Short notes on Seiberg-Witten and applications

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Abstract

We give here a short introduction to the Seiberg-Witten invariants, and two applications of the theory.

Introduction

Mathematics and physics have been being historically deeply connected. One of the last exchange was the use in the 80's by Donaldson of the Yang-Mills theory to give some totally new topological invariants. Then Witten reinterpreted this theory in 1994 in terms of topological quantum field theory, and give physical arguments to conjecture that it was equivalent to an other sytem of EDPs, studied by him and Seiberg. Immediatly, this equations, from far easier to treat with than Donaldson's, recovered the results of the former theory, but give new one.

In this short teaching, we will give an introduction to this equations, the structure of the set of it solutions, and they use to construct topological invariants, the so-called Seiberg-Witten invariants. We will show that this invariants are non trivial on the Kähler manifolds. This will give us examples of homeomorphic but non diffeomorphic manifolds. Moreover, we will establish the Thom's conjecture : in $\mathbb{C}P^2$, the smooth holomorphic curves are genus minimizing amongst real surfaces of the same degree.

1 Seiberg-Witten equations

1.1 Announcement

The primitive setting in this story is (X^4, g) : a compact, orientable, smooth four manifold equiped with a metric g. We will equip it with a spin^c structure σ, which gives us a vector bundle W on X, the spinor bundle. The unknowns of the Seiberg-Witten equations are a section ψ of W and a connection ∇ on W. We will write two equations $SW(\psi, \nabla)$, and the moduli space $\mathcal{M}$ of its solutions has valuable proprieties : it is compact, generically it is smooth, its dimension can be calculated by the Index Theorem, and if the subspace of self-dual harmonic form on X is at least 2, there is a cobordism between two moduli spaces corresponding to two differents metrics. Suppose now that this dimension is nul. Then the parity of the cardinal of $\mathcal{M}$ is the simpliest SW-invariant we can construct. Because this parity does not depend on the metric thanks to cobordism's effect, it is a topological (differential) invariant, the simpliest one in the theory. Even in this simple case, it will bring us two deep applications.

1.2 Around the spinor bundle

Recall (see Frederik Witt's talk) that $\mathrm{SO}(4) \times S^1$ has the group $\mathrm{Spin}(4)^c$ as universal (double) covering, which we call here ρ. There is a concrete way to see this group :

$$\mathrm{Spin}(4)^c = \{ \left(\begin{array}{cc} A^+ & 0 \\ 0 & A^- \end{array} \right),\ A^\pm \in U(2),\ \det A^+ = \det A^- \}.$$

We can see the action on $\mathbb{R}4$ through the identfication of $\mathbb{R}^4$ with the matrices of the form

$$\left(\begin{array}{cc} z & -\bar{w} \\ w & \bar{z} \end{array} \right), (z,w) \in \mathbb{C}^2 = \mathbb{R}^4,$$

by

$$\rho(A_+, A_-)(X) = A_- X A_+,$$

and the associated element of the circle is $\det A_+$.

Thanks to the riemanian structure, we can choose for the tangent bundle TX the transition functions $g_{\alpha\beta}$ in $\mathrm{SO}(4)$. A spin^c structure σ on (X, g) is a way to find a find some $\tilde{g}_{\alpha\beta}$ such that $\rho(\tilde{g}_{\alpha\beta}) = g_{\alpha\beta}$, and the $\tilde{g}_{\alpha\beta}$ verify the same cocycle conditions as the $g_{\alpha\beta}$'s. This is a classical fact that every oriented compact riemanian four manifold has a spin^c-structure.

The determinant bundle of the spin^c structure. Associated to every spin^c structure σ, there is a complex line bundle, the determinant bundle $\det \sigma$, whom transition functions are $\det g^+_{\alpha\beta} = \det g^-_{\alpha\beta}$, if $g_{\alpha\beta}$ is written in the precedent form. If we are given every line bundle L and σ any spin^c structure, $\sigma \otimes L$ is another spin^c structure, whom determinant bundle is $\det \sigma \otimes L^2$.

