Combinatorial interpretations of Jacobi-Stirling numbers

Yoann Gelineau
Université Claude Bernard Lyon 1, France

September 29th, 2009 - 63th SLC

The Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$ satisfy the Jacobi equation :

$$(1-t^2)y''(t)+(\beta-\alpha-(\alpha+\beta+2)t)y'(t)+n(n+\alpha+\beta+1)y(t)=0.$$

The Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$ satisfy the Jacobi equation :

$$(1-t^2)y''(t)+(\beta-\alpha-(\alpha+\beta+2)t)y'(t)+n(n+\alpha+\beta+1)y(t)=0.$$

Let $\ell_{\alpha,\beta}[y](t)$ be the Jacobi differential operator :

$$\ell_{\alpha,\beta}[y](t) = \frac{1}{(1-t)^{\alpha}(1+t)^{\beta}} \left(-(1-t)^{\alpha+1}(1+t)^{\beta+1}y'(t)\right)'$$

So, $P_n^{(\alpha,\beta)}(t)$ is a solution of

$$\ell_{\alpha,\beta}[y](t) = n(n+\alpha+\beta+1)y(t)$$

Everitt and al. gave the expansion of the n-th composite power of $\ell_{\alpha,\beta}$, involving a sequence of positive integers $P^{(\alpha,\beta)}S(n,k)$:

$$(1-t)^{\alpha}(1+t)^{\beta}\ell^{n}_{\alpha,\beta}[y](t) = \sum_{k=0}^{n}(-1)^{k}\left(P^{(\alpha,\beta)}S(n,k)(1-t)^{\alpha+k}(1+t)^{\beta+k}y^{(k)}(t)\right)^{(k)}$$

Everitt and al. gave the expansion of the n-th composite power of $\ell_{\alpha,\beta}$, involving a sequence of positive integers $P^{(\alpha,\beta)}S(n,k)$:

$$(1-t)^{\alpha}(1+t)^{\beta}\ell^{n}_{\alpha,\beta}[y](t) = \sum_{k=0}^{n}(-1)^k\left(P^{(\alpha,\beta)}S(n,k)(1-t)^{\alpha+k}(1+t)^{\beta+k}y^{(k)}(t)\right)^{(k)}$$

Actually, $P^{(\alpha,\beta)}S(n,k)$ depends only on $z=\alpha+\beta+1$. We denote it by $JS_n^k(z)$, the **Jacobi-Stirling number of the second kind**.

The $JS_n^k(z)$ numbers are the relation coefficients in the following formula :

$$X^n = \sum_{k=0}^{n} \mathrm{JS}_n^k(z) \prod_{i=0}^{k-1} (X - i(z+i))$$

The $JS_n^k(z)$ numbers are the relation coefficients in the following formula :

$$X^n = \sum_{k=0}^{n} \mathrm{JS}_n^k(z) \prod_{i=0}^{k-1} (X - i(z+i))$$

Equivalently, these numbers can be defined by the recurrence relation :

$$\mathrm{JS}_n^k(z) = \mathrm{JS}_{n-1}^{k-1}(z) + k(k+z)\mathrm{JS}_{n-1}^k(z), \qquad n, k \geq 1$$

$$\mathrm{JS}_0^0(z) = 1, \qquad \mathrm{JS}_n^k(z) = 0 \quad \text{if } k \notin \{1, \ldots, n\}$$

The **Stirling numbers (of the second kind)** S_n^k are defined by the relation :

$$S_n^k = S_{n-1}^{k-1} + kS_{n-1}^k$$

The **Stirling numbers (of the second kind)** S_n^k are defined by the relation :

$$S_n^k = S_{n-1}^{k-1} + kS_{n-1}^k$$

They count the number of :

- partitions of $[n] := \{1, 2, \ldots, n\}$ in k non-empty blocks.
- supdiagonal quasi-permutations of $[n] := \{1, 2, \ldots, n\}$ with k empty lines.

