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Aspects of Analysis

Curvature Criterion

Isoperimetry

Evolution Equations

MATRIX PRESS
Mathematical Notebooks
Vol III

Published by Matrix Press ltd.

Cataloging in Publication Data
Ivan Gentil, Zhongmin Qian, Cyril Roberto
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Aspects of Analysis Curvature Criterion, Isoperimetry, Evolution Equations.
Version 1.41
ISBN 1-905760-03-5

Mathematical Notebooks Vol.3, Editor B.Zegarlinski

2000 Mathematics Subject Classification : 35K05 60J60 26D10 35Q30
KeyWords: *Curvature Criterion, Isoperimetry, Product Measures*
Navier-Stocks Equations.
InternetSearchKey: **MathX**

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http://matrixpress.co.uk

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Curvature Criterion and Applications

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1 Ornstein-Uhlenbeck semigroup.

We start investigating a particular Markov semigroup: the Ornstein-Uhlenbeck semigroup. First we recall the definition of Markov semigroup.

Definition 1.0.1. A *Markov semigroup* is a family of linear operators $(P_t)_{t\geq 0}$ defined on a Banach space $\big(\mathcal{B}, \|\cdot\|\big)$ and such that

1. $P_0 = Id$,
2. $t \mapsto P_t f$ is continuous, $\forall\, f \in \mathcal{B}$,
3. $P_{t+s} f = P_t(P_s f), \qquad \forall\, f \in \mathcal{B},\ \forall\, t, s \geq 0$,
4. $P_t 1 = 1, \qquad \forall\, t \geq 0$,
5. $f \geq 0 \Rightarrow P_t f \geq 0, \qquad \forall f \in \mathcal{B},\ \ \forall\, t \geq 0$,
6. $\|P_t f\| \leq \|f\|, \qquad \forall f \in \mathcal{B},\ \ \forall\, t \geq 0$.

Note that properties $(1) - (3)$ define a semigroup while properties (4) and (5) tell the Markov property. Moreover we assume property (6) (which in general is not required), which means the contractivity of the semigroup.
The Banach space under our consideration will be in general $\mathcal{B} = L^\infty(\mathbb{R}^n)$ with the usual norm $\|\cdot\|_\infty$.

Definition 1.0.2. For any $f \in L^\infty(\mathbb{R}^n)$ and $t \geq 0$, we define

$$P_t f(x) := \int f\left(\mathrm{e}^{-t} x + \sqrt{1 - \mathrm{e}^{-2t}}\, y\right) d\gamma(y), \tag{1.1}$$

where $d\gamma(y)$ is the standard Gaussian measure in $\mathbb{R}^n$, i.e.

$$d\gamma(y) := \frac{\mathrm{e}^{-\frac{|y|^2}{2}}}{(2\pi)^{\frac{n}{2}}} dy.$$

The semigroup $(P_t)_{t\geq 0}$ is called *the Ornstein-Uhlenbeck semigroup.*

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Remark 1.0.1. The Ornstein-Uhlenbeck semigroup is related to the Ornstein-Uhlenbeck process. More precisely, let us consider the Markov semigroup $(X_t)_{t\geq 0}$ given by the solution of the following stochastic defferential equation:

$$\begin{cases} dX_t = \sqrt{2}dB_t - X_t dt, \\ X_0 = x. \end{cases}$$

where by B_t we indicate the standard Brownian motion in $\mathbb{R}^n$.
Then, by explicit computations, it is possible to show that, for

$$Q_t f(x) := \mathbb{E}_x[f(X_t)],$$

one has $Q_t f(x) = P_t f(x)$, where $\{P_t\}_{t\geq 0}$ is the Ornstein-Uhlenbeck semigroup defined by (1.1).

Proposition 1.0.1. The Ornstein-Uhlenbeck semigroup given by (1.1) is a Markov semigroup in $\mathcal{B} = \big(L^\infty(\mathbb{R}^n), \|\cdot\|_\infty\big)$.

Proof. First we note that property (1) of Definition 1.0.1 holds. In fact

$$P_0 f = \int f(x+0)d\gamma(y) = f(x)\int d\gamma(y) = f(x),$$

since $d\gamma(y)$ is a probability measure.
We want to check property (3) of Definition 1.0.1. For sake of simplicity we set $c_t := \mathrm{e}^{-t}$ and $d_t := \sqrt{1-\mathrm{e}^{-2t}}$. By Remark 1.0.1 the Ornstein-Uhlenbeck semigroup can be written as

$$P_t f(x) = \mathbb{E}[f(c_t x + d_t Y)],$$

where Y is a normally distributed random variable with mean zero and variance 1, i.e. $Y \sim \mathcal{N}(0,1)$. (Note that by $\sim$ we denote that two random variables have the same law.) Hence

$$\begin{aligned} &P_t(P_s f)(x) = \\ &\mathbb{E}[(P_s f)(c_t x + d_t Z)] = \mathbb{E}[f(c_s(c_t x + d_t Z) + d_s Y)] = \mathbb{E}[f(c_{s+t}x + c_s d_t Z + d_s Y)]. \end{aligned}$$

Let $T := c_s d_t Z + d_s Y$, since Y and Z are two independent random variables, then $T \sim \mathcal{N}(0, c_s^2 d_t^2 + d_s^2)$. Since

$$c_s^2 d_t^2 + d_s^2 = \mathrm{e}^{-2s}(1-\mathrm{e}^{-2t}) + 1 - \mathrm{e}^{-2s} = 1 - \mathrm{e}^{-2(t+s)} = d_{t+s}^2,$$

we get $T \sim \mathcal{N}(0, d_{t+s}^2) \sim d_{t+s}\mathcal{N}(0,1) \sim d_{t+s}\,Y$.
Hence we can conclude

$$P_t(P_s f)(x) = \mathbb{E}[f(c_{s+t}x + d_{t+s}\,y)] = P_{t+s}f(x).$$

We omit to verify the other properties: the reader can check the remaining properties as an exercise. □

1.1 The infinitesimal generator for the Ornstein-Uhlenbeck semigroup.

The infinitesimal generator associated to a Markov semigroup is defined as the derivative in time of the semigroup itself.

Let us recall the following notation: $f \in C_b^2(\mathbb{R}^n)$ means $f \in C^2(\mathbb{R}^n)$ bounded with bounded first-order and second-order derivatives.

Proposition 1.1.1. Let $(P_t)_{t\geq 0}$ be the Ornstein-Uhlenbeck semigroup, then for any $f \in C_b^2(\mathbb{R}^n)$ and $t \geq 0$

$$\frac{\partial}{\partial t} P_t f = L(P_t f) = P_t(Lf),$$

where $Lf := \Delta f - x \cdot \nabla f$.
L is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup.

Proof. By definition

$$P_t f(x) = \int f\left(\mathrm{e}^{-t} x + \sqrt{1 - \mathrm{e}^{-2t}}\, y\right) d\gamma(y),$$

which implies

$$\frac{\partial}{\partial t} P_t f(x) = \int \left(-\mathrm{e}^{-t} x + \frac{\mathrm{e}^{-2t}}{\sqrt{1 - \mathrm{e}^{-2t}}}\, y\right) \cdot \nabla f\big(\mathrm{e}^{-t} x + \sqrt{1 - \mathrm{e}^{-2t}}\, y\big) d\gamma(y). \tag{1.2}$$

Since $d\gamma(y)$ is a Gaussian measure, so $\forall\, f \in C_b^1(\mathbb{R}^n)$,

$$\int y_i f(y) d\gamma(y) = \int \partial_i f(y) d\gamma(y), \qquad i = 1, \ldots, n;$$

Therefore, by integration by parts, we can write

$$\int \frac{\mathrm{e}^{-2t}}{\sqrt{1 - \mathrm{e}^{-2t}}}\, y \cdot \nabla f\big(\mathrm{e}^{-t} x + \sqrt{1 - \mathrm{e}^{-2t}}\, y\big) d\gamma(y) = \int \mathrm{e}^{-2t} \Delta\, f\big(\mathrm{e}^{-t} x + \sqrt{1 - \mathrm{e}^{-2t}}\, y\big) d\gamma(y).$$

Using that $\Delta P_t f(x) = \mathrm{e}^{-2t} P_t(\Delta f(x))$, the previous identity yields

$$\int \frac{\mathrm{e}^{-2t}}{\sqrt{1 - \mathrm{e}^{-2t}}}\, y \cdot \nabla f\big(\mathrm{e}^{-t} x + \sqrt{1 - \mathrm{e}^{-2t}}\, y\big) d\gamma(y) = \Delta P_t f(x).$$

To conclude we need just to remark that

$$\nabla P_t f(x) = \mathrm{e}^{-t} P_t\big(\nabla f\big) = \mathrm{e}^{-t} \big(P_t(\partial_i f)\big)_{1 \leq i \leq n}\,.$$

Hence the identity (1.2) gives

$$\frac{\partial}{\partial t}P_tf(x) = -x\cdot\nabla P_tf(x) + \Delta P_tf(x) = L(P_tf)(x).$$

To prove that $\frac{\partial}{\partial t}P_tf(x) = P_t(Lf)$ one can proceed similarly. □

Remark 1.1.1. Let $(P_t)_{t\geq 0}$ be a linear semigroup with generator L defined on a domain $\mathcal{D} = \mathcal{D}(L)$, then the semigroup always commutes with the generator, i.e.

$$P_t(Lf) = L(P_tf),$$

for any $f \in \mathcal{D}$.

