The Transfer-matrix Method

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For the last century, the graph theory has been exploited in order to solve mathematics problems more practically, with aspects more concrete. In his book *Enumerative Combinatorics (Volume 2)*, Richard P. Stanley develops the principle of *Transfer-matrix method* which, using combinatorial and algebraic results, can offer some interesting conclusions on counting issues.

1 Definitions and notations

Let G be a finite **directed graph**, id est a triple (V, E, φ) where :

- $V = \{v_1, v_2, ..., v_p\}$ is a finite set of **vertices** ;
- E is a finite set of directed **edges** ;
- φ is a map from E to $V^2 = V \times V$.

If $\varphi(e) = (u, v)$, then we say that e is an edge **from u to v**, with **initial vertex** u and **final vertex** v, denoted by $u = \mathrm{I}(e)$ and $v = \mathrm{F}(e)$.
In the special case where $u = v$, the edge e is called a **loop**.

A **walk** Γ in G of **length** n from u to v (we'll say a **n-walk** from u to v) is a sequence $e_1e_2...e_n$ of n edges such that $\mathrm{I}(e_1) = u$, $\mathrm{F}(e_n) = v$ and for $1 \leq i < n$, $\mathrm{F}(e_i) = \mathrm{I}(e_{i+1})$.
In the special case where $u = v$, then Γ is called a **closed walk based at u**.

Now we want the edges to be weighted, so we take a **weight function** $\omega : E \longrightarrow R$ with R a commutative ring (we can take $R = \mathbb{C}$ for example).
If $\Gamma = e_1e_2...e_n$ is a walk, then we define the **weight** of Γ by

$$\omega(\Gamma) = \omega(e_1)\omega(e_2)...\omega(e_n).$$

For $i, j \in [p] = \{1, 2..., p\}$ and $n \in \mathbb{N}$, we define

$$A_{ij}(n) = \sum \omega(\Gamma)$$

where the sum is over all n-walks from v_i to v_j.
In particular $A_{ij}(0) = \delta_{ij}$.
The aim of the *Transfer matrix method* is to evaluate $A_{ij}(n)$.

2 Adjacency matrix

We can easily interpret the number $A_{ij}(n)$ as an entry in a $p \times p$ matrix.

Let $A = (A_{ij})_{i,j\in[p]} \in \mathcal{M}_p(R)$ defined by

$$A_{ij} = \sum \omega(e)$$

where the sum is over all edges from v_i to v_j.

We note that

$$\forall i, j \in [p],\ A_{ij} = A_{ij}(1).$$

The matrix A is called the **adjacency matrix** of the graph G with respect to the weight function ω.

Theorem 1 *Let $n \in \mathbb{N}$. Then the (i,j)-entry of the matrix A^n is equal to $A_{ij}(n)$.*

Proof : By induction on n.

For $n = 1$, the propriety is trivially verified (by definition of the adjacency matrix).

For $n \geq 1$, we suppose que the (i,j)-entry of the matrix A^n is equal to $A_{ij}(n)$, id est $A^n_{i,j}$ is the sum of weights on every n-walk from v_i to v_j. A $(n+1)$-walk from v_i to v_j is a n-walk from v_i to a vertice v_k, followed by an edge from v_k to v_j. By induction hypothesis, the sum of all weights on n-walks from v_i to v_k is the number $A_{i,k}(n)$. Moreover, the sum of all weights on edges from v_k to v_j is $A_{k,j}$. Then there are exactly $A_{i,k}(n) \times A_{k,j}$ weights on $(n+1)$-walks from v_i to v_j ending by $(v_k v_j)$. So the sum of weights on $(n+1)$-walks from v_i to v_j is

$$\sum_{k=1}^{p} A_{i,k}(n) \times A_{k,j} = A_{i,j}(n+1).$$

□

3 Generating function

Now we want to analyze the behavior of the function $A_{ij}(n)$. Let's define the generating function of $A_{ij}(n)$:

$$F_{ij}(G, \lambda) = \sum_{n\geq 0} A_{ij}(n)\lambda^n.$$

Theorem 2 *The generating function $F_{ij}(G,\lambda)$ is given by the identity*

$$F_{ij}(G, \lambda) = (-1)^{i+j} \frac{\det((I - \lambda A) \diamond (j,i))}{\det(I - \lambda A)}$$

where $(M \diamond (j,i))$ represents the matrix obtained from M by removing the j-th row and the i-th column of M.