For example,

$$\pi = \big\{\{1,3,6\},\{2,5\},\{4\}\big\}$$

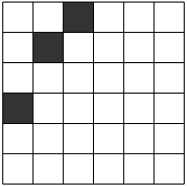
is a partition of [6] in 3 blocks.

For example,

$$\pi = \big\{\{1,3,6\},\{2,5\},\{4\}\big\}$$

is a partition of [6] in 3 blocks.

It corresponds to the following quasi-permutation :



The **central factorial numbers (of the second kind)** U_n^k are defined by the relation :

$$U_n^k = U_{n-1}^{k-1} + k^2 U_{n-1}^k$$

The **central factorial numbers (of the second kind)** U_n^k are defined by the relation :

$$U_n^k = U_{n-1}^{k-1} + k^2 U_{n-1}^k$$

They count the number of :

- ordered pairs (π_1, π_2) of partitions of $[n]$ in k blocks, with $\min(\pi_1) = \min(\pi_2)$.
- ordered pairs (Q_1, Q_2) of supdiagonal quasi-permutations of $[n]$, with Q_1 and Q_2 which have k same empty lines.

For example, if we note

$$\pi_1 = \big\{\{1,3,6\},\{2,5\},\{4\}\big\}, \quad \pi_2 = \big\{\{1\},\{2,3,5\},\{4,6\}\big\},$$

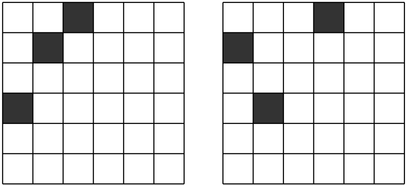
π_1 and π_2 are partitions of [6] in 3 blocks, with $\min(\pi_1) = \min(\pi_2)$

For example, if we note

$$\pi_1 = \{\{1,3,6\},\{2,5\},\{4\}\}, \quad \pi_2 = \{\{1\},\{2,3,5\},\{4,6\}\},$$

π_1 and π_2 are partitions of [6] in 3 blocks, with $\min(\pi_1) = \min(\pi_2)$

The ordered pair (π_1, π_2) corresponds to the folloqing ordered pair of supdiagonal quasi-permutations :



$\Rightarrow$ Let's come back to the Jacobi-Stirling numbers :

$$\mathrm{JS}_n^k(z) = \mathrm{JS}_{n-1}^{k-1}(z) + k(k+z)\mathrm{JS}_{n-1}^k(z), \qquad n, k \geq 1$$

$\Rightarrow$ Let's come back to the Jacobi-Stirling numbers :

$$\mathrm{JS}_n^k(z) = \mathrm{JS}_{n-1}^{k-1}(z) + k(k+z)\mathrm{JS}_{n-1}^k(z), \qquad n, k \geq 1$$

$\mathrm{JS}_n^k(z)$ is a polynomial in z of degree $n-k$:

$$\mathrm{JS}_n^k(z) = a_{n,k}^{(0)} + a_{n,k}^{(1)} z + \cdots + a_{n,k}^{(n-k)} z^{n-k}$$

$\Rightarrow$ Let's come back to the Jacobi-Stirling numbers :

$$\mathrm{JS}_n^k(z) = \mathrm{JS}_{n-1}^{k-1}(z) + k(k+z)\mathrm{JS}_{n-1}^k(z), \qquad n, k \geq 1$$

$\mathrm{JS}_n^k(z)$ is a polynomial in z of degree $n-k$:

$$\mathrm{JS}_n^k(z) = a_{n,k}^{(0)} + a_{n,k}^{(1)} z + \cdots + a_{n,k}^{(n-k)} z^{n-k}$$

Moreover,

$$a_{n,k}^{(n-k)} = S_n^k$$

$$a_{n,k}^{(0)} = U_n^k$$

Definition

*A k-**signed partition** of $[\pm n]_0 = \{0, \pm 1, \pm 2, \dots, \pm n\}$ is a partition of $[\pm n]_0$ in $k+1$ non-empty blocks $B_0, B_1, \dots, B_k$, such that*

- $0 \in B_0$ *et* $\forall i \in [n], \{i, -i\} \not\subset B_0$
- $\forall j \in [k], \forall i \in [n], \{i, -i\} \subset B_j \Leftrightarrow i = \min B_j \cap [n]$

For example,

$$\pi = \{\{0, 2, -5\}, \{\pm 1, -2\}, \{\pm 3\}, \{\pm 4, 5\}\}$$

is a 3-signed partition of $[\pm 5]_0$.