Fundamental example. If X has an almost complex structure J and is equipped with a hermitian metric g, then we can chose the $g_{\alpha\beta}$ in $U(2)$. Now there is a simple and canonical way to send $U(2)$ into $\mathrm{spin}^c(4)^c$:

$$j(g) = \left(\begin{array}{ccc} \begin{array}{cc} 1 & 0 \\ 0 & \det g \end{array} & & 0 \\ & 0 & g \end{array} \right),$$

such that there is a canonical spin^c structure σ_0 on a almost complex manifold. What is its determinant bundle ? Simply K^{-1}, the anticanonical bundle.

The spinor bundle. We see that there is a natural representation of $\mathrm{Spin}(4)^c$ on $\mathbb{C}^2 \oplus \mathbb{C}^2$. The bundle associated W is called the spinor bundle. Moreover, the representation splits into two irreducible one, W^+ and W^-. If we twist the spin^c structure by some line bundle L, then W and W are twisted by L too.

Moreover, if we define the complex Clifford algebra $Cl^c(\mathbb{C}^2)$ as $\mathrm{End}(\mathbb{C}^2 \oplus \mathbb{C}^2)$, we see that there is a Clifford bundle $End(W)$ over X. Then the complexified tangent bundle $TX \otimes \mathbb{C}$ can be identified with the subundle $\mathrm{Hom}(W^+, W^-)$

(just remark that the transition functions are the same), and that the space of exterior products $\Lambda TX \otimes \mathbb{C}$ is in fact $\mathrm{End}(W)$.

The complex case. If X has an almost complex structure J, then $W^+ = \Lambda^{0,2*}TX^*$, and $W^- = \Lambda^{0,2*+1}$ and in the case of dimension 4, that means $\Gamma(W^+) = \Gamma(X, \mathbb{C}) \oplus \Omega^{0,2}(X)$ and $\Gamma(W^-) = \Omega^{0,1}(X)$.

Connections on W. We can prove that given a hermitian connection ∇_A on det (σ), there is a unique hermitian connection ∇ on W that is compatible with the Levi-Civita connection ∇_{LC} on TX and that induces ∇_A. That means

$$\nabla(v.\psi) = (\nabla_{LC}v).\psi + v.\nabla\psi$$

for all vector field v and spinor ψ, and

$$\nabla_A(\psi \wedge \phi) = \nabla\psi \wedge \phi + \psi \wedge \nabla\phi.$$

The Dirac operator is the composition of the two following arrows :

$$\Gamma(X, W^+) \overset{\nabla}{\to} \Gamma(X, TX^* \otimes W) \to \Gamma(X, TX \otimes W) \to \Gamma(X, W),$$

where the second arrow uses the identification through the metric between TX and TX^*, and the last one uses the action of ΛX on W through the action of the Clifford multiplication. If $\{e_1, \cdots e_4\}$ is a local orthonormal framing of TX, then

$$D_A\psi = \sum_{i=1}^{4} e_i \cdot \nabla_{e_i}\psi.$$

The Dirac operator has simple interesting proprieties : it is selfadjoint, and it exchanges the section of W^+ and W^-. We will note $D_A^\pm$ the restriction of $D_A^\pm$ on $\Gamma(W^\pm)$.

The function q. Let define q by the following :

$$\begin{array}{rcl} q : \Gamma(W^+) & \to & \mathrm{End}(W^+) \\ \psi & \mapsto & q(\psi) = -2i(<\psi, \mathrm{Id}\ > \mathrm{Id} - \frac{1}{2}|\psi|^2\mathrm{Id}). \end{array}$$

In fact, q is trace-free and hermitian. It means that as a form, it belongs to Ω^2_+.

The equations. Let μ a real self-dual closed 2-form. The Seiberg-Witten equations (depending on μ) rely a positive spinor to a hermitian connection on $\det\sigma$ through the following pair of equations :

$$\begin{array}{rcl} D_A^+\psi & = & 0 \\ \frac{1}{2}F_A^+ & = & 2q(\psi) + \mu. \end{array}$$

(Sorry, the $1/2$ coefficient is here to have the same equations thant in Moore's book [Mo].)

