a) $f_X(x) = \int_0^\infty f(x,y;\theta)\,dy = \int_0^\infty \theta^{-2} x^{\frac{1}{\theta}-1} e^{-\frac{y}{\theta}}\,dy$

$= \theta^{-2} x^{\frac{1}{\theta}-1} \int_0^\infty e^{-\theta^{-1}y}\,dy = -\theta^{-1} x^{\frac{1}{\theta}-1} e^{-\frac{y}{\theta}}\Big]_0^\infty$

$= \theta^{-1} x^{\frac{1}{\theta}-1} \mathbb{1}_{0<x<1}$ c'est la densité de la loi ~~Exp~~ $\beta\left(\frac{1}{\theta}, 1\right)$

$f_Y(y) = \int_0^x f(x,y;\theta)\,dx = \theta^{-2} e^{-\frac{y}{\theta}} \int_0^1 x^{\frac{1}{\theta}-1}\,dx = \theta^{-1} e^{-\frac{y}{\theta}} \int_0^1 \left(x^{\frac{1}{\theta}}\right)' dx$

$= \theta^{-1} e^{-\frac{y}{\theta}} \mathbb{1}_{y>0}$ c'est la densité de la loi $Exp\left(\frac{1}{\theta}\right)$

b) $Z = -\log(X)$ soit g sa densité et G sa f de rép.

$G(x) = \mathbb{P}[Z < x] = \mathbb{P}[-\log(X) < x] = \mathbb{P}[X > e^{-x}] = 1 - \mathbb{P}(X < e^{-x})$

La f de rép de X :

$0<x<1$ $\quad F_X(x) = \int_0^x \theta^{-1} t^{\frac{1}{\theta}-1}\,dt = + t^{\frac{1}{\theta}}\Big]_0^x = x^{\frac{1}{\theta}}$

Donc :

$$F_X(x) = \begin{cases} 0 & 0 > x \\ x^{\frac{1}{\theta}} & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$\Rightarrow X$ prend que des valeurs positives

$$G(z) = 1 - F_X(e^{-x}) = \begin{cases} 1 - e^{-\frac{x}{\theta}} & , x > 0 \\ 0 & , x < 0 \end{cases}$$

Sa densité :

$g(z) = G'(z) = \theta^{-1} e^{-\frac{z}{\theta}} \mathbb{1}_{z>0} = f_Y(y)$

c) la vraisemblance :

$L_n(\theta) = \prod_{i=1}^n f(X_i, Y_i; \theta) = \theta^{-2n} \left(\prod_{i=1}^n X_i\right)^{\frac{1}{\theta}-1} \cdot e^{-\frac{n\bar{Y}_n}{\theta}}$

$\log L_n(\theta) = -2n \log\theta - \left(\frac{1}{\theta} - 1\right) \cdot n \cdot \bar{Z}_n - \frac{n\bar{Y}_n}{\theta}$

$\frac{\partial \log L_n(\theta)}{\partial\theta} = -\frac{2n}{\theta} + \frac{1}{\theta^2} n\bar{Z}_n + \frac{n\bar{Y}_n}{\theta^2} = 0 \Rightarrow \boxed{\hat{\theta}_n = \frac{\bar{Y}_n + \bar{Z}_n}{2}}$

②

$$\frac{\partial \log L_n(\theta)}{\partial \theta} = \frac{2n}{\theta^2} - \frac{2}{\theta^3} n \bar{Z}_n - \frac{2n}{\theta^3} \bar{Y}_n = \frac{2n}{\theta^3}\left[\theta - \bar{Z}_n - \bar{Y}_n\right]$$

$$\frac{\partial^2 \log L_n(\hat{\theta}_n)}{\partial \theta} = \frac{2n}{\hat{\theta}_n^3}\left[\frac{\bar{Y}_n + \bar{Z}_n}{2} - (\bar{Y}_n + \bar{Z}_n)\right] = -\frac{2n}{\hat{\theta}^3}\frac{(\bar{Y}_n + \bar{Z}_n)}{2}$$

$$= -\frac{2n}{\hat{\theta}_n^2} < 0 \Rightarrow \hat{\theta}_n \text{ point de max.}$$