Thus in particular, $F_{ij}(G,\lambda)$ is a rational function of λ whose degree $< n_0$, with n_0 multiplicity of 0 as an eigenvalue of A.

Proof : Thanks to Theorem 1, we know that $A_{ij}(n) = A^n_{i,j}$. Then $F_{ij}(G,\lambda)$ is the (i,j)-th coefficient of the matrix $\sum_{n\geq 0} \lambda^n A^n$.

On the right hand, since R is a commutative ring, the series $\sum_{n\geq 0} \lambda^n A^n$ is geometric and we know that

$$\sum_{n\geq 0} \lambda^n A^n = (I - \lambda A)^{-1}.$$

On the other hand, we know that if B is any invertible matrix (ie it exists B^{-1} as $BB^{-1} = B^{-1}B = I$) then we know that $B^{-1} = \frac{1}{\det B}\, {}^t\mathrm{com}B$ where the matrix ${}^t\mathrm{com}B$ is the co-matrix of B (ie matrix with (i,j)-th term equal to $(-1)^{i+j}\det(B \diamond (i,j))$)

$$\implies \quad B^{-1}_{ij} = (-1)^{i+j}\,\frac{\det(B \diamond (j,i))}{\det B}.$$

Let's suppose that $A \in \mathcal{M}_p(R)$. We know that $\det(A - \lambda I)$ is the characteristic polynomial of A and we note n_0 the multiplicity of zero as root of the characteristic polynomial :

$$\det(A - \lambda I) = (-1)^p(\lambda^p + \alpha_1\lambda^{p-1} + ... + \alpha_{p-n_0}\lambda^{n_0}).$$

Then

$$\det(I - \lambda A) = 1 + \alpha_1\lambda + ... + \alpha_{p-n_0}\lambda^{p-n_0}$$

ie $\det(I - \lambda A)$ is a polynomial in λ which verified $\deg\det(I - \lambda A) = p - n_0$. And since $\deg\det(I - \lambda A \diamond (j,i)) \leq p - 1$, it follows that

$$\deg F_{ij}(G,\lambda) \leq p - 1 - (p - n_0) < n_0.$$

$\square$

Special case

Let

$$C_G(n) = \sum \omega(\Gamma)$$

where the sum is over all closed n-walks in G.

Clearly, we have $C_G(1) = \mathrm{tr}(A)$.

Corollary 1 *Let $Q(\lambda) = \det(I - \lambda A)$. Then*

$$\sum_{n\geq 1} C_G(n)\lambda^n = \frac{-\lambda\, Q'(\lambda)}{Q(\lambda)}.$$

Proof : Thanks to Theorem 1, we have $C_G(n) = \sum_{i=1}^{p} A_{ii}(n) = \mathrm{tr}(A^n)$.

We can call $\omega_1, \omega_2, ...\ \omega_q$ the eigenvalues of A which are not null. Then $\mathrm{tr}(A^n) = \omega_1^n + ... + \omega_q^n$, so

$$\sum_{n\geq 1} C_G(n)\lambda^n = \frac{\omega_1\,\lambda}{1 - \omega_1\lambda} + ... + \frac{\omega_q\,\lambda}{1 - \omega_q\lambda}$$

and the result follows putting all over the denominator $(1-\omega_1\lambda)...(1-\omega_q\lambda) = Q(\lambda)$ since the numerator becomes $-\lambda Q'(\lambda)$.