Theorem

$a_{n,k}^{(i)}$ is equal to the number of k-signed partitions of $[\pm n]_0$ with i negative values in the block that contains 0.

Theorem

$a_{n,k}^{(i)}$ is equal to the number of k-signed partitions of $[\pm n]_0$ with i negative values in the block that contains 0.

Proof : Since $\mathrm{JS}_n^k(z)$ verify the relation :

$$\mathrm{JS}_n^k(z) = \mathrm{JS}_{n-1}^{k-1}(z) + k(k+z)\,\mathrm{JS}_{n-1}^k(z),$$

it suffices to check that the wanted numbers verify the recurrence :

$$a_{n,k}^{(i)} = a_{n-1,k-1}^{(i)} + k a_{n-1,k}^{(i-1)} + k^2 a_{n-1,k}^{(i)}$$

Theorem

$a_{n,k}^{(i)}$ is equal to the number of k-signed partitions of $[\pm n]_0$ with i negative values in the block that contains 0.

Proof : Since $\mathrm{JS}_n^k(z)$ verify the relation :

$$\mathrm{JS}_n^k(z) = \mathrm{JS}_{n-1}^{k-1}(z) + k(k+z)\,\mathrm{JS}_{n-1}^k(z),$$

it suffices to check that the wanted numbers verify the recurrence :

$$a_{n,k}^{(i)} = \underbrace{a_{n-1,k-1}^{(i)}}_{B_k=\{\pm n\}} + \underbrace{k a_{n-1,k}^{(i-1)}}_{-n\in B_0} + \underbrace{k^2 a_{n-1,k}^{(i)}}_{\text{other cases}}$$

$a_{n,k}^{(i)} = \sharp\{k - \text{signed partitions of } [\pm n]_0 \text{ with } i \text{ values} < 0 \text{ in } B_0\}$

$a_{n,k}^{(i)} = \sharp\{k -$ signed partitions of $[\pm n]_0$ with i values < 0 in $B_0\}$

$\Rightarrow$ For $i = n - k$, we recover the interpretation of S_n^k.

$$\pi = \big\{\{0, -3, -5, -6\}, \{\pm 1, 3, 6\}, \{\pm 2, 5\}, \{\pm 4\}\big\}$$

$$\Downarrow$$

$$\pi' = \big\{\{1, 3, 6\}, \{2, 5\}, \{4\}\big\}$$

$a_{n,k}^{(i)} = \sharp\{k-$ signed partitions of $[\pm n]_0$ with i values < 0 in $B_0\}$

$\Rightarrow$ <u>For $i = n - k$</u>, we recover the interpretation of S_n^k.

$$\pi = \big\{\{0, -3, -5, -6\}, \{\pm 1, 3, 6\}, \{\pm 2, 5\}, \{\pm 4\}\big\}$$

$$\Downarrow$$

$$\pi' = \big\{\{1, 3, 6\}, \{2, 5\}, \{4\}\big\}$$

$\Rightarrow$ <u>For $i = 0$</u>, we recover the interpretation of U_n^k.

$$\pi = \big\{\{0, 3, 6\}, \{\pm 1, -3, \}, \{\pm 2, -5\}, \{\pm 4, 5, -6\}\big\}$$

$$\Downarrow$$

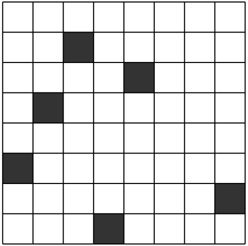
$$\pi_1 = \big\{\{1, 3\}, \{2\}, \{4, 5, 6\}\big\}, \quad \pi_2 = \big\{\{1, 3\}, \{2, 5\}, \{4, 6\}\big\}$$