2 A Seiberg-Witten invariant

In this section we study the space of solutions, and we will use it to produce the famous invariants.

2.1 The moduli space of solutions

The first thing one could notice is that the gauge group $\Gamma(X, S^1)$ acts on the space of solutions by

$$f.(\psi, \nabla) = (f\psi, \nabla + f^{-1}df).$$

So the moduli space $\mathcal{M}(g, \sigma, \mu)$ is the quotient of the space of solutions by this action. It inherits of the quotient topology, and we can prove that it is a separated space. Moreover :

Proposition 1 *The space $\mathcal{M}$ is compact.*

Proof. One need the fundamental following fact :

Lemme 1 *Let (A, ψ) a non trivial solution of the equations. Then at a maximum of $|\psi|$, we have*

$$|\psi|^2 \leq -\frac{1}{4}s + \mu,$$

where s is the scalar curvature of g.

One need for the 1-form another lemma :

Lemme 2 *Choose a fixed hermitian connection ∇_0 on $\det \sigma$. If X is simply connected, then for any other hermitian connexion ∇ on $\det \sigma$, there is a unique gauge equivalent connection ∇' such that $\nabla' = \nabla_0 - ia$, where a is a real 1-form such that $d^*a = 0$.*

Proofof this lemma. The new connection can be written $\nabla = \nabla_0 - ia'$, so we just have to solve the equation : $d^*(g^{-1}dg) = d^*a'$, where $g \in \Gamma(X, S^1)$. If X is simply connected, there is $f \in \Gamma(X, \mathbb{R})$ such that $g = e^{if}$, such that we have to solve the equation :

$$\Delta f = d^*a'.$$

This have a solution thanks to Hodge theory : $a' = h + d\alpha + d^*\beta$, such that $d^*a' = d^*d\alpha$. For the function α, $\Delta = d^*d$, so that we can take $f = \alpha$. We won't prove the uniqueness. □

Now, let (∇_{A_n}, ψ_n) a sequence of solutions. The first lemma shows that the sequence of ψ_n's solutions is bounded in L^p. Now, we know for elliptic operators L like $d \oplus d^*$ that if u is not in the kernel of L, its distance Pu from ker L is well-controlled, i.e for all positive integer m and every real $p > 1$, there is a constant

$$||u - Pu||_{1+m,p} \leq C||Lu||_{m,p}.$$

The kernel of $L = d^+ \oplus d^*$ on the 1-form is trivial (to be checked!), and we have due to the SW equations and the coclosed condition for a_n : $L(a_n) = -F_0^+ + 2q(\psi_n) + 2\mu$. So thanks to the former estimation, we see that a_n is bounded in the $L^{1,p}$ norm. Thanks to a classical bootstrap and the Sobolev embedding theorem, we can prove that the two estimates give us the compactness. □

Proofof the lemma 1. . The most important ingredient of this estimate is the Weitzenböck's formula :

$$D_A^2 = \nabla_A^* \nabla_A + \frac{1}{4}s + \frac{1}{2}F_A.$$

Moreover, it's easy to prove that

$$\Delta|\psi|^2 \leq 2\Re < \nabla_A^* \nabla_A \psi, \psi > .$$

So if x is a maximum of $|\psi|$, $\Delta|\psi|^2 \leq 0$, and ψ a solution of the SW equations, we get :

$$0 \leq -\frac{1}{4}s|\psi|^2 - < F_A \cdot \psi, \psi > .$$

As $F_A^+ = \mu + \sigma(\psi)$, and $F_A^+ \cdot \psi = F \cdot \psi$ because ψ is a $+$ spinor, we get easily the result. $\square$

Smoothness. Let (A, ψ) a solution of the SW equations. Choosing as in lemma 2 a special connection equivalent to A let still a S^1-action on ψ. So we have to quotient by this action to get the moduli space. So we can set :

$$\begin{aligned} \tilde{\mathcal{M}} &= \{(A = A_0 + ia, \psi, \mu) \text{ s.t } SW(A, \psi, \mu) = 0 \text{ and } d^*a = 0\}, \\ \tilde{\mathcal{M}}_\mu &= \{(A = A_0 + ia, \psi) \text{ s.t } SW(A, \psi, \mu) = 0 \text{ and } d^*a = 0\}, \\ \mathcal{M}_\mu &= \tilde{\mathcal{M}}_\mu / S^1. \end{aligned}$$