4 Counting words on a finite alphabet without fixed factors

Let $\mathcal{A}$ be the 3-letter alphabet $\mathcal{A} = \{1, 2, 3\}$. Let $\mathcal{A}^*$ be the set of finite words on $\mathcal{A}$.

Problem 1 : we want to count how many words $u = a_1 a_2 ... a_n \in \mathcal{A}^*$ exist such that neither 11 nor 23 is a factor of u (is neither 11 nor 23 appear as consecutive terms $a_i a_{i+1}$ in u).

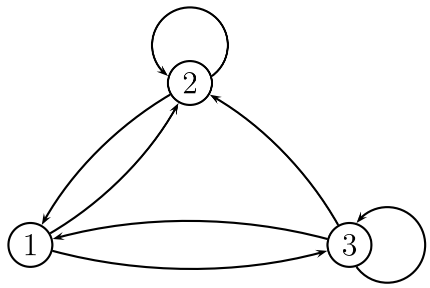

Figure 1

Let $f(n)$ be the wanted number.

Let G be the directed graph given by Figure 1. G clearly represents the situation : if there is an edge between i and j, the factor ij is allowed in the word u. The words of length n are the $(n-1)$-walks on G and if we set $\omega(e) = 1$ for every edge e, then

$$f(n) = \sum_{i,j \in \{1,2,3\}} A_{ij}(n-1).$$

We have $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

By simple calculation, we obtain $Q(\lambda) = \det(I - \lambda A) = 1 - 2\lambda - \lambda^2 + \lambda^3$.

Setting $Q_i j(\lambda) = \det((I - \lambda A) \diamond (j, i))$, from Theorem 2 we deduce that

$$F(\lambda) = \sum_{n \geq 0} f(n+1)\lambda^n = \sum_{i,j \in \{1,2,3\}} \frac{(-1)^{i+j} Q_{ij}(\lambda)}{Q(\lambda)}.$$

Theorem 2 affirms that $\deg F(\lambda) < 0$ so, since the denominator is a 3-degree polynomial, the numerator has to be a polynomial of degree at most 2. The determination of the numerator only needs 3 particular values : $f(1) = 3$, $f(2) = 7$, $f(3) = 16$. It follows that

$$F(\lambda) = \frac{3 + \lambda - \lambda^2}{1 - 2\lambda - \lambda^2 + \lambda^3}.$$

Problem 2 : now, among the results above, we want to impose that $a_n a_1$ is also different of 11 or 23. How many words are suitable ?

Let $g(n)$ be the wanted number.

Clearly, $g(n)$ represents the number of closed n-walks in G. By Corollary 1, we can affirm that

$$\sum_{n \geq 0} g(n)\lambda^n = \frac{-\lambda\, Q'(\lambda)}{Q(\lambda)} = \frac{-\lambda(-2 - 2\lambda + 3\lambda^2)}{1 - 2\lambda - \lambda^2 + \lambda^3}.$$

Problem 3 : we want to count how many words $u = a_1a_2...a_n \in \mathcal{A}^*$ exist such that neither 12 nor 213 , 222 , 231, or 313 is a factor of u (is neither 11 nor 213 , 222 , 231, or 313 appear as consecutive terms a_ia_{i+1} or $a_ja_{j+1}a_{j+2}$ in u).

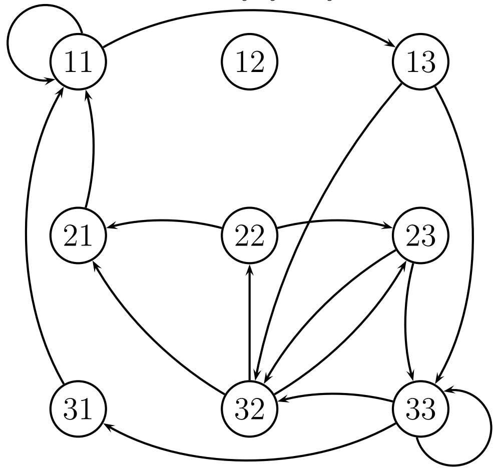

Figure 2

Let $h(n)$ be the wanted number.