Proof. Let $f \in \mathcal{D}$ and recall that the infinitesimal generator is defined as $L(P_tf) := \frac{\partial}{\partial t}P_tf$, which in particular implies for $t = t_0$

$$L(P_{t_0}f) := \frac{\partial}{\partial t}P_tf\Big|_{t=t_0}.$$

By the semigroup property we know that $P_{t+t_0}f = P_{t_0}(P_tf)$. Using the linearity and the continuity of the semigroup, we can conclude

$$L(P_{t_0}f) = \frac{\partial}{\partial t}P_{t+t_0}f\Big|_{t=0} = \frac{\partial}{\partial t}P_{t_0}(P_tf)\Big|_{t=0} = P_{t_0}\frac{\partial}{\partial t}P_tf\Big|_{t=0} = P_{t_0}(Lf).$$

□

Remark 1.1.2 (Properties of the Ornstein-Uhlenbeck semigroup)**.** Let $(P_t)_{t\geq 0}$ be the Ornstein-Uhlenbeck semigroup and γ the standard Gaussian measure, then

(i) *P_t is γ-ergodic*, that is

$$\lim_{t\to+\infty} P_tf(x) = \int f d\gamma.$$

(ii) *The Gaussian measure γ is invariant under P_t*, that means for any $t \geq 0$

$$\int P_tf d\gamma = \int f d\gamma,$$

which is equivalent to say that for any $f \in \mathcal{D}(L)$

$$\int Lf d\gamma = 0;$$

(iii) *L is a self-adjoint operator in $L^2(d\gamma)$*, i.e.

$$\int fLg d\gamma = \int gLf d\gamma = -\int \nabla g\nabla f d\gamma.$$

1.2 The Poincaré inequality.

As we have already remarked in the proof of Proposition 1.1.1, that for the Ornstein-Uhlenbeck semigroup the following identity holds

$$\nabla P_t f(x) = \mathrm{e}^{-t} P_t(\nabla f)(x), \quad \forall\, x \in \mathbb{R}^n,\ \forall\, t \geq 0. \tag{1.3}$$

From the identity above and by the Jensen's inequality, we can deduce

$$|\nabla P_t f|(x) \leq \mathrm{e}^{-t} P_t\big(|\nabla f|\big)(x), \tag{1.4}$$

where $|\cdot|$ is the standard norm in $\mathbb{R}^n$. Inequality (1.4) is called *local inequality.*

Proposition 1.2.1 (The Poincaré inequality)**.** Let f be smooth enough. Then

$$\mathrm{Var}_\gamma(f) \leq \int |\nabla f|^2 d\gamma, \tag{1.5}$$

where

$$\mathrm{Var}_\gamma(f) := \int f^2 d\gamma - \left(\int f d\gamma\right)^2.$$

Proof. Note that, by ergodicity and using the Fubini theorem and the definition of infinitesimal generator, we have

$$\begin{aligned}\mathrm{Var}_\gamma(f) = -\int \left(\int_0^{+\infty} \frac{\partial}{\partial t}(P_t f)^2 dt\right) d\gamma &= -\int_0^{+\infty} \left(\int \frac{\partial}{\partial t}(P_t f)^2 d\gamma\right) dt \\ &= -2\int_0^{+\infty} \left(\int L(P_t f)\, P_t f d\gamma\right) dt.\end{aligned}$$

We apply now the property (iii) given in Remark 1.1.2. Then

$$\mathrm{Var}_\gamma(f) = -2\int_0^{+\infty} \left(\int L(P_t f)\, P_t f d\gamma\right) dt = 2\int_0^{+\infty} \left(\int |\nabla P_t f|^2 d\gamma\right) dt$$

By the local inequality (1.4) and Jensen's inequality, we can conclude that in fact

$$\begin{aligned}\mathrm{Var}_\gamma(f) &= 2\int_0^{+\infty} \left(\int |\nabla P_t f|^2 d\gamma\right) dt \leq 2\int_0^{+\infty} \left(\int \mathrm{e}^{-2t}\big(P_t(|\nabla f|)\big)^2 d\gamma\right) dt \\ &\leq 2\int_0^{+\infty} \mathrm{e}^{-2t} \left(\int P_t(|\nabla f|^2) d\gamma\right) dt = 2\int_0^{+\infty} \mathrm{e}^{-2t} \left(\int |\nabla f|^2 d\gamma\right) dt \\ &= \int |\nabla f|^2 d\gamma \;\; 2\int_0^{+\infty} \mathrm{e}^{-2t} dt = \int |\nabla f|^2 d\gamma,\end{aligned}$$

where we used $2\int_0^{+\infty} \mathrm{e}^{-2t} dt = 1$. □

Remark 1.2.1. The constant 1 is the optimal constant in the inequality (1.5).

An interesting application of the Poincaré inequality is given by the following corollary. In fact, we already know that the Ornstein-Uhlenbeck semigroup is ergodic (see Remark 1.1.2, property (i)). So a natural question is to understand the exponential rate of convergence to the mean, which is answered by the following corollary.

Corollary 1.2.1. Under the assumption of the Poincaré inequality

$$\mathrm{Var}_\gamma(P_t f) \leq \mathrm{e}^{-2t}\mathrm{Var}_\gamma(f).$$

Proof. Since γ is an invariant measure under P_t, we have

$$U(t) := \mathrm{Var}_\gamma(P_t f) = \int (P_t f)^2 d\gamma - \left(\int f d\gamma\right)^2,$$

and

$$U'(t) = -2\int |\nabla P_t f|^2 d\gamma.$$

The Poincaré inequality tells $U'(t) \leq -2U(t)$. Therefore, by the Gronwall's Lemma, we can conclude

$$\mathrm{Var}_\gamma(P_t f) \leq \mathrm{e}^{-2t}\mathrm{Var}_\gamma(f).$$

□

1.3 The Logarithmic-Sobolev inequality.

We are now going to prove the following result.

Proposition 1.3.1 (The Logarithmic-Sobolev inequality)**.** For f smooth enough,

$$\mathrm{Ent}_\gamma(P_t f) \leq \frac{1}{2}\int \frac{|\nabla f|^2}{f} d\gamma, \tag{1.6}$$

where $\mathrm{Ent}_\gamma(f)$ is called *entropy of* f and it is defined for any $f \geq 0$ as

$$\mathrm{Ent}_\gamma(f) := \int f \log f d\gamma - \int f d\gamma \, log \int f d\gamma.$$

Proof. To get (1.6), we apply the previous calculation to the entropy instead of to the variance. Note that

$$\mathrm{Ent}_\gamma(f) = -\int_0^{+\infty} \left(\int \frac{\partial}{\partial t} P_t f \log(P_t f) d\gamma \right) dt = -\int_0^{+\infty} \left(\int L(P_t f) \log(P_t f) d\gamma \right) dt \tag{1.7}$$

where we have used that γ is invariant under P_t, which means $\int L(P_t f) d\gamma = 0$. Using that L is self-adjoint, we get

$$\int L(P_t f) \log(P_t f) d\gamma = -\int \nabla P_t f \; \nabla\big(\log(P_t f)\big) d\gamma = \int \frac{|\nabla P_t f|^2}{P_t f} d\gamma.$$

