Corrigé Partiel 23 nov. 2011

Exercice 1

$$f_\theta(x) = \theta^2 x \exp(-\theta x) \mathbb{1}_{x \geq 0}$$

1) $\mathbb{E}(X) = \int_{\mathbb{R}} x f_\theta(x)\,dx = \theta^2 \int_0^\infty x^2 \exp(-\theta x)\,dx = \frac{1}{\theta}\int_0^\infty y^2 \exp(-y)\,dy$

$\theta x = y \Rightarrow dx = \frac{dy}{\theta}$

$$= -\frac{1}{\theta} y^2 \exp(-y)\Big]_0^\infty + \frac{2}{\theta}\int_0^\infty y \exp(-y)\,dy$$

$$= -\frac{2}{\theta} y \exp(-y)\Big|_0^\infty + \frac{2}{\theta}\int_0^\infty \exp(-y)\,dy = -\frac{2}{\theta} e^{-y}\Big|_0^\infty = \frac{2}{\theta}$$

$\mathbb{E}(X^2) = \int_{\mathbb{R}} x^2 f_\theta(x)\,dx = \theta^2 \int_0^\infty x^3 \exp(-\theta x)\,dx = \frac{1}{\theta^2}\int_0^\infty y^3 \exp(-y)\,dy$

$$= -\frac{1}{\theta^2} y^3 \exp(-y)\Big]_0^\infty + \frac{3}{\theta^2}\int_0^\infty y^2 \exp(-y)\,dy = \frac{6}{\theta^2}$$

$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) = \frac{6}{\theta^2} - \frac{4}{\theta^2} = \frac{2}{\theta^2}$

2) $\bar{X}_n = \frac{2}{\theta} \Rightarrow \hat{\theta}_n^{(1)} = \frac{2}{\bar{X}_n} = \frac{2n}{\sum_{i=1}^n X_i}$

3) $\bar{X}_n \xrightarrow{p.s} \mathbb{E}(X) = \frac{2}{\theta} \Rightarrow \hat{\theta}_n^{(1)}$ fortement conv.

4) Soit $F_\theta(x)$ la f de rep de X et $G(y)$ pour Y.

$G(y) = \mathbb{P}[Y \leq y] = \mathbb{P}\left[\frac{1}{X} < y\right] = 1 - F_\theta\left(\frac{1}{y}\right)$

$g(y) = \frac{1}{y^2} f_\theta\left(\frac{1}{y}\right) = \frac{\theta^2}{y^3}\exp\left(-\frac{\theta}{y}\right)\mathbb{1}_{y \geq 0} = \exp\left(-\frac{\theta}{y} + 2\log\theta - 3\ldots\right.$

5) Pour Y:

$\Rightarrow C_1(\theta) = -\theta,\ T_1(y) = \frac{1}{y},\ D_1(\theta) = 2\log\theta,\ S_1(y) = \exp(-3\log\ldots$

Pour X:

$$f_\theta(x) = \exp\left(C_2(\theta)\cdot T_2(x) + D_2(\theta)\right)\cdot \tilde{S}_2(x)$$

$$C_2(\theta) = -\theta,\quad T_2(x) = x,\quad D_2(\theta) = 2\log\theta$$

$$\tilde{S}_2(x) = \exp(\log x)\cdot \mathbb{1}_{x>0}$$

(2)

6) $L_n(\theta) = \theta^{2n} \prod_{i=1}^{n} X_i \exp(-\theta \sum_{i=1}^{n} X_i)$

$\log L_n(\theta) = 2n \log \theta + \sum_{i=1}^{n} \log X_i - \theta n \overline{X_n}$

$\frac{\partial}{\partial \theta} \log L_n(\theta) = \frac{2n}{\theta} - n\overline{X_n} = 0 \Rightarrow \frac{2}{\theta} = \overline{X_n} \Rightarrow \hat{\theta}_n = \frac{2}{\overline{X_n}}$

$\frac{\partial^2}{\partial \theta^2} \log L_n(\theta) = -\frac{2n}{\theta^2} < 0 \Rightarrow \hat{\theta}_n$ point de max.

7) $f_{\sum X_i}(x) = \frac{\theta^{2n}}{(2n-1)!} x^{2n-1} \cdot \exp(-\theta x) \mathbb{1}_{x \geq 0}$

$\mathbb{P}[T_n < z] = \mathbb{P}\left[\frac{2n}{\sum X_i} < z\right] = 1 - P\left[\sum_{i=1}^{n} X_i < \frac{2n}{z}\right]$

$f_{T_n}(z) = \frac{2n}{z^2} \cdot f_{\sum X_i}\left(\frac{2n}{z}\right) = \frac{2n}{z^2} \cdot \frac{\theta^{2n}}{(2n-1)!} \cdot \left(\frac{2n}{z}\right)^{2n-1} \cdot \exp\left(-\frac{2n\theta}{z}\right)$

$= \theta^{2n} \cdot \frac{(2n)^{2n}}{(2n-1)!} \cdot z^{-2n-1} \cdot \exp\left(-\frac{2n\theta}{z}\right) \mathbb{1}_{z \geq 0}$