$$E(Y) = \frac{1}{\theta}\int_0^\infty y e^{-\frac{y}{\theta}} dy = -y e^{-\frac{y}{\theta}}\Big]_0^\infty + \int_0^\infty e^{-\frac{y}{\theta}} dy = \int_0^\infty e^{-\frac{y}{\theta}} dy$$

$$= -\theta e^{-\frac{y}{\theta}}\Big]_0^\infty = \theta \Rightarrow \boxed{E(Z) = \theta}$$

$$E(Y^2) = \frac{1}{\theta}\int_0^\infty y^2 e^{-\frac{y}{\theta}} dy = -y^2 e^{-\frac{y}{\theta}}\Big]_0^\infty + 2\int_0^\infty y e^{-\frac{y}{\theta}} dy$$

$$= 2\theta E(Y) = 2\theta^2 \Rightarrow \boxed{Var(Y) = Var(Z) = \theta^2}$$

<u>Propriétés</u> :

$$E(\hat{\theta}_n) = \frac{E(Y) + E(Z)}{2} = \theta \Rightarrow \hat{\theta}_n \text{ sans biais}$$

Convergence :

$$\left.\begin{array}{l} \bar{Y}_n \xrightarrow{p.s} E(Y) = \theta \\ \bar{Z}_n \xrightarrow{p.s} E(Y) = \theta \end{array}\right\} \Rightarrow \hat{\theta}_n \xrightarrow{p.s} \theta$$

fort. consist

<u>Efficacité</u>. X_i et Y_i indép $\Rightarrow Z_i$ et Y_i indép $\Rightarrow \bar{Z}_n$ et $\bar{Y}_n$ indép

$$Var(\hat{\theta}_n) = \frac{1}{4}\left[Var(\bar{Y}_n) + Var(\bar{Z}_n)\right] = \frac{Var(\bar{Y}_n)}{2} = \frac{\theta^2}{2n}$$

L'information de Fisher :

$$I_1(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\log f(X, Y; \theta)\right] = -E\left[\frac{\partial^2}{\partial\theta^2}\left(-2\log\theta + \left(\frac{1}{\theta} - 1\right)\log X - \frac{Y}{\theta}\right)\right] = -E\left[\frac{2}{\theta^2} - \frac{2}{\theta^3} Z - \frac{2Y}{\theta^3}\right] = \frac{2}{\theta^3}\left[E(Z) + E(Y) - \theta\right] = \frac{2}{\theta^2}$$

$$\Rightarrow Var(\hat{\theta}_n) = \frac{1}{n I(\theta)} \Rightarrow \hat{\theta}_n \text{ efficace.}$$

valle de conf. asymptotique

indép

Par le TLC : $\sqrt{n}\,\frac{\bar{Y}_n-\theta}{\theta}\xrightarrow[n\to\infty]{\mathcal{L}}\mathcal{N}(0,1)$ et $\sqrt{n}\,\frac{\bar{Z}_n-\theta}{\theta}\xrightarrow{\mathcal{L}}\mathcal{N}(0,1)$

Alors : $S_1=\sqrt{n}\,\frac{\bar{Y}_n+\bar{Z}_n}{\theta}-2\xrightarrow[n\to\infty]{\mathcal{L}}\mathcal{N}(0,2)$

$$=2\sqrt{n}\,\frac{\hat{\theta}_n}{\theta}-2=2\left(\sqrt{n}\,\frac{\hat{\theta}_n}{\theta}-1\right)$$

Donc : $S_2=\sqrt{2}\left(\sqrt{n}\,\frac{\hat{\theta}_n}{\theta}-1\right)\xrightarrow[n\to\infty]{\mathcal{L}}\mathcal{N}(0,1)$