Hence, (1.7) becomes

$$\mathrm{Ent}_\gamma(f) = \int_0^{+\infty} \left(\int \frac{|\nabla P_t f|^2}{P_t f} d\gamma \right) dt = \int_0^{+\infty} \mathrm{e}^{-2t} \left(\int \frac{P_t(|\nabla f|^2)}{P_t f} d\gamma \right) dt. \tag{1.8}$$

The Cauchy-Schwartz inequality tells that

$$P_t(|\nabla f|^2) = P_t \left(\frac{|\nabla f|}{\sqrt{f}} \sqrt{f} \right)^2 \leq P_t \left(\frac{|\nabla f|^2}{f} \right) P_t f,$$

which implies

$$\frac{P_t(|\nabla f|^2)}{P_t f} \leq P_t \left(\frac{|\nabla f|^2}{f} \right).$$

Applying this inequality to (1.8), we get

$$\begin{aligned} \mathrm{Ent}_\gamma(f) \leq \int_0^{+\infty} \mathrm{e}^{-2t} \left(\int P_t \left(\frac{|\nabla f|^2}{f} \right) d\gamma \right) dt &= \int \frac{|\nabla f|^2}{f} d\gamma \int_0^{+\infty} \mathrm{e}^{-2t} dt \\ &= \frac{1}{2} \int \frac{|\nabla f|^2}{f} d\gamma. \end{aligned}$$

□

Remark 1.3.1. The constant $\frac{1}{2}$ is the optimal constant in the inequality (1.6).

As an application of the Logarithmic-Sobolev inequality, we show that the Lyapunov function given by the relative entropy is exponentially decreasing.

Corollary 1.3.1. For any $t \geq 0$ and f smooth enough,

$$\mathrm{Ent}_\gamma(P_t f) \leq \mathrm{e}^{-2t} \mathrm{Ent}_\gamma(f).$$

1.4 The Ornstein-Uhlenbeck semigroup and the Fokker-Planck equation.

The Fokker-Planck equation is the linear PDE defined by

$$\frac{\partial}{\partial t}u(t,x) = \nabla \cdot \big(\nabla u + xu\big) = \mathrm{div}(\nabla u) + \mathrm{div}(xu) = \Delta u + \mathrm{div}(xu) =: G(u). \quad (1.9)$$

For this equation in general there exists a unique solution. So one can define an associated semigroup, that we indicate by $(G_t)_{t\geq 0}$, which is a linear semigroup but it is not a Markov semigroup. In fact $G(1) \neq 0$.
To look at a steady state means to study the behavior of $u(t,x)$ as $t \to +\infty$.
The corresponding equation is

$$\nabla U(x) + xU(x) = 0,$$

which holds if and only if $U(x) = C\, \mathrm{e}^{-\frac{|x|^2}{2}}$ for some constant C.
The equation conserves the mass so, if we consider an initial datum $u(0,x) = u_0(x)$ such that $\int u_0(x)dx = 1$, then $\int u(t,x)dx = 1$ for any $t > 0$.
We assume that $u(t,x) \to u_\infty(x) = U(x)$, as $t \to +\infty$, then the constant C has to satisfy $\int u_\infty(x)dx = 1$. This implies

$$u_\infty(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\mathrm{e}^{-\frac{|x|^2}{2}},$$

i.e. the steady state is given by $u_\infty(x)dx = \gamma(dx)$.
We would like to prove that a solution u of (1.9) converges to u_∞ as $t \to +\infty$.

Lemma 1.4.1. $L^* = G$ in $L^2(dx)$.

Proof. Integrating by parts

$$\int fG(g)dx = \int f\Delta g dx + \int f\nabla\cdot(xg)dx = \int g\Delta f dx - \int g(x\cdot\nabla f)dx = \int gL(f)dx.$$

□

Proposition 1.4.1. Let $u(t,x)$ be the solution of (1.9), then $u(t,x) \to u_\infty(x)$ in $L^2(dx)$ as $t \to +\infty$.

Proof. We define

$$U(t) := \int \left|\frac{u(t,x)}{u_\infty(x)} - 1\right|^2 u_\infty(x)dx = \int \left(\frac{u(t,x)}{u_\infty(x)}\right)^2 u_\infty(x)dx - 1,$$

since $\int u_\infty(x)dx = 1 = \int u(t,x)dx$.
Using that $u(t,x)$ is a solution of (1.9), that means $\frac{\partial}{\partial t}u(t,x) = Gu(t,x) = L^* u(t,x)$, we can deduce

$$U'(t) = 2\int \frac{u(t,x)}{u_\infty(x)} L^*(u(t,x))dx = 2\int L\left(\frac{u(t,x)}{u_\infty(x)}\right)\frac{u(t,x)}{u_\infty(x)} d\gamma(x),$$

where we have used $dx = \frac{d\gamma(x)}{u_\infty(x)}$. To conclude, we have just to observe that

$$U'(t) = -2\int \left|\nabla\left(\frac{u(t,x)}{u_\infty(x)}\right)\right|^2 d\gamma(x) \le -2U(t).$$

This implies

$$\int \left|\frac{u(t,x)}{u_\infty(x)} - 1\right|^2 d\gamma(x) \le \mathrm{e}^{-2t}\int \left|\frac{u_0(x)}{u_\infty(x)} - 1\right|^2 d\gamma(x),$$

which converges to 0 whenever $t \to +\infty$. □

A last remark about the connections between the Fokker-Planck equation and the Ornstein-Uhlenbeck semigroup $(P_t)_{t\ge 0}$ is the following. First we need to recall Remark 1.0.1, that tells

$$P_t f(x) = \mathbb{E}[f(X_t)],$$

where X_t solves the SODE

$$dX_t = \sqrt{2}\, dB_t - X_t dt.$$

By an explicit calculation, one can show that

$$\int P_t f(x) d\nu(x) = \int f(x)\mathcal{L}(X_t)(dx) = \int f(x) P_t^*\nu(dx),$$

where ν is an absolutely continuous probability measure, $\mathcal{L}(X_t)(dx) = u(t,x)dx$ and $u(t,x)$ is a solution of the Fokker-Planck equation (1.9) with initial condition $u(0,t) = u_0(x) = \frac{d\nu}{dx}$.

2 The $\mathcal{CD}(\rho,\infty)$ criterion.

We now study operators of the form:

$$Lf(x) = \sum_{i,j=1}^{n} D_{ij}(x)\partial_{ij}f(x) - \sum_{i=1}^{n} a_i(x)\partial_i f(x), \quad x \in \mathbb{R}^n \tag{2.1}$$

where ∂_i and ∂_{ij} denote $\frac{\partial}{\partial x_i}$ and $\frac{\partial^2}{\partial x_i \partial x_j}$, respectively, while $D(x) = \big(D_{ij}(x)\big)_{1\le i,j\le n}$ is a symmetric, non-negative, $n \times n$ matrix and $a(x) = \big(a_i(x)\big)_{1\le i,j\le n}$ is a vector in $\mathbb{R}^n$.
We assume the following conditions:

1. $L1 = 0$, which is equivalent to requirement that $P_t 1 = 1$, where P_t is the semigroup that has the operator L as infinitesimal generator.
   This assumption implies that $(P_t)_{t\ge 0}$ is a Markov semigroup.

2. There exists $p_t(x,dy)$ (Markov kernel) such that

$$P_t f(x) = \int f(y) p_t(x,dy).$$

Remark 2.0.1. Let $(X_t)_{t\ge 0}$ be solution of the SODE

$$dX_t = \sigma(X_t)dB_t - a(X_t)dt,$$

where $\frac{1}{2}\sigma(x)\sigma^T(x) = D(x)$. Then

$$P_t f(x) = \mathbb{E}_x[f(X_t)];$$

in fact by the Itô's formula

$$\partial_t \mathbb{E}_x[f(X_t)] = L\,\mathbb{E}_x[f(X_t)],$$

where L is given by (2.1), and this shows that they have the same generator.
This remark generalizes what we have already observed in the case of the Ornstein-Uhlenbeck semigroup.

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Definition 2.0.1 (Carré du Champ)**.** The Carré du Champ form is a quadratic form, which we denote by Γ, defined as follows

$$\Gamma(f) = \Gamma(f,f) := \frac{1}{2}L(f^2) - fLf.$$

The associated bilinear form is given by

$$\Gamma(f,g) = \frac{1}{2}\big(L(fg) - fLg - gLf\big).$$

Proposition 2.0.2. For any f, $\Gamma(f) \geq 0$.