If we write $F_\mu(a, \psi) = SW(\nabla_0 - ia, \psi, \mu) \oplus d^*(a)$, we would like that for some μ, dF_μ should be surjective, such that $\mathcal{M}_\mu$ is smooth thanks to the local inversion theorem. One other problem will be the dependance with μ of $\mathcal{M}_\mu$.

Theorem 1 *If $b^+ > 0$, then a generic choice of μ, $\mathcal{M}_\mu$ is either empty, or a smooth compact submanifold of dimension*

$$d(\sigma) = \frac{1}{4}(< c_1(L)^2, [X] > -(2\chi(X) + 3\tau(X)).$$

Remark. If σ is the canonical spin^c structure σ_0 on a complex manifold, then $d(\sigma_0) = 0$. In this case we have $L = K^{-1}$. Moreover, the first Pontryagin class of TX is $p_1(TX) = c_1(TX)^2 - 2c_2(TX)$. We have

$$\chi(X) = \int e(TX) = \int c_2(TX),$$

$\tau(X) = \int p_1(X)$ and $c_1(TX) = -c_1(K^{-1})$, so unifying all that gives $d = 0$.

Proof. First, the action of S^1 on $\tilde{M}_\mu$ is free when there is no solution of the SW-invariant with $\psi \equiv 0$, and so M_μ is smooth if $\tilde{M}_\mu$ is, with the same dimension minus 1. If there is such solution, that implies that $F_A^+ = \mu$. But the set of F_A^+'s when A covers the set of hermitian connections is an affine subspace of direction $d^+\Omega^1$. But $\mu \in \Omega_+^2$ is orthogonal to this vector space, it is orthogonal to $d\Omega^1$, so it is d^*-closed. As a self-dual form, it is then harmonic. So the "bad subspace" of μ's is of codimension b^+. Now if $b^+ > 0$, we call Λ the "good space" Λ, and for $\mu \in \Lambda$, $\tilde{\mathcal{M}}_\mu$ is smooth if $\mathcal{M}_\mu$ is. We use the latitude given by the parameter μ in the following way : let $\pi : \tilde{\mathcal{M}} \to \Lambda$.

Fact 1 : dF_μ as well as π are Fredholm operators, with the same index.

Fact 2 : There is a dense, open subset $\mathcal{U} \subset \Omega^2_+$, such that dF is surjective for all solution (A, ψ, μ) with $\mu \in \mathcal{U}$.

Proofof the fact 2. . We have

$$\begin{aligned} dF(A,\psi,\mu) : \Omega^1 \oplus \Gamma(W^+) \oplus \Omega^2_+ &\rightarrow \Gamma(W_-) \oplus \Omega^2_+ \oplus \tilde{\Omega}^0 \\ (a,\phi,\nu) &\mapsto (D_A\phi - ia.\psi, d^+a + D_\psi q\phi + \nu, d^*a), \end{aligned}$$

where $\tilde{\Omega}^0$ is the set of functions whose integral on X vanishes. We see immediatly that dF is onto on the last component, and on Ω^2_+ thanks the ν's. So let now a section ψ' orthogonal to the image of the first component of dF. Then due to the term $ia.\psi$, ψ' is vanishing on the open set where $\psi \neq 0$. But then, it means that it's orthogonal to all the $D_A\phi$'s, which implies that $D^-_A\psi'$ is vanishing. Because ψ' vanishes on an open set and D^- is elliptic, $\psi' \equiv 0$. □

Fact 2. proves that $\tilde{\mathcal{M}}$ is a smooth manifold. But now, Fact 1. and the Sard-Smale theorem implies that for a generic μ, the fiber of π $\tilde{\mathcal{M}}_\mu$ is a smooth manifold.