Let H be the directed graph given by `Figure 2`. H clearly represents the situation : if there is an edge between ab and ac, the factor abc is allowed in the word u. The words of length n are the $(n-2)$-walks on H and if we set $\omega(e) = 1$ for every edge e, then

$$h(n) = \sum_{ab,cd \in \{H\}} A_{ab,cd}(n-2).$$

Here we have $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$.

Thus, $\sum_{n\geq 0} h(n)\lambda^n$ is a rational function with denominator $Q(\lambda) = \det(I - \lambda A)$.

Proposition 1 *Let $\mathcal{A}$ be a finite alphabet, $\mathcal{A}^*$ the set of finite words on $\mathcal{A}$. Let $\mathcal{F}$ be a finite subset of $\mathcal{A}^*$. Let $f(n)$ be the number of words $a_1a_2...a_n \in \mathcal{A}^*$ such that no factor $a_ia_{i+1}...a_{i+j}$ is in $\mathcal{F}$.*
Then

$$\sum_{n\geq 0} f(n)\lambda^n \in \mathbb{C}(\lambda).$$

5 Counting permutations such that $|\sigma(i) - i| \leq 1$

Let $\mathcal{S}_n$ be the set of permutations of $[n] = \{1, 2, ..., n\}$

<u>Problem 1</u> : we want to count how many permutations $\sigma \in \mathcal{S}_n$ exist such that

$$\forall i, |\sigma(i) - i| \leq 1.$$

Let $f(n)$ be the wanted number.

For $i \in [n]$, we only have three possibilities to $\sigma(i)$: $i-1$, i, or $i+1$. Furthermore, the values for $\sigma(i)$ depend on the choices made for $\sigma(i-1)$ and $\sigma(i-2)$. Let's try and represent the situation by a graph.

Set G be a directed graph whose vertices consist of pairs $(a, b) \in \{0, \pm 1\}$ for which it is possible to have $\left| \begin{array}{l} \sigma(i) - i = a \\ \sigma(i+1) - i - 1 = b \end{array} \right.$. An edge will connect (a, b) to (b, c) if it is possible to have $\left| \begin{array}{l} \sigma(i) - i = a \\ \sigma(i+1) - i - 1 = b \\ \sigma(i+2) - i - 2 = c \end{array} \right.$. We deduce that the only possible vertices are $v_1 = (-1, -1)$, $v_2 = (-1, 0)$, $v_3 = (-1, 1)$, $v_4 = (0, 0)$, $v_5 = (0, 1)$, $v_6 = (1, -1)$, $v_7 = (1, 1)$.

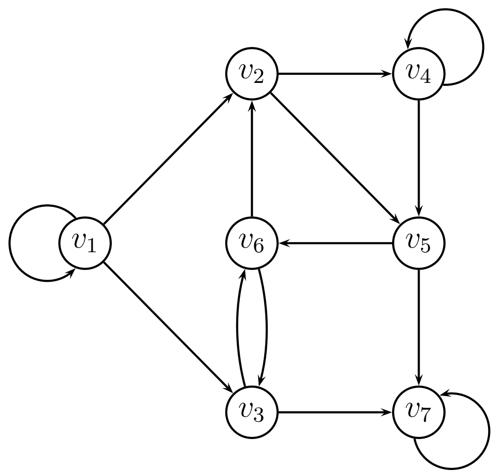

Figure 3

Then, a n-walk in G is a sequence $((\alpha_1, \alpha_2), (\alpha_2, \alpha_3)), ((\alpha_2, \alpha_3), (\alpha_3 \alpha_4)), ..., ((\alpha_n, \alpha_{n+1}), (\alpha_{n+1}, \alpha_n))$ which represents the permutation $\sigma \in \mathcal{S}_{n+2}$ such that $\forall i \in [n+2], \sigma(i) = i + \alpha_i$.

