Corrigé – Partiel 24 Novembre 2010

Exercice 1

1) $\mathbb{E}(X) = \int_{\mathbb{R}} x\, f_\theta(x)\,dx = (1-\theta)\int_{-\frac{1}{2}}^{0} x\,dx + (1+\theta)\int_{0}^{\frac{1}{2}} x\,dx = \frac{1-\theta}{8} + \frac{1+\theta}{8} = \frac{\theta}{4}$

$\mathbb{E}(X^2) = (1-\theta)\int_{-\frac{1}{2}}^{0} x^2\,dx + (1+\theta)\int_{0}^{\frac{1}{2}} x^2\,dx = \frac{1-\theta}{24} + \frac{1+\theta}{24} = \frac{1}{12}$

$Var(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) = \frac{1}{12} - \frac{\theta^2}{16} = \frac{4-3\theta^2}{48}$

2) $\overline{X}_n = \frac{\theta}{4} \Rightarrow \hat{\theta}_n^{(1)} = 4\overline{X}_n$.

3) $\overline{X}_n \xrightarrow{p.s} \mathbb{E}(X)$, donc $\hat{\theta}_n^{(1)} \xrightarrow{p.s} 4\,\mathbb{E}(X) = \theta \Rightarrow \hat{\theta}_n^{(1)}$ fortement conv.

$\mathbb{E}[\hat{\theta}_n^{(1)}] = 4\,\mathbb{E}(\overline{X}_n) = 4\cdot\mathbb{E}(X) = \theta \Rightarrow \hat{\theta}_n^{(1)}$ sans biais.

$Var(\hat{\theta}_n^{(1)}) = 16\,\frac{Var(X)}{n} = \frac{4-3\theta^2}{3n}$

4) $Y_i \sim \mathcal{B}(p)$ avec $p = \mathbb{P}[0 \le X_i < \frac{1}{2}] = \int_0^{1/2} f_\theta(x)\,dx = \frac{1+\theta}{2}$

$S_n = \sum_{i=1}^{n} Y_i \sim \mathcal{B}(n,p)$

5) $L_n(\theta) = (1-\theta)^{\sum_{i=1}^{n} \mathbb{1}_{-\frac{1}{2}<X_i<0}} \cdot (1+\theta)^{\sum_{i=1}^{n} \mathbb{1}_{0\le X_i<\frac{1}{2}}} = (1-\theta)^{n-S_n}\cdot(1+\theta)^{S_n}$

Mais $n = \sum_{i=1}^{n} \mathbb{1}_{-\frac{1}{2}<X_i<0} + \sum_{i=1}^{n} \mathbb{1}_{0\le X_i<\frac{1}{2}} \Rightarrow \sum_{i=1}^{n} \mathbb{1}_{-\frac{1}{2}<X_i<0} = n - S_n$

$\log L_n(\theta) = (n-S_n)\log(1-\theta) + S_n \log(1+\theta)$

Rq: la fonction $f_\theta(x)$ est continue, dérivable en θ (pas en x!)

$0 = \frac{\partial}{\partial\theta}\log L_n(\theta) = -\frac{n-S_n}{1-\theta} + \frac{S_n}{1+\theta} \Rightarrow S_n + \theta S_n - n - n\theta + S_n - \cancel{\theta S_n} = 0$

$\Rightarrow 2S_n - n - n\theta = 0 \Rightarrow \hat{\theta}_n^{(2)} = 2\frac{S_n}{n} - 1 = 2\overline{Y}_n - 1$

$\left.\begin{array}{l}\frac{\partial^2}{\partial\theta^2}\log L_n(\theta) = -\frac{S_n}{(1+\theta)^2} + \frac{S_n - n}{(1-\theta)^2} \\ S_n \in \{0,1,\dots,n\}\end{array}\right\} \Rightarrow \frac{\partial^2}{\partial\theta^2}\log L_n(\theta) \le 0$

$\Rightarrow \hat{\theta}_n^{(2)}$ point de max.