Definition

A **simply hooked** k**-quasi-permutation of** $[n]$ *(k-SHQP of $[n]$) is a part Q of a tableau $[n] \times [n]$ such that :*

- *Q is contained in the graph of a permutation σ without fix points,*
- *each diagonal hook contains at most one elemnt,*
- *there are k empty diagonal hooks.*

For example,

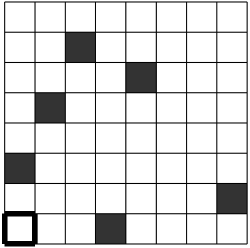


is a 2-SHQP of [8].

Definition

A **simply hooked** k-**quasi-permutation of** $[n]$ *(k-SHQP of $[n]$) is a part Q of a tableau $[n] \times [n]$ such that :*

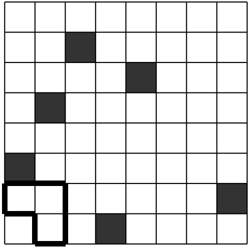
- *Q is contained in the graph of a permutation σ without fix points,*
- *each diagonal hook contains at most one elemnt,*
- *there are k empty diagonal hooks.*

For example,  is a 2-SHQP of [8].

Definition

A **simply hooked** k**-quasi-permutation of** $[n]$ *(k-SHQP of $[n]$) is a part Q of a tableau $[n] \times [n]$ such that :*

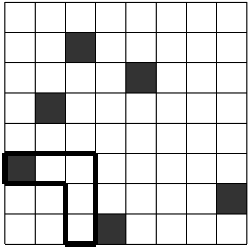
- *Q is contained in the graph of a permutation σ without fix points,*
- *each diagonal hook contains at most one elemnt,*
- *there are k empty diagonal hooks.*

For example,  is a 2-SHQP of [8].

Definition

A **simply hooked** k**-quasi-permutation of** $[n]$ *(k-SHQP of $[n]$) is a part Q of a tableau $[n] \times [n]$ such that :*

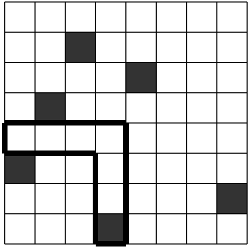
- *Q is contained in the graph of a permutation σ without fix points,*
- *each diagonal hook contains at most one elemnt,*
- *there are k empty diagonal hooks.*

For example,  is a 2-SHQP of [8].

Definition

A **simply hooked** k**-quasi-permutation of** $[n]$ *(k-SHQP of $[n]$) is a part Q of a tableau $[n] \times [n]$ such that :*

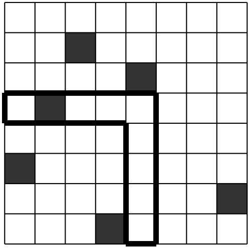
- *Q is contained in the graph of a permutation σ without fix points,*
- *each diagonal hook contains at most one elemnt,*
- *there are k empty diagonal hooks.*

For example,  is a 2-SHQP of [8].

Definition

A **simply hooked** k**-quasi-permutation of** $[n]$ *(k-SHQP of $[n]$) is a part Q of a tableau $[n] \times [n]$ such that :*

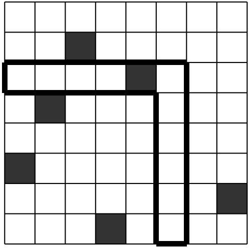
- *Q is contained in the graph of a permutation σ without fix points,*
- *each diagonal hook contains at most one elemnt,*
- *there are k empty diagonal hooks.*

For example,  is a 2-SHQP of [8].

Definition

A **simply hooked** k**-quasi-permutation of** $[n]$ *(k-SHQP of $[n]$) is a part Q of a tableau $[n] \times [n]$ such that :*

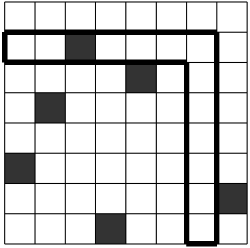
- *Q is contained in the graph of a permutation σ without fix points,*
- *each diagonal hook contains at most one elemnt,*
- *there are k empty diagonal hooks.*

For example,  is a 2-SHQP of [8].