8) $\mathbb{E}[T_n] = \int_{\mathbb{R}} x f_{T_n}(x) dx = \frac{(2n)^{2n}}{(2n-1)!} \theta^{2n} \int_0^{\infty} x^{-2n} \exp\left(-\frac{2n\theta}{x}\right) dx$

$\frac{2n\theta}{x} = t \Rightarrow x = \frac{2n\theta}{t} \qquad dx = -\frac{2n\theta}{t^2} dt$

$= -\frac{(2n)^{2n}}{(2n-1)!} \theta^{2n} \int_{\infty}^{0} \frac{(2n\theta)^{-2n}}{t^{-2n}} \cdot e^{-t} \frac{2n\theta}{t^2} dt$

$= \frac{2n\theta}{(2n-1)!} \int_0^{\infty} t^{2n-1-1} e^{-t} dt = \frac{2n\theta}{(2n-1)!} \Gamma(2n-1)$

$= \frac{2n\theta}{(2n-1)!} \cdot (2n-2)! = \frac{2n}{2n-1} \theta$

9) $\mathbb{E}[T_n^2] = \frac{(2n)^2 \theta^2}{(2n-1)!} \int_0^{\infty} t^{2n-3} e^{-t} dt = \frac{(2n)^2 \theta^2}{(2n-1)!} \Gamma(2n-2)$

$= \frac{(2n)^2 \theta^2}{(2n-1)!} (2n-3)! = \frac{2n^2}{(n-1)(2n-1)} \theta^2$

③

$$\mathrm{Var}(T_n) = \frac{2n^2\theta^2}{(n-1)(2n-1)} - \frac{4n^2}{(2n-1)^2}\theta^2 = \frac{2n^2\theta^2}{(n-1)(2n-1)^2}$$

Estimateur sans biais :

$$\tilde{T}_n = \frac{2n-1}{2n} T_n$$

$$\mathrm{Var}(\tilde{T}_n) = \frac{(2n-1)^2}{(2n)^2} \cdot \frac{2n^2}{(n-1)(2n-1)^2}\theta^2 = \frac{2\theta^2}{n-1}$$

$$I_1(\theta) = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f_\theta(X)\right] = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}(2\log\theta + \log X - \theta X)\right]$$

$$= -\mathbb{E}\left[\frac{\partial}{\partial\theta}\left(\frac{2}{\theta} - X\right)\right] = \frac{2}{\theta^2}$$

$$I_n(\theta) = nI_1(\theta) = \frac{2n}{\theta^2} \neq \mathrm{Var}(\tilde{T}_n) \Rightarrow \tilde{T}_n \text{ pas effic}$$

Exercice 2

$$f(x) = \mathbb{P}[X = x] = C_m^x\, p^x (1-p)^{m-x}, \quad x \in \{0, 1, \dots,$$

Supposons qu'il existe un estimateur T_n sans biais

$$\mathbb{E}[T_n] = \sum_{(x_1,\dots,x_n)\in\Omega^n} T(x_1,\dots,x_n)\cdot\mathbb{P}[X_1 = x_1, \dots X_n = x_n]$$

$$= \sum_{(x_1,\dots,x_n)\in\Omega^n} T(x_1,\dots,x_n)\, C_m^{x_1}\dots C_m^{x_n}\, p^{\sum_{i=1}^n x_i}\cdot(1-p)^{mn-\sum_{i=1}^n x_i}$$

$$= \frac{1}{p^2} \qquad \Omega = \{0, 1, \dots, m\}$$

$$\Rightarrow 1 = \sum_{(x_1,\dots,x_n)\in\Omega^n} T(x_1,\dots,x_n)\, C_m^{x_1}\dots C_m^{x_n}\, p^{\sum x_i + 2}(1-p)^{n\dots}$$

$\forall p \in]$

si $p \to 0$ alors $0 = 1$ faux

Donc, il n'existe pas d'estimateur sans biais pour $\frac{1}{p^2}$.

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1) $f(x) = \frac{\theta^{-p}}{\Gamma(p)} e^{-\frac{x}{\theta}} . x^{p-1} \mathbb{1}_{x>0}$ $\qquad \lambda = \frac{1}{\theta}, \ s = p$

$L_n(\theta) = \frac{\theta^{n \cdot p}}{\Gamma^n(p)} e^{-\frac{1}{\theta}\sum_{i=1}^n x_i} \prod_{i=1}^n x_i^{p-1} \mathbb{1}_{x_i>0}$

$\log L_n(\theta) = -np \log \theta - \frac{\sum x_i}{\theta} + \dots$

$\frac{\partial}{\partial\theta} \log L_n(\theta) = -\frac{np}{\theta} + \frac{n\overline{X_n}}{\theta^2} = 0 \Rightarrow \hat{\theta}_n = \frac{\overline{X_n}}{p}$

$\frac{\partial^2}{\partial\theta^2} \log L_n(\theta) = \frac{np}{\theta^2} - \frac{2n\overline{X_n}}{\theta^3}$

$\frac{\partial^2}{\partial\theta^2} \log L_n(\hat{\theta}_n) = \frac{np^3}{(\overline{X_n})^2} - \frac{2n\overline{X_n}p^3}{(\overline{X_n})^3} = -\frac{np^3}{(\overline{X_n})^2} < 0$

$\Rightarrow \hat{\theta}_n$ point de max.