Proof. By the definition (2.1) of the operator L, we have

$$L(f^2) = \sum D_{ij}\partial_i(2f\partial_j f) - \sum a_i 2f\partial_i f = 2\sum D_{ij}\big(\partial_i f\partial_j f + f\partial_{ij} f\big) - 2\sum a_i f\partial_i f.$$

On the other hand,

$$-2fLf = -2\sum D_{ij} f\partial_{ij} f + 2\sum a_i f\partial_i f,$$

hence

$$L(f^2) - 2fLf = 2\sum D_{ij}\partial_i f\partial_j f = 2\nabla f \cdot D\nabla f \geq 0,$$

since by definition D is assumed to be non-negative. $\square$

Remark 2.0.2. Since L is the infinitesimal generator for the Markov semigroup $(P_t)_{t\geq 0}$, then

$$\Gamma(f) = \lim_{t\to 0^+} \frac{P_t(f^2) - (P_t f)^2}{2t}.$$

In fact by assumption (2) P_t can be given using a Markov kernel, this remark gives an alternative proof of Proposition 2.0.2 (i.e. $\Gamma(f) \geq 0$).

Definition 2.0.2. We define

$$\Gamma_2(f) := \frac{1}{2}L(\Gamma(f)) - 2\Gamma(f, Lf).$$

Analogously we can define Γ_n for any $n \geq 0$, setting $\Gamma_0(f) := f^2$ and

$$\Gamma_{n+1}(f) = \frac{1}{2}L(\Gamma_{n-1}(f)) - \Gamma_n(f, Lf).$$

Example 2.0.1.

1. Let $Lf = \Delta f - x \cdot \nabla f$ be the infinitesimal generator associated to the Ornstein-Uhlenbeck semigroup, then
$$\Gamma_2(f) = \|\partial_{ij} f\|^2_{H-S} + |\nabla f|^2,$$
where $\|\partial_{ij} f\|^2_{H-S} = \sum(\partial_{ij} f)^2$ is the Hilbert-Schmidt norm.

2. Let $Lf = \Delta f - \nabla\Psi \cdot \nabla f$, where $\Psi : \mathbb{R}^n \to \mathbb{R}$ is a C^2-function, then
$$\Gamma_2(f) = \|\partial_{ij} f\|^2_{H-S} + \nabla f \cdot \mathrm{Hess}(\Psi)\nabla f,$$
where Hess indicates the Hessian matrix of Ψ (i.e. the matrix of the second-derivatives of the scalar function Ψ).

3. Let L be given by (2.1), in this case the Carré du Champ form is easy to calculate, in fact
$$\Gamma(f) = \nabla f \cdot D\nabla f.$$

4. If $D(x) = Id$ and $a \in C^1(\mathbb{R}^n)$, then
$$\Gamma_2(f) = \|\partial_{ij} f\|^2_{H-S} + \nabla f \cdot J(a)\nabla f,$$
where $J(a)$ is the Jacobian-matrix of the vector $a = a(x)$, i.e.
$J(a) = \big(\partial_i a_j\big)_{1\le i,j\le n}$.

5. Let $D(x) = D$ be a constant matrix, then
$$\Gamma_2(f) = \frac{1}{2}\mathrm{Tr}\big((D\,\mathrm{Hess}(f))^2\big) + \nabla f \cdot J(a)\nabla f.$$

6. Let $D = b(x)Id$ with $b : \mathbb{R}^n \to \mathbb{R}$. Then a formula for Γ_2 exists, but it is very complicated to write down, so we omit this.

Definition 2.0.3 ($\mathcal{CD}(\rho,\infty)$ criterion)**.** We say that the operator L (or equivalently the semigroup $(P_t)_{t\ge0}$) satisfies the $\mathcal{CD}(\rho,\infty)$ criterion if
$$\Gamma_2(f) \ge \rho\,\Gamma(f), \quad \forall\, f \in \mathcal{D}(L).$$

Example 2.0.2.

1. Let L be given by (2.1) with $D(x) = Id$ and $a(x) = \nabla\Psi$ for some $\Psi \in C^2(\mathbb{R}^n)$. Then

$$\Gamma_2(f) = \|\text{Hess}(f)\|^2_{H-S} + \langle \nabla f, \text{Hess}(\Psi)\nabla f \rangle,$$

which implies that L satisfies $\mathcal{CD}(\rho,\infty)$ if and only if $\text{Hess}(\Psi) \geq \rho\, Id$.

2. If $D(x) = Id$ and $a \in C^1(\mathbb{R}^n)$ is a general vector field, then L satisfies $\mathcal{CD}(\rho,\infty)$ if and only if

$$\frac{J(a) + J(a)^T}{2} \geq \rho\, Id.$$

3. Let $D(x) = D$ be constant and $a(x) \in C^1(\mathbb{R}^n)$ be a vector field, then L satisfies $\mathcal{CD}(\rho,\infty)$ if and only if

$$\frac{DJ(a) + \big(DJ(a)\big)^T}{2} \geq \rho\, Id.$$

2.1 Φ-entropy and linear inequalities.

Given a probability measure μ, we define the Φ-entropy as

$$\text{Ent}^{\Phi}_{\mu}(f) := \int \Phi(f)d\mu - \Phi\left(\int f d\mu\right).$$

E.g. if $\Phi(x) = x^2$, $\text{Ent}^{\Phi}_{\mu}(f) = \text{Var}_{\mu}(f)$, while if $\Phi(x) = x\log x$, $\text{Ent}^{\Phi}_{\mu}(f) = \text{Ent}_{\mu}(f)$. Note that the Φ-entropy is well-defined under suitable assumptions on the function Φ which we specify in the following definition.

Definition 2.1.1. Let I be a real interval, then $\Phi : I \to \mathbb{R}$ is *admissible* if and only if $\Phi \in C^2(I)$ is strictly convex and the function $-\frac{1}{\Phi''}$ is convex.

We define the Φ-entropy only for admissible functions.

Example 2.1.1.

1. If we assume $\Phi \in C^4(I)$, then

$$-\frac{1}{\Phi''} \text{ is convex } \iff \left(-\frac{1}{\Phi''}\right)'' \geq 0.$$

   This means that $\Phi \in C^4(I)$ is admissible for $\Phi(x) = x^2$ or $\Phi(x) = x \log x$.

2. Let $\Phi(x) = x^p$, then Φ is admissible in $\mathbb{R}$ for $p \in (1,2)$.
   If $p = 1$ (i.e. $\Phi(x) = x$), then Φ it is admissible but only in $\mathbb{R}^+ := (0, +\infty)$.

3. Let $\Phi(x) = (x + a)^\alpha \log(x + a)^\beta$ with $\alpha \in (1, 2)$ and $\beta \in \mathbb{R}$.
   Then for any $a >> 1$, Φ is admissible in $\mathbb{R}^+$.

Remark 2.1.1. The set of admissible functions is a cone, that means

$$\forall\, \lambda, \mu > 0 \;\forall \text{and } \Phi \text{ and } \Psi \text{ admissible } \implies \lambda\Phi + \mu\Psi \text{ is admissible.}$$

Theorem 2.1.1 (Chafai 2004)**.** Let $\Phi \in C^2(I)$, then Φ is admissible in I if and only if one of the following properties holds:

(i) Definition 2.1.1;

(ii) $(x, y) \mapsto \frac{\Phi(x)}{y}$ is convex in $I \times \mathbb{R}^+$;

(iii) $\mathrm{Ent}^\Phi_{\mu_1 \oplus \mu_2}(f) \leq \mathbb{E}_{\mu_1}(\mathrm{Ent}^\Phi_{\mu_2}(f)) + \mathbb{E}_{\mu_2}(\mathrm{Ent}^\Phi_{\mu_1}(f))$.

Remark 2.1.2. By using property (ii) in the theorem above, Remark 2.1.1 follows immediately: in fact a linear combination with positive coefficients of convex functions is always convex.