Definition

A **simply hooked** k**-quasi-permutation of** $[n]$ *(*k*-SHQP of* $[n]$*) is a part* Q *of a tableau* $[n] \times [n]$ *such that :*

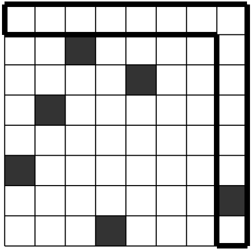
- Q *is contained in the graph of a permutation* σ *without fix points,*
- *each diagonal hook contains at most one elemnt,*
- *there are* k *empty diagonal hooks.*

For example,  is a 2-SHQP of [8].

Definition

A **simply hooked** k**-quasi-permutation of** $[n]$ *(k-SHQP of* $[n]$*) is a part Q of a tableau* $[n] \times [n]$ *such that :*

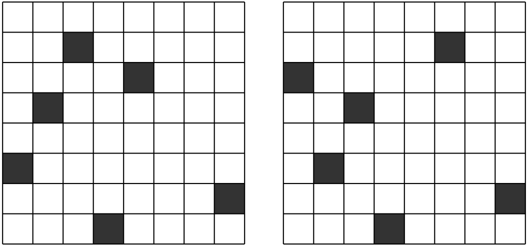
- *Q is contained in the graph of a permutation* σ *without fix points,*
- *each diagonal hook contains at most one elemnt,*
- *there are k empty diagonal hooks.*

For example,  is a 2-SHQP of [8].

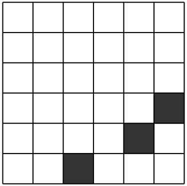
Theorem

$a_{n,k}^{(i)}$ *counts also the number of ordered pairs* (Q_1, Q_2) *of* k*-SHQP of* $[n]$ *such that*

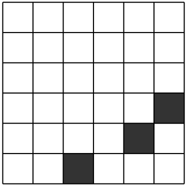
- Q_1 *and* Q_2 *have* k *empty lines (the same),*
- Q_1 *and* Q_2 *have identical subdiagonal parts,*
- Q_1 *and* Q_2 *have* i *filled boxes in their subdiagonal parts.*



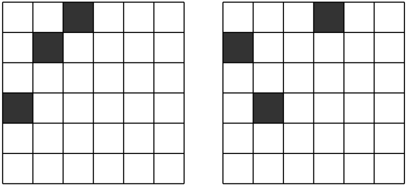
⇒ Pour $i = n - k$, we recover the interpretation of S_n^k.



⇒ Pour $i = n - k$, we recover the interpretation of S_n^k.



⇒ For $i = 0$, we recover the interpretation of U_n^k.



The **Jacobi-Stirling numbers (of the first kind)** $\mathrm{js}_n^k(z)$ are defined by inversing the relation on the $\mathrm{JS}_n^k(z)$ numbers :

$$X^n = \sum_{k=0}^{n} \mathrm{JS}_n^k(z) \prod_{i=0}^{k-1} (X - i(z+i))$$

$$\prod_{i=0}^{n-1} (X - i(z+i)) = \sum_{k=0}^{n} (-1)^{n-k} \mathrm{js}_n^k(z) X^k$$

The **Jacobi-Stirling numbers (of the first kind)** $\mathrm{js}_n^k(z)$ are defined by inversing the relation on the $\mathrm{JS}_n^k(z)$ numbers :

$$X^n = \sum_{k=0}^{n} \mathrm{JS}_n^k(z) \prod_{i=0}^{k-1} (X - i(z+i))$$

$$\prod_{i=0}^{n-1} (X - i(z+i)) = \sum_{k=0}^{n} (-1)^{n-k} \mathrm{js}_n^k(z) X^k$$