$E(X_i) = p\theta \Rightarrow E(\hat{\theta}_n) = \frac{E(X)}{p} = \theta \rightarrow \hat{\theta}_n$ sans biais

$\overline{X_n} \xrightarrow{P.S} E(X) \Rightarrow \hat{\theta}_n \xrightarrow{P.S} \theta \Rightarrow \hat{\theta}_n$ fort. conv.

2) $Y = \frac{2\overline{X_n}}{\theta} = \frac{2n}{\theta} Z$ avec $Z = \sum_{i=1}^n X_i \sim \gamma(np, \frac{1}{\theta})$.

$P[Y \leq y] = P[\frac{2nZ}{\theta} \leq y] = F(\frac{\theta}{2n} y)$

$g(y) = \frac{\theta}{2n} f(\frac{\theta}{2n} y) = \frac{\theta}{2n} \cdot \frac{1}{\theta^p} e^{-\frac{\theta}{2n} \cdot \frac{y}{\theta}} (\frac{\theta}{2n} y)^{p-1} \mathbb{1}_{y>}$

$= \frac{1}{(2n)^p} \cdot e^{-\frac{y}{2}} \cdot y^{p-1} \mathbb{1}_{y>0}$.

Donc, la loi de Y ne dép pas de θ.

3) Un estimateur sans biais de θ : $\hat{\theta}_n = \frac{\overline{X_n}}{p}$

$1-\alpha = P[a < \frac{2\overline{X_n}}{\theta} < b]$ avec $a = u_{\frac{\alpha}{2}}$ fractile de la

$$1-\alpha = \mathbb{P}\left[\frac{2\overline{X}_n}{u_{1-\frac{\alpha}{2}}} \leq \theta \leq \frac{2\overline{X}_n}{u_{\frac{\alpha}{2}}}\right]$$

$$A_n = \frac{2\overline{X}_n}{u_{1-\frac{\alpha}{2}}}, \quad B_n = \frac{2\overline{X}_n}{u_{\frac{\alpha}{2}}}$$

4) $H_0 : \theta = \theta_0 \qquad H_1 : \theta > \theta_0$.

$$L_1(\theta) = \frac{\theta^{np}}{\Gamma^n(p)} e^{-\frac{n\overline{X}_n}{\theta}} \cdot \prod_{i=1}^{n} X_i^{p-1} \mathbb{1}_{x_i > 0}$$

$$L_0 = L(\theta_0) = \frac{\theta_0^{np}}{\Gamma^n(p)} \cdot \exp\left(-\frac{n\overline{X}_n}{\theta_0}\right) \prod_{i=1}^{n} X_i^{p-1} \mathbb{1}_{x_i > 0}$$

Il faut det k :

$$\alpha = \mathbb{P}\left[L_1 > k L_0 \mid \theta = \theta_0\right]$$

$$= \mathbb{P}\left[\left(\frac{\theta}{\theta_0}\right)^{np} \cdot \exp\left(-\left(\frac{1}{\theta} - \frac{1}{\theta_0}\right) n\overline{x}_n\right) > k_1 \mid \theta = \theta_0\right]$$

$$= \mathbb{P}\left[\exp \frac{\theta - \theta_0}{\theta_0 \theta} n\overline{X}_n > k_2 \mid \theta = \theta_0\right]$$

$$= \mathbb{P}\left[\frac{n\overline{X}_n}{\theta} > k_3 \mid \theta = \theta_0\right]$$

$$= \mathbb{P}\left[\frac{2\overline{X}_n}{\theta} > k_4 \mid \theta = \theta_0\right]$$

$$= \mathbb{P}\left[\frac{2\overline{X}_n}{\theta_0} > k_4\right]$$ avec k_4 le fractile d'ordre $1-\alpha$ de la loi g.

Donc, la zone de rejet de H_0 est :

$$R = \left\{(x_1, \dots, x_n), \ \frac{2\overline{X}_n}{\theta} > u_{1-\alpha}\right\}$$

Exercice 4

$X \sim \mathcal{N}(m, \sigma^2)$ σ inconnu. $m_0 = 20$

$H_0 : m = m_0$ $H_1 : m > m_0$

$\bar{x}_n = 20,89$ $s_n^* = 1,193$

Stat de test $Z_n = \sqrt{n} \dfrac{\bar{X}_n - m_0}{S_n^*} \underset{H_0}{\sim} t(8)$

Zone de rejet : $R = \{ Z_n > u_{1-\alpha} \} = \{ Z_n > \cancel{2,306} \ 1,86 \}$

Val stat de test :

$$Z_n = 3 \cdot \frac{20,89 - 20}{1,193} = 2,238 \in R$$

Donc H_1 acceptée, H_0 rejetée.