Calculation of $d(\sigma)$. The 0-order term $a.\phi$ in dF is a compact operator, so the index of dF doesn't change if we drop it. But now this is the same as $D_A \oplus d^* \oplus d^+$. The kernel of $d^* \oplus$ can be proved to be $\mathcal{H}^1(X)$, and we already saw that the cokernel of d^+ is $\mathcal{H}^2_+$. Now, if a function is orthogonal to $d^*\Omega^1$, it's a constant, so coker $d^* = \mathbb{R}$. Summing up,

$$\mathrm{Ind}(d^* \oplus d^+) = b^1 - b^+.$$

On the other hand, the Index Theorem gives us the complex index of the Dirac operator :

$$\mathrm{Ind}\ D_A = -\frac{1}{8}\tau + \frac{1}{8} < c_1(\det\sigma)^2, X > .$$

Summing up all this, we get that the real index of the total operator is

$$\mathrm{Ind\ DF} = \frac{1}{4}(< c_1(\det\sigma)^2, X > -2\chi(X) - 3\tau(X)) + b_0,$$

which is the dimension of $\tilde{\mathcal{M}}_\mu$. In fact, this is the good dimension even when X is not simply connected. Now, with $b_0 = 1$, we get the result for $\mathcal{M}_\mu$. □

2.2 One simple invariant

Suppose that $b^+ > 1$. Then if μ_1 and μ_2 are two parameters, we can find a path γ in Ω^2_+ joigning both chooses and staying in the good parameter subspace Λ. Then again the Sard-Smale theorem says that $\mathcal{W} = \pi^{-1}(\gamma)$ is a manifold for γ generic, and $\partial\mathcal{W} = \mathcal{M}_{\mu_2} \cup \mathcal{M}_{\mu_1}$. In this notes, we will suppose that we are in the case where $d(\sigma) = 0$. The preceeding considerations show that in this case, $\mathcal{M}_{\mu_i}$ are both compact, 0-dimensional manifolds. They parity is then the same, and we can define :

$$SW(X, \sigma) = \text{Card } \mathcal{M}_\mu \text{ mod } 2.$$

Of course, if $b^+ > 1$, the same argument prove that SW does not depend on the choosen metric g.

3 Invariants of Kähler manifolds

Given a almost complex structure J, there is a canonical spin^c structure σ_0 whose determinant line bundle is the anticanonical bundle K^{-1}. Every other spin^c structure can be written as $\sigma_0 \otimes L = \sigma(L)$, where L is any complex line bundle. So $\det \sigma(L) = K^{-1} \otimes L^2$.

Theorem 2 *Let (X, ω, J) a Kähler manifold, with $b_2^+ > 1$. Then $SW(\sigma_0) = 1$.*

Proof. In this case, the spinor's space is $W^+ = \Theta \oplus \Lambda^{0,2}$ and $W^- = \Lambda^{0,1}$, and the Dirac operator is just :

$$D_A^+ = \sqrt{2}(\overline{\partial}_A + \overline{\partial}_A^*).$$

If ∇_0 is the canonical connexion compatible with the Kähler structure on K, then any other connection can be written as $\nabla_A = \nabla_0 + ia$, and $\overline{\partial}_A = \overline{\partial} + ia^{0,1}$. Let ω be the Kähler form. The complex self-dual set of forms is

$$\Omega_+^2 \otimes \mathbb{C} = \Gamma(X, \mathbb{C})\omega \oplus \Omega^{0,2}(X, \mathbb{C}) \oplus \Omega^{2,0}(X, \mathbb{C}),$$

and the one with coefficients in $i\mathbb{R}$ are of the form : $if\omega + \nu - \bar{\nu}$, where $\nu \in \Omega^{0,2}(X, \mathbb{C})$ and f a real function. As a parameter, let's choose $\mu = F_{A_0}^+ + it\omega$ for some real t. The SW-equations reduce, if we put $\psi = \alpha + \beta$, where α is a complex function, and β a $(0,2)$-form, to the following equations :

$$\begin{aligned} \overline{\partial}_A \alpha + \overline{\partial}_A^* \beta &= 0 \\ (d^+ a)^{1,1} &= \frac{i}{4}(|\alpha|^2 - |\beta|^2 - t)\omega \\ (da)^{0,2} &= \frac{1}{2}\bar{\alpha}\beta. \end{aligned}$$