It follows that they verify the following relation :

$$\mathrm{js}_n^k(z) = \mathrm{js}_{n-1}^{k-1}(z) + (n-1)(n-1+z)\,\mathrm{js}_{n-1}^k(z)$$

The **Stirling numbers (of the first kind)** s_n^k are defined by the relation :

$$s_n^k = s_{n-1}^{k-1} + (n-1)s_{n-1}^k$$

The **central factorial numbers (of the first kind)** u_n^k are defined by the relation :

$$u_n^k = u_{n-1}^{k-1} + (n-1)^2 u_{n-1}^k$$

The **Stirling numbers (of the first kind)** s_n^k are defined by the relation :

$$s_n^k = s_{n-1}^{k-1} + (n-1)s_{n-1}^k$$

The **central factorial numbers (of the first kind)** u_n^k are defined by the relation :

$$u_n^k = u_{n-1}^{k-1} + (n-1)^2 u_{n-1}^k$$

s_n^k is the number of permutations of $[n]$ with k cycles.
u_n^k had still no interpretation until present.

$\mathrm{js}_n^k(z)$ is a polynomial in z of degree $n-k$:

$$\mathrm{js}_n^k(z) = b_{n,k}^{(0)} + b_{n,k}^{(1)} z + \cdots + b_{n,k}^{(n-k)} z^{n-k}$$

$\text{js}_n^k(z)$ is a polynomial in z of degree $n-k$:

$$\text{js}_n^k(z) = b_{n,k}^{(0)} + b_{n,k}^{(1)} z + \cdots + b_{n,k}^{(n-k)} z^{n-k}$$

Moreover,

$$b_{n,k}^{(n-k)} = s_n^k$$

$$b_{n,k}^{(0)} = u_n^k$$

Definition

Let $w = w(1)w(2)\dots w(\ell)$ be a word on the alphabet $[n]$. A letter $w(j)$ is a **record** *of w if*

$$w(j) < w(k), \qquad \forall k = 1 \dots j-1$$

We denote by $rec(w)$ the number of records of w and $rec_0(w) = rec(w) - 1$.

For example, for

$$w = 5\,7\,4\,8\,6\,2\,3\,1\,9,$$

the records are

$$5, 4, 2, 1.$$

Thus, $\mathrm{rec}(w) = 4$ and $\mathrm{rec}_0(w) = 3$.

Theorem

$b_{n,k}^{(i)}$ *is the number of ordered pairs* (σ, τ) *with :*

- σ *is a permutation of* $[n] \cup \{0\}$ *with* k *cycles,*
- τ *is a permutation of* $[n]$ *with* k *cycles,*
- σ *and* τ *have the same cyclic minimas,*
- $1 \in Orb_\sigma(0)$ *and* $rec_0(w) = i$
  where $w = \sigma(0) \ldots \sigma^\ell(0)$ *with* $\sigma^{\ell+1}(0) = 0$.

Theorem

$b_{n,k}^{(i)}$ is the number of ordered pairs (σ, τ) with :

- *σ is a permutation of $[n] \cup \{0\}$ with k cycles,*
- *τ is a permutation of $[n]$ with k cycles,*
- *σ and τ have the same cyclic minimas,*
- *$1 \in Orb_\sigma(0)$ and $rec_0(w) = i$ where $w = \sigma(0) \ldots \sigma^\ell(0)$ with $\sigma^{\ell+1}(0) = 0$.*

Idea : interpret the formula

$$b_{n,k}^{(i)} = b_{n-1,k-1}^{(i)} + (n-1) b_{n-1,k}^{(i-1)} + (n-1)^2 b_{n-1,k}^{(i)}$$

$\Rightarrow$ For $i = n - k$, we recover the interpretation of s_n^k.

⇒ For $i = n - k$, we recover the interpretation of s_n^k.

⇒ For $i = 0$, we recover the interpretation of u_n^k.

u_n^k is the number of ordered pairs of permutations (σ_1, σ_2) of $[n]$ with k cycles, with same cyclic minimas.

Thanks for your attention.