big[f_n(\omega_0^P)\, f_n(\omega_t^P)\big] \quad = \quad \sum_{|S|=|S'|} \widehat{f}_n(S)\widehat{f}_n(S')\,\mathbf{P_t(S,S')}$$

$$\text{"}=\text{"} \quad \Big\langle \hat{f}_n\,,\, P_t \star \hat{f}_n \Big\rangle$$

We would like to prove that for large scale n, the vectors $\{\hat{f}_n(S)\}_S$ and $\{P_t \star \hat{f}_n(S)\}_S$ are almost orthogonal.

$$\mathbb{E}\big[f_n(\omega_0^P)\, f_n(\omega_t^P)\big] \quad = \quad \sum_{|S|=|S'|} \widehat{f}_n(S)\widehat{f}_n(S')\, \mathbf{P_t(S, S')}$$

$$\text{"} = \text{"} \quad \Big\langle \hat{f}_n \,,\, P_t \star \hat{f}_n \Big\rangle$$

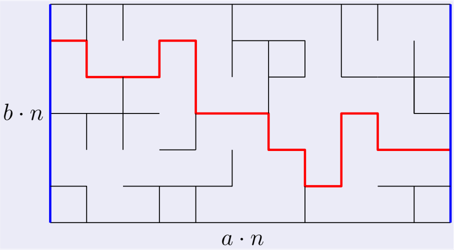
We would like to prove that for large scale n, the vectors $\{\hat{f}_n(S)\}_S$ and $\{P_t \star \hat{f}_n(S)\}_S$ are almost orthogonal.

Unfortunately, we know much more on the vector $\{\hat{f}_n(S)^2\}$ than on $\{\hat{f}_n(S)\}$:

The spectral measure ν_{f_n}

Definition

Recall that f_n is the Boolean function:



Define the spectral measure of f_n as follows:

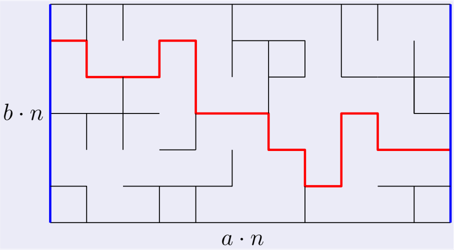
$$\nu_{f_n}(\mathscr{S} = S) := \hat{f}_n(S)^2$$

In particular $\mathscr{S}$ can be considered as a **random** subset of $[0, an] \times [0, bn]$.

The spectral measure ν_{f_n}

Definition

Recall that f_n is the Boolean function:



Define the spectral measure of f_n as follows:

$$\nu_{f_n}(\mathscr{S} = S) := \hat{f}_n(S)^2$$

In particular $\mathscr{S}$ can be considered as a **random** subset of $[0, an] \times [0, bn]$.

We can prove the following:

Proposition (asymptotic singularity)

For any medium-range exponent $\alpha > 0$ and any fixed $t > 0$: as $n \to \infty$, the measures ν_{f_n} and $P_t \star \nu_{f_n}$ are asymptotically mutually **singular**

Why does this imply noise sensitivity ?

Fact

- *If ϕ^2 and ψ^2 are the densities of two probability measures on $\mathbb{R}$, then*

$$\int \phi\, \psi \leq 2 \sqrt{\int \phi^2 \wedge \psi^2}$$

- *In particular, if the two corresponding probability measures are almost singular with respect to each other then $\int \phi\psi$ is small.*

Why does this imply noise sensitivity ?

Fact

- *If ϕ^2 and ψ^2 are the densities of two probability measures on $\mathbb{R}$, then*

$$\int \phi\,\psi \leq 2\sqrt{\int \phi^2 \wedge \psi^2}$$

- *In particular, if the two corresponding probability measures are almost singular with respect to each other then $\int \phi\psi$ is small.*

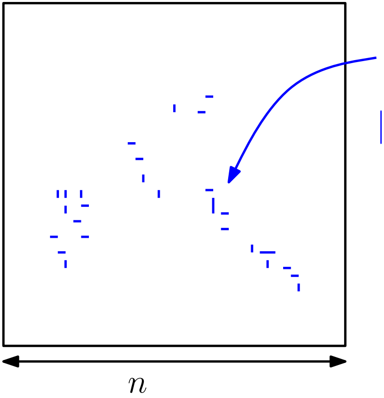
$$\text{Take } \begin{cases} \phi^2 \equiv \hat{f}_n(S)^2 \\ \psi^2 \equiv P_t\Big[(\hat{f}_n)^2\Big](S) \end{cases} \text{ this gives that}$$

$$\Big\langle \sqrt{\hat{f}_n(S)^2}, \sqrt{P_t\Big[(\hat{f}_n)^2\Big](S)} \Big\rangle \quad \text{is small.}$$

By Cauchy-Schwartz, one concludes that $\Big\langle \hat{f}_n(S), P_t \star \hat{f}_n(S) \Big\rangle$ is small.

Singularity in the medium-range case ($\alpha > 0$)

(Recall $P(x,y) \asymp \frac{1}{\|x-y\|^{2+\alpha}}$)



$\mathcal{S}_{f_n} \sim \nu_{f_n}$

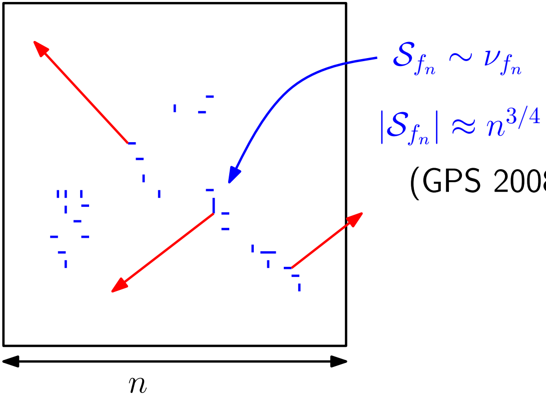
$|\mathcal{S}_{f_n}| \approx n^{3/4}$

(GPS 2008)

Singularity in the medium-range case ($\alpha > 0$)

(Recall $P(x,y) \asymp \frac{1}{\|x-y\|^{2+\alpha}}$)

If the exponent α is small:

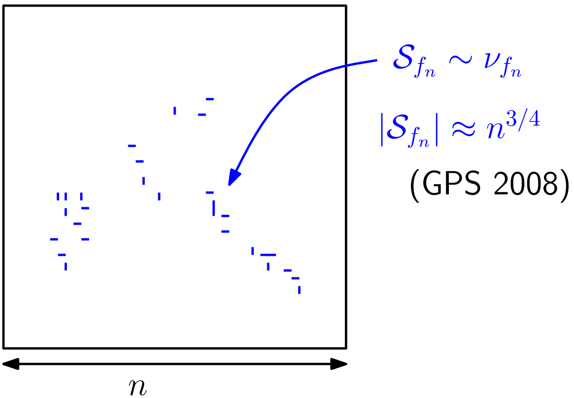


(GPS 2008)

Singularity in the medium-range case ($\alpha > 0$)

(Recall $P(x, y) \asymp \frac{1}{\|x-y\|^{2+\alpha}}$)

If the exponent α is large...



Question

What about the nearest-neighbor case ?