But now we have

$$0 = \int da \wedge \omega = \int (da)^{(1,1)} \wedge \omega = \int (|\alpha|^2 - |\beta|^2 - t)\omega \wedge \omega).$$

On the other hand, we get

$$\overline{\partial}_A \overline{\partial}_A \alpha + \overline{\partial}_A \overline{\partial}_A^* \beta = F_A^{0,2} \alpha + \overline{\partial}_A \overline{\partial}_A^* \beta.$$

Let's take the scalar product with β, use the last equation and integrate. We have then :

$$\int |\bar{\alpha}\beta|^2 + |\partial_A^* \beta|^2 = 0.$$

So for t big enough, the first consequence and the first part of the second show that $\beta \equiv 0$. That implies $F_A^{0,2} = 0$, and α is an holomorphic function, so is a constant, which has to be $\alpha = \sqrt{t}$. Summing up, $\psi = \sqrt{t} \oplus 0$, and $a = 0$ is the only solution to the equations. Assuming that this solution is regular, we have proved that $SW(X, \sigma_0) = 1$. □

4 Applications

We will present here two applications of the theory. The first was known from Donaldson, and handle with the topology of the 4-manifolds. The second concerns the topology of curves in $\mathbb{C}P^2$.

4.1 Application to the topology of 4-manifold

For smooth manifolds of dimension ≥ 5, Smale proved that being homeomorphic is the same as being diffeomorphic. In dimension 4, Donaldson proved the following :

Theorem 3 (Donaldson) *There is a pair of smooth 4-manifolds homeomorphic but not diffeomorphic.*

The powerfool tool for studying the topology of 4-manifolds is the intersection form $q : H_2(X, \mathbb{Z}) \times \to H_2(X, \mathbb{Z}) \to \mathbb{Z}$.

Theorem 4 (Freedman) *Two smooth simply connected 4-manifolds are homeomorphic if they have the same intersection form.*

Proofof the Donaldson theorem. . We begin by the following fact : for all integers p and q, the Seiberg-Witten invariant of $p\mathbb{C}P^2 \# q\overline{\mathbb{C}P^2}$ vanishes. Indeed, Gromov and Lawson proved that the connected sum of two manifolds with positive scalar curvature has itself positive scalar curvature. But then the lemma 1 shows that in this case, there are no solutions of the SW equations. Now, the intersection form of $p\mathbb{C}P^2 \# q\overline{\mathbb{C}P^2}$ is $I_p \oplus -I_q$. In fact, there is a Kähler manifold, so with non trivial SW invariant, and with the same intersection form with $p =$. Let's take X a smooth complex surface in $\mathbb{C}P^3$ of degree d. This smooth manifold has $b_1 = 0$ by the Lefschetz theorem, and we can calculate its b^+ :

$$b^+ = \frac{1}{3}(d-1)(d-2)(d-3) + 1 > 1$$

for $d \geq 4$. So the intersection form is $I_{b^+} \oplus I_{b^-}$, the same as the precedent with the appropriate p and q. So every smooth hypersurface of P^3 is homeomorophic, but never diffeomorphic, to an other smooth manifold. □

Remark. $\mathbb{C}P^2 \# q\overline{\mathbb{C}P^2}$ is the blow-up at q points of $\mathbb{C}P^2$, and so is Kähler. Is it something wrong ? No, because it's b^+ is one, so its SW invariant is not well-defined.