We need to impose $\alpha_1 \neq -1$ and $\alpha_{n+2} \neq 1$, thus $f(n+2)$ represents the number of n-walks in G from $v_i \in \{v_4, v_5, v_6, v_7\}$ to $v_j \in \{v_1, v_2, v_4, v_6\}$. If we set $\omega(e) = 1$ for every edge $e \in G$, then

$$f(n+2) = \sum_{i=4,5,6,7} \sum_{j=1,2,4,6} (A^n)_{ij}.$$

By a calculation, we obtain $Q(\lambda) = \det(I - \lambda A) = (1 - \lambda^2)(1 - \lambda - \lambda^2)$.

Like in example 1, $(1 - \lambda^2)(1 - \lambda - \lambda^2) \sum_{n \geq 0} f(n+2)\lambda^n$ has a degree less larger than $\deg Q(\lambda) + 3 = 6$: we just need the initial values $f(0), f(1), ...f(6)$. By this method, we find :

$$\sum_{n \geq 0} f(n)\lambda^n = \frac{1}{1 - \lambda - \lambda^2}$$

so $f(n) = F_{n+1}$ where F_n is the **n-th Fibonnaci number** defined by $\left\{ \begin{array}{l} F_1 = 1, \ F_2 = 1 \\ \forall n, F_{n+2} = F_{n+1} + F_n \end{array} \right.$

Problem 2 : we want to count how many permutations $\sigma \in \mathcal{S}_n$ exist such that

$$\forall i, \sigma(i) - i \equiv 0, \pm 1 \pmod{n}.$$

Let $g(n)$ be the wanted number.

It is indeed the same problem than before, but we can allow that $a_1 = n$ and $a_n = 1$. Thus, $g(n)$ is just the number of closed n-walks of the form

$$((\alpha_1, \alpha_2), (\alpha_2, \alpha_3)), ((\alpha_2, \alpha_3), (\alpha_3, \alpha_4)), ..., ((\alpha_{n-1}, \alpha_n), (\alpha_n, \alpha_1)), ((\alpha_n, \alpha_1), (\alpha_1, \alpha_2)).$$

By Corollary 1, we can affirm that

$$\sum_{n \geq 0} g(n)\lambda^n = \frac{-\lambda/Q'(\lambda)}{Q(\lambda)} = \frac{2\lambda}{1-\lambda} + \frac{\lambda(1+2\lambda)}{1-\lambda-\lambda^2},$$

so $g(n) = L_n + 2$ where L_n is the **n-th Lucas number** defined by $\begin{cases} L_1 = 1, \ L_2 = 3 \\ \forall n, L_{n+2} = L_{n+1} + L_n \end{cases}$

Proposition 2 *Let A be a finite subset of $\mathbb{Z}$. Let's note $f_A(n)$ the number of permutations $\sigma \in \mathcal{S}_n$ such that $\forall i \in \mathbb{N}_n, \sigma(i) - i \in A$.*
Then

$$\sum_{n \geq 0} f_A(n)\lambda^n \in \mathbb{C}(\lambda).$$

Proposition 3 *Let A be a finite subset of $\mathbb{Z}$. Let's note $g_A(n)$ the number of permutations $\sigma \in \mathcal{S}_n$ such that $\forall i \in \mathbb{N}_n, \exists j \in A \ / \ \sigma(i) - i \equiv j \pmod{n}$.*
Then there is a polynomial $Q(\lambda) \in \mathbb{C}[\lambda]$ for which

$$\sum_{n \geq 0} g_A(n)\lambda^n = \frac{-\lambda Q'(\lambda)}{Q(\lambda)}.$$

6 Conclusion

The transfer-matrix method has simple theoretical foundations but however implies results which are very appreciable : first the combinatorial aspect which links the number $A_{ij}(n)$ to the powers of the adjacency matrix, and also the algebraic part which gives the rationality of the generating function of $A_{ij}(n)$, a result which is not obvious at first sight.
Thus, concrete tools like graphs can help reduce combinatorial problems in easier results, and the transfer-matrix method is a good example of applications they can have in mathematics nowadays.

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