Theorem 2.1.2 (Bakry-Emery 1985, Chafai 2004, Bakry 2006, and others)**.**
Let $p \in \mathbb{R}$ and let us assume that Φ is admissible in I, then the following statments are equivalent:

(i) L satisfies $\mathcal{CD}(\rho, \infty)$ criterion;

(ii) $\mathrm{Ent}^\Phi_{P_t}(f) := P_t(\Phi(f)) - \Phi(P_t f) \leq \frac{1-\mathrm{e}^{-2\rho t}}{2\rho} P_t(\Phi''(f)\Gamma(f))$, for all I-valued functions f and $t \geq 0$;

(iii) $\mathrm{Ent}^\Phi_{P_t}(f) \geq \frac{\mathrm{e}^{2\rho t}-1}{2\rho} \Phi''(P_t f)\Gamma(P_t f)$, for all f I-valued functions and $t \geq 0$;

(iv) $\Gamma(P_t f) \leq \mathrm{e}^{-2\rho t} P_t(\sqrt{\Gamma(f)})^2$, for all I-valued functions f and $t \geq 0$;

(v) $\Gamma(P_t f) \le \mathrm{e}^{-2\rho t} P_t(\Gamma(f))$, for all I-valued functions f and $t \ge 0$;

(where we have assumed $\rho > 0$).

Remark 2.1.3. We can note that

$$\lim_{\rho\to 0^+} \frac{1-\mathrm{e}^{-2\rho t}}{2\rho} = t,$$

so Theorem 2.1.2 is still true for $\rho = 0$ and it can be proved as in the case $\rho > 0$.

Example 2.1.2. Let $L = \Delta$, then $\Gamma_2(f) = \|\mathrm{Hess}(f)\|_2^2$. In this case $\rho = 0$.

Before proving the theorem we need to state some lemmas.

Lemma 2.1.1. For any $\Phi \in C^2(\mathbb{R}^n)$, we have

$$L(\Phi(g)) = \Phi'(g)Lg + \Phi''(g)\Gamma(g), \quad \Gamma(\Phi(g), f) = \Phi'(g)\Gamma(g,f). \tag{2.2}$$

Moreover

$$\Gamma_2(\Phi(g)) = \big(\Phi'(g)\big)^2\Gamma_2(g) + \Phi'(g)\Phi''(g)\Gamma(g,\Gamma(g)) + \big(\Phi''(g)\big)^2\Gamma(g). \tag{2.3}$$

Lemma 2.1.2. Let $\Phi \in C^4(\mathbb{R}^n)$ be a function with nonvanishing second derivatives, set

$$\Psi(s) := P_s(\Phi(P_{t-s}f))$$

then the function $\Psi(s)$ is twice differentiable in $[0,t]$.
Moreover setting $g = P_{t-s}f$, the first and second derivatives can be written as

$$\Psi'(s) = P_s\left(\frac{1}{\Phi''(g)}\Gamma(\Phi'(g))\right) = P_s\big(\Phi''(g)\Gamma(g)\big), \tag{2.4}$$

and

$$\Psi''(s) = 2P_s\left(\frac{1}{\Phi''(g)}\Gamma_2(\Phi'(g))\right) + P_s\left(\left(-\frac{1}{\Phi''(g)}\right)''\left(\frac{1}{\Phi''(g)}\Gamma(\Phi'(g))\right)^2\right). \tag{2.5}$$

Note that, by definition of Ψ, $\Psi(0) = \Phi(P_t(f))$ and $\Psi(t) = P_t(\Phi(f))$.

Proof. A simple calculation shows that

$$\Psi'(s) = P_s(\Phi''(g)\Gamma(g)) = P_s\left(\frac{\big(\Phi''(g)\big)^2}{\Phi''(g)}\Gamma(g)\right) = P_s\left(\frac{1}{\Phi''(g)}\Gamma(\Phi'(g))\right). \tag{2.6}$$

where we have used the second identity in (2.2) i.e.$\Gamma\big(\Phi'(g),\Phi'(g)\big)=\big(\Phi''(g)\big)^2\Gamma(g)$ and our assumption $g=P_{t-s}f$.

To get (2.5), we differentiate (2.6) w.r.t. $s\in[0,t]$, to get

$$\Psi''(s)=P_s\big(L(\Phi''(g)\Gamma(g))-Lg\Phi''(g)\Gamma(g)-2\Phi''(g)\Gamma(g,Lg)\big).$$

Recalling that $2\Gamma(f,g)=L(fg)-fLg-gLf$. Thus the previous identity becomes

$$\Psi''(s)=$$
$$=P_s\big(2\Gamma(\Phi''(g),\Gamma(g))+\Phi''(g)L(\Gamma(g))+\Gamma(g)L(\Phi''(g))+Lg\Phi''(g)\Gamma(g)-2\Phi''(g)\Gamma(g,Lg)\big).$$

Using (2.3), we get

$$\begin{aligned}\Psi''(s)&=P_s\big(2\Phi''(g)\Gamma_2(g)+2\Phi'''(g)\Gamma(g,\Gamma(g))+\big(\Gamma(g)\big)^2\Phi'''(g)\big)\\&=2P_s\left(\frac{1}{\Phi''(g)}\left[\Gamma_2(\Phi'(g))+\left(-\big(\Phi'''(g)\big)+\frac{\Phi'''(g)\Phi''(g)}{2}\right)\big(\Gamma(g)\big)^2\right]\right)\\&=2P_s\left(\frac{1}{\Phi''(g)}\Gamma_2(\Phi'(g))\right)+P_s\left(\left[\frac{-2\big(\Phi'''(g)\big)^2}{\Phi''(g)}+\Phi'''(g)\right]\big(\Gamma(g)\big)^2\right),\end{aligned}$$

which gives (2.5). □

Proof of Theorem 2.1.2. We do not prove property (iv) since the way to prove this is by using a method very different from the one we are going to use to show all the others characterizations.

Let Ψ be as in Lemma 2.1.2 and let us assume property (i), i.e. $\Gamma_2(f)\geq\rho\,\Gamma(f)$. Then

$$\Psi''(s)\leq 2\rho\,\Psi'(s),$$

i.e.

$$\big(\Psi'(s)\mathrm{e}^{-2\rho s}\big)'\geq 0\implies\Psi'(s)\mathrm{e}^{-2\rho s}\geq\Psi'(0).\tag{2.7}$$

Let $\Phi(x)=\frac{x^2}{2}$ so that $\Phi''(x)=1$. Using (2.2) with $g=P_tf$, we can write

$$\Psi'(t)\mathrm{e}^{-2\rho t}=P_t\big(\Gamma(f)\big)\mathrm{e}^{-2\rho t}\geq\Psi'(0)=\Gamma(P_tf),$$

which gives property (v).
Integrating (2.7) over $[0,t]$, we get

$$\int_0^t\Psi'(s)\geq\Psi'(0)\int_0^t\mathrm{e}^{2\rho s}ds,$$

which using that $\Phi'' \equiv 1$ implies

$$\Psi(t) - \Psi(0) = P_t(\Phi(f)) - \Phi(P_t f) = \mathrm{Ent}^{\Phi}_{P_t}(f) \geq \frac{\mathrm{e}^{2\rho t} - 1}{2\rho}\Gamma(P_t f),$$

which proves property (iii).

Analogously we can deduce property (ii). In fact if $s \in [0,t]$ we can apply (2.7) to two points t and s such that $t > s > 0$ and integrate the corresponding inequality over $[0,t]$. This gives

$$\Psi'(t)\mathrm{e}^{-2\rho t} \geq \Psi'(s)\mathrm{e}^{-2\rho s} \iff \int_0^t \Psi'(s) \leq \Psi'(t)\int_0^t \mathrm{e}^{-2\rho(t-s)}ds.$$

The previous inequality implies (ii) by using (2.4): in fact

$$\Psi(t) - \Psi(0) = \mathrm{Ent}^{\Phi}_{P_t}(f) \leq \frac{1 - \mathrm{e}^{2\rho t}}{2\rho}P_t(\Phi''(f)\Gamma(f)).$$