4.2 Application to the Thom's conjecture

In $\mathbb{C}P^2$, the genus of a smooth complex curve can be computed from its degree d :

$$g(C) = \frac{1}{2}(d-1)(d-2).$$

Theorem 5 (Thom's conjecture) *For all smooth curve, not necessarly complex, representing the class* $d[H]$*, then*

$$g(C) \geq \frac{1}{2}(d-1)(d-2).$$

Proof. We will give the ideas of the proof by Kronheimer and Mrowka. Let's be a real smooth surface C in $\mathbb{C}P^2$. We first suppose that $C.C = 0$. Then a tubular neighborhood V of C is trivial, so is diffeomorphic to $C \times D^2$. We identify a neighborhood of ∂V with $C \times S^1 \times [0, 1]$. We will "separate" C from the rest of $\mathbb{C}P^2$, in the following sense : we take a metric g, such that on $C \times [0,1]$, g is just

$$g = g_L = g_C \oplus dt^2 \oplus \ L^2 dx^2,$$

where g_C is a metric on C with constant curvature K, so that

$$\int_C K dVol_g = 4\pi\chi(C).$$

We can suppose that g_L is C^2-bounded indepently of L outside this neighborhood. Now the SW equations intervene. We have $b^+(\mathbb{C}P^2) = 1$, so the SW invariant is a priori not well defined, because the image of Ω^1 by $a \mapsto F^+_{A_0} + id^+a$ is of codimension 1 in Ω^2_+, so splits this space in two parts for all metrics g. But for μ's in the same semi-space, the SW invariant is well-defined. But we can prove (not here) that for the canonical spin^c structure σ_0 and the metric g_L, the associated SW invariants are nontrivial, so for all L, there is a solution (ψ, A) to the equations. We have $F^+_A = q(\sigma)$, and we saw that $|\psi|^2 \leq -s$, so the L^2-norm (with the metric g_L) of F^+_A is less than

$$|K| \ \mathrm{Vol}_{g_L}(\text{long tube}) + o(L) = |\chi(C)|L + o(L)$$

. On the other side, we have

$$c_1(L)^2 = \int F_A^2 = ||F_A^+||^2 - ||F_A^-||^2,$$

and so

$$||F_A||^2 = ||F_A^+||^2 + ||F_A^-||^2 = 2||F_A^+||^2 + \text{topogical term}.$$

Now,

$$||F_A||^2 \geq ||F_A||^2_{Tube} = ||F_A||^2_{L^2(C)} \mathrm{Vol}_{g_L}(S^1 \times [0,1]) \geq | < c_1(L), C > |L.$$

So we can see when $L \to \infty$ that $| < c_1(L), C > | \leq |\chi(C)|$.

Now, let C a surface in $\mathbb{C}P^2$ which represent $d[H]$, so that $C.C = d^2$. We blow-up $\mathbb{C}P^2$ in d^2 points outside C. In $M = \mathbb{C}P^2 \sharp d^2 \overline{\mathbb{C}P^2}$, the b^1 and b^+ are the same, and M is Kähler. But now

$$K_M = \pi^* K_{\mathbb{C}P^2} - \sum_i E_i,$$

where the E_i are the exceptionnal divisors of the blow-up. In M, we join C to the E_i's with long tubes to form a new surface $\tilde{C}$, such that

$$\tilde{C}.\tilde{C} = C.C - d^2 = 0.$$

We apply the precedent reflexion, and we get :

$$|c_1(M).\tilde{C}| \leq |\chi(\tilde{C})| = |\chi(C)|.$$

But the left side is $| < c_1(\mathbb{C}P^2), C > - \sum_i E_i.E_j| = | - 3d + d^2|$. So we have

$$g \geq \frac{1}{2}(d^2 - 3d + 2) = \frac{1}{2}(d-1)(d-2),$$

which is the genus of a complex curve of degree d. □

Références

[Mo] J. D. MOORE *Lectures on Seiberg-Witten invariants*, Lectures notes in Mathematics 1629, Springer 2001.

[Mo] J. W. MORGAN *The Seiberg-Witten equations and applications tot the topology of smooth four-manifolds*, Math. Notes, Princeton university press, 1996.

[Ni] L. I. NICOLAESCU *Notes on Seiberg-Witten theory*, Graduate Studies in Mathematics AMS, 2000.

[Wi] E. WITTEN *Monopoles and four-manifolds*, Math. Research Letters 1 (1994), 769-796.

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