②

6) $\bar{Y}_n \xrightarrow{P.S} \mathbb{E}(Y) = p = \frac{1+\theta}{2} \Rightarrow \hat{\theta}_n^{(2)} = 2\bar{Y}_n - 1 \xrightarrow[n\to\infty]{P.S} \theta \Rightarrow \hat{\theta}_n^{(2)}$ fort consist

$\mathbb{E}(\bar{Y}_n) = \mathbb{E}(Y) = p = \frac{1+\theta}{2} \Rightarrow \mathbb{E}(\hat{\theta}_n^{(2)}) = \theta \Rightarrow \hat{\theta}_n^{(2)}$ sans biais.

$$Var(\hat{\theta}_n^{(2)}) = 4\,Var(\bar{Y}_n) = 4\,\frac{Var(Y)}{n} = 4\,\frac{p(1-p)}{n} = \frac{4}{n}\,\frac{1+\theta}{2}\cdot\frac{1-\theta}{2} = \frac{1-\theta}{n}$$

7) $Var(\hat{\theta}_n^{(1)}) - Var(\hat{\theta}_n^{(2)}) = \frac{4-3\theta^2}{3n} - \frac{1-\theta^2}{n} = \frac{4-3\theta^2 - 3 + 3\theta^2}{3n} = \frac{1}{3n} > 0$

$\Rightarrow Var(\hat{\theta}_n^{(1)}) > Var(\hat{\theta}_n^{(2)})$ donc $\hat{\theta}_n^{(1)}$ est moins précis que $\hat{\theta}_n^{(2)}$

$\Rightarrow$ on choisit $\hat{\theta}_n^{(2)}$.

8) Un estimateur sans biais de θ : $\hat{\theta}_n^{(2)} = 2\bar{Y}_n - 1$

On a le TCL pour une loi de Bernoulli :

$$\sqrt{n}\,\frac{\bar{Y}_n - p}{\sqrt{p(1-p)}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1) \quad \text{avec } p = \frac{1+\theta}{2}$$

D'autre part, un estimateur pour $p(1-p)$ est : ~~$\hat{\theta}_n^{(2)}(1-\hat{\theta}_n^{(2)})$~~

$$= \cancel{(2\bar{Y}_n - 1)(1 - 2\bar{Y}_n - 1)} \qquad \frac{1+\hat{\theta}_n^{(2)}}{2}\left(1 - \frac{1+\hat{\theta}_n^{(2)}}{2}\right) = \frac{1-(\hat{\theta}_n^{(2)})^2}{4} =$$

$$= \frac{1-(2\bar{Y}_n - 1)^2}{4} = \frac{1 - 4(\bar{Y}_n)^2 + 4\bar{Y}_n - 1}{4} = \bar{Y}_n(1-\bar{Y}_n)$$

Donc : $$\sqrt{n}\,\frac{\bar{Y}_n - p}{\sqrt{\bar{Y}_n(1-\bar{Y}_n)}} = \underbrace{\sqrt{n}\,\frac{\bar{Y}_n - p}{\sqrt{p(1-p)}}}_{\xrightarrow{\mathcal{L}} \mathcal{N}(0,1)} \cdot \underbrace{\sqrt{\frac{p(1-p)}{\bar{Y}_n(1-\bar{Y}_n)}}}_{\xrightarrow{PS} 1} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1)$$

La définition de l'estimateur par intervalle (asymptotique) :

$$1-\alpha = \mathbb{P}\left[a \le \sqrt{n}\,\frac{\bar{Y}_n - p}{\sqrt{\bar{Y}_n(1-\bar{Y}_n)}} \le b\right] \quad \text{avec} \begin{cases} a = u_{\frac{\alpha}{2}} = -u_{1-\frac{\alpha}{2}} \\ b = u_{1-\frac{\alpha}{2}} \end{cases}$$

$$1-\alpha = \mathbb{P}\left[\bar{Y}_n - \sqrt{\frac{\bar{Y}_n(1-\bar{Y}_n)}{n}}\cdot u_{1-\frac{\alpha}{2}} \le p \le \bar{Y}_n + \sqrt{\frac{\bar{Y}_n(1-\bar{Y}_n)}{n}}\, u_{1-\frac{\alpha}{2}}\right]$$