We rest to prove the reverse implications, i.e. we have to deduce $\mathcal{CD}(\rho,\infty)$ criterion by the properties $(i), (ii), (iii)$ and (v). This is indeed the most difficult part of the theorem. First we show that

$$\mathrm{Ent}^{\Phi}_{\mu}(1+\varepsilon g) \approx \frac{\varepsilon^2}{2}\mathrm{Var}_{\mu}(g)\Phi''(g),$$

as $\varepsilon \to 0^+$. In fact

$$\begin{aligned}
\mathrm{Ent}^{\Phi}_{\mu}(1+\varepsilon g) &= \int \Phi(1+\varepsilon g)d\mu - \Phi\left(\int (1+\varepsilon)d\mu\right) = \Phi(1) - \varepsilon\Phi'(1)\int g d\mu \\
&+ \frac{\varepsilon^2}{2}\Phi''(1)\int g^2 d\mu - \Phi(1) - \varepsilon\Phi'(1)\int g d\mu - \frac{\varepsilon^2}{2}\Phi''(1)\left(\int g d\mu\right)^2 + o(\varepsilon^2) \\
&= \frac{\varepsilon^2}{2}\Phi''\left(\int g^2 d\mu - \left(\int g d\mu\right)^2\right) + o(\varepsilon^2) = \frac{\varepsilon^2}{2}\mathrm{Var}_{\mu}(g)\Phi''(g) + o(\varepsilon^2).
\end{aligned}$$

For sake of simplicity let us assume that $\Phi''(1) \neq 0$.
Property (ii) with $f = 1 + \varepsilon g$ implies

$$\Phi''(1)\frac{\varepsilon^2}{2}\mathrm{Var}_{P_t}(g) + o(\varepsilon^2) \leq \frac{1 - \mathrm{e}^{-2\rho t}}{2\rho}P_t\big(\Phi''(1)\varepsilon^2\Gamma(g)\big),$$

that gives

$$\mathrm{Var}_{P_t}(g) \leq \frac{1 - \mathrm{e}^{-2\rho t}}{2\rho}P_t(\Gamma(g)).$$

Property *(iii)* instead implies

$$\mathrm{Var}_{P_t}(g) \geq \frac{\mathrm{e}^{2\rho t} - 1}{2\rho} \Gamma(P_t(g)).$$

Now we show that property *(v)* implies $\mathcal{CD}(\rho, \infty)$.
To get the other implications, one can proceed in a similar way.

If we write property *(v)* with $t = 0$, we get a trivial identity since $P_0 f = f$ for any function f. Then

$$\frac{\partial}{\partial t}\Gamma(P_t f)\Big|_{t=0} \leq \frac{\partial}{\partial t}\Big(\mathrm{e}^{-2\rho t} P_t(\Gamma(f))\Big)\Big|_{t=0},$$

which implies

$$2\Gamma(f, Lf) \leq -2\rho\,\Gamma(f) + L\big(\Gamma(f)\big) \implies \rho\,\Gamma(f) \leq \Gamma_2(f).$$

□

Example 2.1.3 (The Laplacian case)**.** If $L = \Delta$ then

$$P_t f(x) = \int \frac{f(y)\mathrm{e}^{-\frac{\|x-y\|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}} dy,$$

is a solution of $\partial_t u = \Delta u$ with initial condition $u(0, x) = f(x)$.
Note that for $t = \frac{1}{2}$ we have

$$P_{\frac{1}{2}} f(0) = \int f(y) \frac{\mathrm{e}^{-\frac{\|y\|^2}{2}}}{(2\pi)^{\frac{n}{2}}} dy = \int f(y) d\gamma(y),$$

where $d\gamma(y)$ is the standard Gaussian measure.
Recall that if $L = \Delta$ then $\mathcal{CD}(\rho, \infty)$ holds with $\rho = 0$. By Remark 2.1.3 in the case $\rho = 0$ property *(ii)* of Theorem 2.1.2 becomes

$$\mathrm{Ent}^{\Phi}_{P_t} \leq t\, P_t\big(\Phi''(f)\Gamma(f)\big).$$

Writing the previous inequality at $t = \frac{1}{2}$, we find

$$\mathrm{Ent}^{\Phi}_{\gamma}(f) \leq \frac{1}{2} \int \Phi''(f)\Gamma(f) d\gamma. \tag{2.8}$$

Now we want to introduce in a general setting some definitions already introduced in the particular case of the Ornstein-Uhlenbeck semigroup (see Remark 1.1.2).

Definition 2.1.2.

1. Given a probability measure μ, $(P_t)_{t\geq 0}$ is μ-*ergodic* if and only if
$$\lim_{t\to+\infty} P_t f(x) = \mu(f) := \int f d\mu,$$
in $L^2(d\mu)$.
2. A probability measure μ is *invariant* under P_t if and only if
$$\int P_t f d\mu = \int f d\mu, \quad \text{for any } \ t \geq 0,$$
or equivalently
$$\int Lf d\mu = 0, \quad \forall\, f \in \mathcal{D}(L).$$
3. A probability measure μ is *reversible* if and only if
$$\int fLg d\mu = \int gLf d\mu.$$

Note that μ ergodic implies μ invariant. In fact the semigroup property $\int P_t(P_s f)d\mu = \int P_{t+s}(f)d\mu$ and ergodicity imply that the first quantity converges to $\mu(P_t(f))$ as $s \to +\infty$, while the second one converges to $\mu(f)$: hence the invariance comes. Moreover also reversibility implies invariance simply by taking $g = Id$ in the property (3) of Definition 2.1.2.

Remark 2.1.4. If μ is a reversible probability measure, then
$$\int fLg d\mu = \frac{1}{2}\int \big(fLg + gLf - L(fg)\big)d\mu = -\int \Gamma(f,g)d\mu,$$
which gives an integration by parts.

Corollary 2.1.1. If $(P_t)_{t\geq 0}$ is μ-ergodic and L satisfies $\mathcal{CD}(\rho,\infty)$ with $\rho > 0$, then
$$\mathrm{Ent}_\mu^\Phi(f) \leq \frac{1}{2\rho}\int \Phi''(f)\Gamma(f)d\mu.$$

Example 2.1.4. Let $d\mu(x) = \mathrm{e}^{\psi(x)}dx$ with $\psi \in C^2(\mathbb{R}^n)$ and $L = \Delta - \nabla\psi\cdot\nabla$. If we assume $\mathrm{Hess}(f) \geq \rho Id$ with $\rho > 0$, then
$$\mathrm{Ent}_\mu^\Phi(f) \leq \frac{1}{2\rho}\int \Phi''(f)\Gamma(f)d\mu,$$
for any admissible function Φ.

2.2 Applications.

Let $(P_t)_{t\geq 0}$ be satisfying $\mathcal{CD}(\rho,\infty)$ for some $\rho>0$ and let us assume that μ is an invariant measure for $(P_t)_{t\geq 0}$. We define $\Psi(t) := \mathrm{Ent}_\mu^\Phi\big(P_t f\big)$ then

$$\Psi'(t) = \frac{d}{dt}\left(\mathrm{Ent}_\mu^\Phi\big(P_t f\big)\right) = \frac{d}{dt}\left(\int \Phi(P_t f)d\mu - \Phi\left(\int P_t f d\mu\right)\right) = \int \Phi'(P_t f)L(P_t f)d\mu. \tag{2.9}$$

Lemma 2.2.1. Let φ be a bijective differentiable function. Then

$$\int L(\varphi(f))f d\mu = -\int \Gamma(f,\varphi(f))d\mu = -\int \varphi'(f)\Gamma(f)d\mu.$$

Proof. Let us denote $g := \varphi(f)$ and $\psi = \varphi^{-1}$, since μ is invariant, $\mu(Lf) = 0$ for any function $f \in \mathcal{D}(L)$, which implies

$$\int Lg\psi(g)d\mu = \int \big(Lg\psi(g) - L(\Psi(g))\big)d\mu,$$

where Ψ is a primitive of ψ. Therefore

$$\begin{aligned}\int L(\varphi(f))f d\mu = \int Lg\psi(g)d\mu = \int \big(Lg\psi(g) - \psi(g)Lg - \psi'(g)\Gamma(g)\big)d\mu \\ = -\int \frac{1}{\varphi'(f)}\Gamma(\varphi(f))d\mu = -\int \varphi'(f)\Gamma(f),\end{aligned}$$

by definition of g and ψ. □

By using Lemma 2.2.1 in (2.9), we can conclude

$$\psi'(t) = -\int \frac{1}{\Phi''(P_t f)}\Gamma(\Phi'(P_t f))d\mu = -\int \Phi''(P_t f)\Gamma(P_t f)d\mu.$$

Theorem 2.2.1. If $(P_t)_{t\geq 0}$ is μ-ergodic and $\mathcal{CD}(\rho,\infty)$ holds for some $\rho>0$, then

$$\mathrm{Ent}_\mu^\Phi(P_t f) \leq \mathrm{e}^{-2\rho t}\mathrm{Ent}_\mu^\Phi(f).$$

Remark 2.2.1. If μ is ergodic but not reversible, in general we do not have an integration by parts, so Lemma 2.2.1 plays the role of the reversibility property.