Pour un niveau α fixé :

$$\mathbb{P}\left[|S_2|<u_{1-\frac{\alpha}{2}}\right]=1-\alpha$$

$u_{1-\frac{\alpha}{2}}$ la fractile d'ordre $1-\frac{\alpha}{2}$ de la loi $\mathcal{N}(0,1)$

$$\mathbb{P}\left[-u_{1-\frac{\alpha}{2}}<\sqrt{2}\left(\sqrt{n}\,\frac{\hat{\theta}_n}{\theta}-1\right)<u_{1-\frac{\alpha}{2}}\right]=1-\alpha$$

$$\mathbb{P}\left[n^{-1/2}\left(1-\frac{u_{1-\alpha/2}}{\sqrt{2}}\right)<\frac{\hat{\theta}_n}{\theta}<n^{-1/2}\left(1+\frac{u_{1-\alpha/2}}{\sqrt{2}}\right)\right]=1-\alpha$$

$$\mathbb{P}\left[\frac{n^{1/2}}{\hat{\theta}_n}\left(1+\frac{u_{1-\alpha/2}}{\sqrt{2}}\right)^{-1}<\theta<\frac{n^{1/2}}{\hat{\theta}_n}\cdot\left(1-\frac{u_{1-\alpha/2}}{\sqrt{2}}\right)^{-1}\right]=1-\alpha$$

Donc, l'intervalle de confiance asymptotique pour θ est :

$$\left[\frac{2\sqrt{n}}{\bar{Y}_n+\bar{Z}_n}\left(1+\frac{u_{1-\alpha/2}}{\sqrt{2}}\right)^{-1};\ \frac{2\sqrt{n}}{\bar{Y}_n+\bar{Z}_n}\left(1-\frac{u_{1-\alpha/2}}{\sqrt{2}}\right)^{-1}\right]$$

$\dots_i + \varepsilon_{ij} \quad \to \hat{m}_i = \overline{Y_{i\cdot}} = \frac{1}{n_i} \sum_{j=1}^{n} Y_{ij} = m_i + \overline{\varepsilon_{i\cdot}}$ (4)

$\frac{1}{\sqrt{n}} \sum_{i=1}^{k} (\xi_i - m_i)^2 \quad , \quad \overline{\varepsilon_{i\cdot}} \sim \mathcal{N}(0, \frac{1}{n_i}) \Rightarrow \xi_i = \sqrt{n_i} \cdot \overline{\varepsilon_{i\cdot}} = \sqrt{n} \cdot \overline{\varepsilon_{i\cdot}}$

$\cancel{\hat{m}_i = m_i + \sqrt{n_i} \cdot \xi_i} \qquad \hat{m}_i = m_i + \frac{\xi_i}{\sqrt{n_i}} = m_i + \frac{\xi_i}{\sqrt{n}}$

$T_n = \sqrt{n}\left(\sum_{i=1}^{k} \left(m_i + \frac{\xi_i}{\sqrt{n}} \right)^2 - 1 \right) = \sqrt{n} \sum m_i^2 + 2 \sum_{i=1}^{k} \xi_i m_i + \frac{1}{\sqrt{n}} \sum_{i=1}^{4} \xi_i$

c) $\frac{1}{\sqrt{n}} \sum_{i=1}^{k} \xi_i^2 = \sqrt{n} \sum_{i=1}^{k} (\overline{\varepsilon_{i\cdot}})^2 = \frac{1}{n\sqrt{n}} \sum_{i=1}^{k} \left(\sum_{j=1}^{n} \varepsilon_{ij} \right)^2$

On a que $\overline{\varepsilon_{i\cdot}} \xrightarrow{P} E$

Parce que k est fixe, il suffit de montrer que: $\frac{1}{\sqrt{n}} \xi_i \xrightarrow{P} 0$

On applique B-Tchebychev.

Sous H_0 : $\sum m_i^2 - 1 = 0 \Rightarrow T_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{4} \xi_i^2 + 2 \sum_{i=1}^{k} \xi_i m_i$

$\sum_{i=1}^{k} \xi_i m_i \sim \mathcal{N}(0, \sum m_i^2) \Rightarrow T_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4)$

d) $\sqrt{n} (\sum m_i^2 - 1) \longrightarrow \infty \Rightarrow T_n \to \infty$

e) ~~Stat de test~~ : On calcule :

$T_n = \frac{1}{\sqrt{n}} \cancel{\sum_{i=1}^{k}} \sqrt{n} \left(\sum_{i=1}^{k} \overline{Y}_{i\cdot}^2 - 1 \right) \xrightarrow[H_0]{\mathcal{L}} \mathcal{N}(0, 4)$