③

Exercice 2 $\quad y_i = x_i^k$, $\quad \mathbb{P}[y_i = 0] = \mathbb{P}[X_i = 0] = p$

1) les valeurs de y_i sont 0 et 1,

Donc $y_i \sim \mathcal{B}(p)$, $\sum_{i=1}^n y_i \sim \mathcal{B}(n, p)$

2) $\mathbb{P}[\overline{X}_n = 0] = \mathbb{P}[\sum_{i=1}^n x_i = 0] = (1-p)^n$

$\mathbb{P}[\overline{Y}_n = 0] = \mathbb{P}[\sum_{i=1}^n y_i = 0] = (1-p)^n$

$\mathbb{P}[\overline{X}_n = \overline{Y}_n = 0] = \mathbb{P}[\overline{X}_n = 0] = (1-p)^n$

Exercice 3 $\quad f_\theta(x) = (2\pi)^{-1/2}\theta^{-1/2}\exp\left(-\frac{(x-\theta)^2}{2\theta}\right)$

$$\log L_n(\theta) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\theta - \frac{1}{2\theta}\sum_{i=1}^n (X_i-\theta)^2$$

$$0 = \frac{\partial}{\partial\theta}\log L_n(\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^n (X_i-\theta)^2 + \frac{1}{\theta}\sum_{i=1}^n (X_i-\theta)$$

$$0 = -n\theta + \sum_{i=1}^n X_i^2 - 2n\theta\overline{X}_n + n\theta^2 + 2n\theta\overline{X}_n - 2n\theta^2$$

$$0 = \sum_{i=1}^n X_i^2 - n\theta^2 - n\theta \Rightarrow \theta^2 + \theta = U_2^n \Rightarrow \theta^2 + \theta + \frac{1}{4} = U_2^n + \frac{1}{4}$$

$$\Rightarrow \left(\theta + \frac{1}{2}\right)^2 = U_2^n + \frac{1}{4} \Rightarrow \theta + \frac{1}{2} = \left(U_2^n + \frac{1}{4}\right)^{1/2} \Rightarrow \hat{\theta}_n = \left(U_2^n + \frac{1}{4}\right)^{1/2} - \frac{1}{2}$$

$$\frac{\partial^2}{\partial\theta^2}\log L_n(\theta) = \frac{n}{2\theta^2} - \frac{1}{\theta^3}\sum_{i=1}^n (X_i-\theta)^2 + \frac{n\overline{X}_n}{\theta^2}$$

$$= \frac{1}{2\theta^2}\left[n - \frac{1}{\theta}\left(\sum X_i^2 - 2n\theta\overline{X}_n + n\theta^2\right) - 2n\overline{X}_n\right]$$

$$= \frac{1}{2\theta^2}\left[n - \frac{1}{\theta}\sum X_i^2 + n\theta\right] = \frac{1}{2\theta^3}\left[n\theta - \sum X_i^2 + n\theta^2\right]$$

2) $U_2^n \xrightarrow{P.S} \mathbb{E}(X^2) = Var(X) + \mathbb{E}^2(X) = \theta + \theta^2$

$\Rightarrow \hat{\theta}_n \xrightarrow{P.S} \left(\frac{1}{4} + \theta + \theta^2\right)^{1/2} - \frac{1}{2} = \left(\left(\theta + \frac{1}{2}\right)^2\right)^{1/2} - \frac{1}{2} = \theta \Rightarrow \hat{\theta}_n$ fort conv.