2.3 The linear case: the Fokker-Planck equation.

The aim of this section is to study

$$\frac{\partial}{\partial t} P_t f = L(P_t f), \tag{2.10}$$

in the particular linear case of the Fokker-Planck equation.
Therefore let us consider the following problem:

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = \nabla \cdot \big(D(x)(\nabla u + u a(x)) \big), \\ u(0,x) = u_0(x) \geq 0 \text{ such that } \displaystyle\int u_0(x)\, dx = 1, \end{cases} \tag{2.11}$$

with $D(x) = \big(D_{i\,j}(x)\big)_{1\leq i,j\leq n}$ symmetric $n\times n$-matrix such that $D \geq 0$ and $a(x)$ is a vector in $\mathbb{R}^n$.

We have already remarked that the equation conserves the mass, which means that for any $t \geq 0$

$$\int u(t,x) dx = \int u_0(x)\, dx = 1.$$

Moreover

$$L^*(u) = \nabla \cdot \big(D(x)(\nabla u + u a(x) \big), \quad L^*(u) \neq 0.$$

We may assume that there exists a steady state u_∞ with $u_\infty(x)dx = \mu_\infty(dx)$ and

$$\int u_\infty(x)\, dx = 1.$$

Note that

$$\begin{aligned} \frac{d}{dt}\left(\mathrm{Ent}^{\Phi}_{\mu_\infty(dx)} \left(\frac{u}{u_\infty} \right) \right) &= \frac{d}{dt}\left[\int \Phi\left(\frac{u}{u_\infty} \right) u_\infty dx - \Phi\left(\int \frac{u}{u_\infty} u_\infty dx \right) \right] \\ &= \int \Phi'\left(\frac{u}{u_\infty} \right) \frac{L^*(u)}{u_\infty} u_\infty dx = \int \Phi'\left(\frac{u}{u_\infty} \right) L^*(u) dx. \end{aligned} \tag{2.12}$$

Lemma 2.3.1. If L is the operator associated to the Fokker-Planck equation then

$$\int f L^* g dx = \int g \big(\nabla \cdot (D \nabla f) - \langle Da, \nabla f \rangle \big).$$

Proof. Integrating by parts

$$\int fL^*g dx = \int f\nabla\cdot\big(D(\nabla g+ga)\big)dx = -\int\langle\nabla f, D\nabla g\rangle dx - \int g\big\langle Da,\nabla f\big\rangle dx$$
$$= \int\langle\nabla f, D\nabla g\rangle dx - \int g\big\langle Da,\nabla f\big\rangle dx = \int g\big(\nabla\cdot(D\nabla f) - \big\langle Da,\nabla f\big\rangle\big),$$

where we have omitted the dependence of f, g, D, a on $x\in\mathbb{R}^n$. $\square$

Recall that $Lf(x) := \nabla\cdot\big(D(x)\nabla f(x)\big) - D(x)a(x)\nabla f(x)$. Therefore by using Lemma 2.3.1 in (2.12) we can deduce

$$\frac{d}{dt}\left(\mathrm{Ent}^{\Phi}_{\mu_\infty(dx)}\left(\frac{u}{u_\infty}\right)\right) = \int L\Phi'\left(\frac{u}{u_\infty}\right)\frac{u}{u_\infty}u_\infty dx = \int\Gamma\left(\Phi'\left(\frac{u}{u_\infty}\right),\frac{u}{u_\infty}\right)\mu_\infty(dx).$$

Proposition 2.3.1. Assume that L satisfies $\mathcal{CD}(\rho,\infty)$ for some $\rho>0$ and $(P_t)_{t\geq 0}$ is μ-ergodic for some probability measure μ then

$$\mathrm{Ent}^{\Phi}_{\mu}\left(\frac{u}{u_\infty}\right)\leq \mathrm{e}^{-2\rho t}\mathrm{Ent}^{\Phi}_{\mu}\left(\frac{u_0}{u_\infty}\right).$$

Example 2.3.1.

1. Let $n=1$ and let us assume that $a(x)$ is a smooth vector field, so there exists $b(x)$ primitive of $a(x)$ and we can write
$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) - a(x)\frac{\partial}{\partial x}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) - b'(x)\frac{\partial}{\partial x}u(t,x).$$
The previous computation implies $\mu(dx) = \mathrm{e}^{-b(x)}dx$, which means that u is a reversible measure associated to u.

2. Let $n=2$ and $D(x) = Id$ and
$$a(x,y) = \begin{pmatrix} x+\frac{1}{2}y \\ y\end{pmatrix}.$$
In this case $Lf(x) = \Delta f(x) - a(x)\cdot\nabla f(x)$ and $\Gamma(f) = |\nabla f|^2$.
Therefore $\Gamma_2(f)\geq\rho\Gamma(f)$ if and only if
$$\frac{1}{2}\left(\begin{pmatrix}1 & \frac{1}{2}\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\ \frac{1}{2} & 1\end{pmatrix}\right)\geq\rho\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$
which is equivalent to
$$\begin{pmatrix}1-\rho & \frac{1}{2}\\ \frac{1}{2} & 1-\rho\end{pmatrix}\geq 0,$$
i.e. $(1-\rho)^2-\frac{1}{4}\geq 0$, which can be easily solved (e.g. $\rho=\frac{1}{2}$ is a solution).

3 Nonlinear case

Recall that $\Psi(s) := P_s\big(\Phi(P_{t-s}f)\big)$ and

$$\Psi''(s) = 2P_s\left(\frac{1}{\Phi''(g)}\Gamma_2(\Phi'(g))\right) + P_s\left(\left(-\frac{1}{\Phi''}\right)''(g)\left(\frac{1}{\Phi''(g)}\Gamma(\Phi'(g))^2\right)\right),$$

where $g = P_{t-s}f$.

The model to keep in mind is $\Phi_p(x) = \frac{x^p}{p(p-1)}$.

Theorem 3.0.1. Let $p \in (1,2)$ then the following properties are equivalent:

(i) $(P_t)_{t\geq 0}$ satisfies $\mathcal{CD}(\rho,\infty)$ criterion for some $\rho > 0$.

(ii) For all $t \geq 0$

$$\frac{1}{(p-1)^2}\left[P_t(f^p) - \big(P_tf\big)^p\left(\frac{P_t(f^p)}{\big(P_tf\big)^p}\right)^{\frac{2}{p}-1}\right] \leq \frac{1-\mathrm{e}^{-2\rho t}}{\rho}P_t\big(f^{p-2}\Gamma(f)\big).$$

(iii) For all $t \geq 0$

$$\frac{1}{(p-1)^2}\left[P_t(f^p) - \big(P_tf\big)^p\left(\frac{P_t(f^p)}{\big(P_tf\big)^p}\right)^{\frac{2}{p}-1}\right] \geq \frac{\mathrm{e}^{2\rho t}-1}{\rho}\left(\frac{P_t(f^p)}{\big(P_tf\big)^p}\right)^{\frac{2}{p}-1}\big(P_tf\big)^{p-2}\Gamma(P_tf).$$

Passing to the limit as $t \to +\infty$ in the property (ii) in Theorem 3.0.1, one can get the following corollary.