3) $\log f_\theta(x) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\theta - \frac{(x-\theta)^2}{2\theta}$

$$\frac{\partial}{\partial\theta}\log f_\theta(x) = -\frac{1}{2\theta} + \frac{x-\theta}{\theta} + \frac{(x-\theta)^2}{2\theta^2} = -\frac{1}{2\theta} + \cancel{\frac{x}{\theta}} - 1 + \frac{x^2}{2\theta^2}$$

$$- \cancel{\frac{x}{\theta}} + \frac{1}{2} = \frac{x^2}{2\theta^2} - \frac{1}{2\theta} - \frac{1}{2}$$

$$\frac{\partial^2}{\partial \theta^2} \log f_\theta(x) = -\frac{x^2}{\theta^3} + \frac{1}{2\theta^2}$$

$$I(\theta) = -E\left[\frac{\partial^2}{\partial \theta^2} \log f_\theta(X)\right] = -E\left[\frac{1}{2\theta^2} - \frac{X^2}{\theta^3}\right] = \frac{E(X^2)}{\theta^3} - \frac{1}{2\theta^2} = \frac{\theta+\theta^2}{\theta^3} - \frac{1}{2\theta^2}$$

$$= \frac{2\theta+2\theta^2-\theta}{2\theta^3} = \frac{2\theta^2+\theta}{2\theta^3} = \frac{2\theta+1}{2\theta^2}$$

Exercice 4

1) $T_1 = \frac{1}{\sigma_1^2} \sum_{i=1}^{n_1} (X_i - \overline{X}_{n_1})^2 \sim \chi^2(n_1 - 1)$

$T_2 = \frac{1}{\sigma_2^2} \sum_{i=1}^{n_2} (Y_i - \overline{Y}_{n_2})^2 \sim \chi^2(n_2 - 1)$

Puisque X_i et Y_i sont indép, alors T_1 et T_2 le sont aus

Donc, $Z_{n_1,n_2} = \frac{T_1/(n_1-1)}{T_2/(n_2-1)} \sim \frac{\chi_1^2(n_1-1)/(n_1-1)}{\chi^2(n_2-1)/(n_2-1)} \sim F(n_1-1, n_2-1)$

2) $E[Z_{n_1,n_2}] = E[F(n_1-1, n_2-1)] = \frac{n_2-1}{n_2-3}$

$Z_{n_1,n_2} = \frac{1}{\sigma_1^2} S_{X,n_1}^{*2} \cdot \frac{\sigma_2^2}{S_{Y,n_2}^{*2}} = \frac{\sigma_2^2}{\sigma_1} \cdot \frac{S_{X,n_1}^{*2}}{S_{Y,n_2}^{*2}}$

avec $S_{X,n_1}^{*2} = \frac{1}{n_1-1} \sum_{i=1}^{n_1} (X_i - \overline{X}_{n_1})^2$

Donc $E\left[\frac{S_{X,n_1}^{*2}}{S_{Y,n_2}^{*2}}\right] = \frac{n_2-1}{n_2-3} \cdot \frac{\sigma_1^2}{\sigma_2^2}$

Donc, un estimateur sans biais pour $\frac{\sigma_1^2}{\sigma_2^2}$.

$$H_{n_1,n_2} = \frac{n_2-3}{n_2-1} \cdot \frac{S_{X,n_1}^{*2}}{S_{Y,n_2}^{*2}}$$

3) $\frac{n_2-3}{n_2-1} \xrightarrow[n_2\to\infty]{} 1$

$S^{*2}_{X,n_1} \xrightarrow[n_1\to\infty]{p.s.} Var(X) = \sigma_1^2$

$S^{*2}_{Y,n_2} \xrightarrow[n_2\to\infty]{p.s} Var(Y) = \sigma_2^2$

Donc $H_{n_1,n_2} \xrightarrow[n_1,n_2\to\infty]{p.s} \frac{\sigma_1^2}{\sigma_2^2}$ $\Rightarrow H_{n_1,n_2}$ fortement consistant pour $\frac{\sigma_1^2}{\sigma_2^2}$.