Corollary 3.0.1. Let us assume $\rho > 0$ and $(P_t)_{t\geq 0}$ is μ-ergodic. Then

$$\frac{1}{(p-1)^2}\left[\mu(f^p) - \big(\mu(f)\big)^p\left(\frac{\mu(f^p)}{\big(\mu(f)\big)^p}\right)^{\frac{2}{p}-1}\right] \leq \frac{1}{\rho}\mu\big(f^{p-2}\Gamma(f)\big) = \frac{1}{\rho}\left(\frac{4}{p^2}\Gamma\big(f^{\frac{p}{2}}\big)\right). \tag{3.1}$$

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Writing $g = f^{\frac{p}{2}}$ and dividing (3.1) by $\frac{2}{p}$, we get

$$\frac{p}{2(p-1)^2}\left[\mu(g^2) - \left(\mu(g^{\frac{2}{p}}\right)^p\left(\frac{\mu(g^2)}{\left(\mu(g^{\frac{2}{p}})\right)^p}\right)^{\frac{2}{p}-1}\right] \le \frac{2}{\rho\, p}\mu(\Gamma(g)). \tag{3.2}$$

We want to recall that a similar inequality was proved by Beckner.
More precisely, the Beckner's inequality is:

$$\frac{\mu(g^2) - \left(\mu(g^{\frac{2}{p}})\right)^p}{p-1} \le \frac{2}{\rho\, p}\mu(\Gamma(g)), \quad p \in (1,2). \tag{3.3}$$

Note that inequality (3.3) with $p = 2$ is the Poincaré inequality while if $p = 1$, inequality (3.3) gives the Logarithmic-Sobolev inequality.
Moreover for any $g \ge 0$ and $p \in (1,2)$, it is possible to show that the function

$$p \longmapsto \frac{\mu(g^2) - \left(\mu(g^{\frac{2}{p}})\right)^p}{p-1}$$

is decreasing for $p \in (1,2)$.
That means, in some sense, that the Logarithmic-Sobolev inequality is the strongest form of inequality (3.3), while the Poincaré inequality is the weakest form.
The same decreasing property holds for inequality (3.2).

Proposition 3.0.2. For any $g \ge 0$ and $p \in (1,2)$ the function

$$p \longmapsto \frac{p}{2(p-1)}\left[\mu(g^2) - \left(\mu(g^{\frac{2}{p}}\right)^p\left(\frac{\mu(g^2)}{\left(\mu(g^{\frac{2}{p}})\right)^p}\right)^{\frac{2}{p}-1}\right]$$

is decreasing.

The proof of this proposition is not difficult but very involved, so we prefer to omit it (see [6], Proposition 11).

Proof of Theorem 3.0.1. Let us start with assuming property (i), we consider

$$\Psi_p(s) = P_s(\Phi_p(P_{t-s}f)),$$

with $\Phi_p(x) = \frac{x^p}{p(p-1)}$. Then

$$\Psi''(s) = 2P_s\left(\frac{\Gamma_2\big(\Phi_p'(g)\big)}{\Phi_p''(g)}\right) + (2-p)(p-1)P_s\left(\left(\frac{\Gamma\big(\Phi_p'(g)\big)}{\Phi_p''(g)}\right)^2\frac{1}{g^p}\right), \tag{3.4}$$

with $g = P_{t-s}f$. Let us set $G := \frac{\Gamma\left(\Phi_p'(g)\right)}{\Phi_p''(g)}$ and $H := g^p$, by $\mathcal{CD}(\rho,\infty)$ identity (3.4) becomes

$$\Psi''(s) \leq 2\rho P_s(G) + (2-p)(p-1)P_s\left(\frac{G^2}{H}\right).$$

Note that $P_s(G) = \Phi'(s)$ while $P_s(H) = p(p-1)\Phi(s)$.
Moreover $P_s\left(\frac{G^2}{H}\right) \geq \frac{P_s\left(G^2\right)}{P_s(H)}$, which allow us to conclude

$$\Psi''(s) \geq 2\rho\Psi'(s) + \frac{2-p}{p}\frac{\left(\Psi'(g)\right)^2}{\Psi(s)}.$$

Dividing by $\Psi'(s)$, and applying the Gronwall's Lemma to the function $F(s) := \frac{\Psi'(s)}{\Psi(s)}$, we get

$$\frac{d}{dt}\Bigg(\log\left(\Psi'(t)\mathrm{e}^{-2\,\rho t}\right)\Bigg) \geq \left(\frac{2-p}{p}\log\Psi(t)\right),$$

which is equivalent to

$$\frac{d}{dt}\left(\log\left(\frac{\Psi'(t)\mathrm{e}^{-2\,\rho t}}{\Psi(t)^{\frac{2-p}{p}}}\right)\right) \geq 0. \tag{3.5}$$

Integrating inequality (3.5) over $[0,s]$ and using that the logarithm is an increasing function, we get

$$\frac{\Psi'(s)}{\Psi(s)^{\frac{2-p}{p}}} \geq \frac{\Psi'(0)}{\Psi(0)^{\frac{2-p}{p}}}\ \mathrm{e}^{2\rho s},$$

which can be written also as

$$\frac{d}{dt}\left(\frac{1}{-\frac{2-p}{p}+1}\Psi(s)^{-\frac{2-p}{p}+1}\right) \geq \frac{d}{dt}\left(\frac{\Psi'(0)}{\Psi(0)^{\frac{2-p}{p}}}\ \frac{\mathrm{e}^{2\rho s}}{2\rho}\right).$$

Note that $-\frac{2-p}{p}+1 = 2\left(\frac{p-1}{p}\right)$. Therefore integrating over $[0,t]$, we get

$$\frac{2}{2(p-1)}\left(\Psi(t)^{\frac{2}{2(p-1)}} - \Psi(0)^{\frac{2}{2(p-1)}}\right) \geq \frac{\mathrm{e}^{2\rho\,t}-1}{2\rho}\frac{\Psi'(0)}{\Psi(0)^{\frac{2-p}{p}}}. \tag{3.6}$$

By recalling that $\Psi(t) = P_t\left(\frac{f^p}{p(p-1)}\right)$, the inequality (3.6) gives exactly property (iii).
To get property (ii), we proceed in the same way but integrating inequality (3.5) over $[s,t]$, i.e.

$$\mathrm{e}^{-2\rho t}\frac{\Psi'(t)}{\Psi(t)^{\frac{2-p}{p}}} \geq \frac{\Psi'(s)}{\Psi(s)^{\frac{2-p}{p}}}\ \mathrm{e}^{-2\rho s},$$

that implies

$$\frac{\Psi'(s)}{\Psi(s)^{\frac{2-p}{p}}} \leq \mathrm{e}^{-2\rho(t-s)} \frac{\Psi'(t)}{\Psi(t)^{\frac{2-p}{p}}},$$

or equivalently

$$\frac{d}{ds}\left(\frac{\Psi(s)^{\frac{2(p-1)}{p}}}{\frac{2(p-1)}{p}}\right) \leq \frac{d}{ds}\left(\frac{\Psi'(t)}{\Psi(t)^{\frac{2-p}{p}}} \frac{1}{2\,\rho} \mathrm{e}^{-2\rho(t-s)}\right).$$

Integrating over $[0,t]$, we get property (ii).

It remains to prove the reverse implications, i.e. (ii) or $(iii) \implies \mathcal{CD}(\rho,\infty)$. Both the proofs are very similar to the corresponding proofs given in the linear case, so we omit these. □

As for the Fokker-Planck case, we now state the following corollary.

Corollary 3.0.2. Let $(P_t)_{t\geq 0}$ be μ-ergodic and $\rho > 0$, then μ satisfies (3.1).

In the next result we show that $\mathcal{CD}(\rho,\infty)$ criterion is not necessary for (3.1). In fact we can also get (3.1) assuming a weaker inequality.

Proposition 3.0.3. Let us assume that

$$\mu\left(g^{\frac{2-p}{p-1}} 2\Gamma_2(g)\right) \geq \rho\, \mu\left(g^{\frac{2-p}{p-1}} 2\Gamma(g)\right), \quad \forall\, g \geq 0 \;\text{ and }\; p \in (1,2).$$

Let μ be an invariant measure w.r.t. L and $\rho > 0$, then (3.1) holds.

Proof. The idea of the proof is to calculate the derivatives of the following function

$$\Psi(t) := \mu(\Phi_p(P_t f))) = \int \Phi_p(P_t f))d\mu,$$

and then to find a suitable bound for $\Psi''(t)$.
More details on the proof can be found in [6] (Proposition 14). □

3.1 Remarks on the case of non-admissible functions

We want now to understand what happens when $p \notin [1,2)$. In this case the function $\Phi_p(x) = \frac{x^p}{p(p-1)}$ is not admissible. Nevertheless we can get some similar results. In the positive case, i.e. $p \in (0,1) \cup [2,+\infty)$, we can get almost the same characterization that we have found for admissible Φ_p, i.e. for $p \in (1,2)$.

Theorem 3.1.1. Let $p \in (0,1) \cup [2,+\infty)$ and $\rho \in \mathbb{R}$. Then the following properties are equivalent:

(i) L satisfies $\mathcal{CD}(\rho,\infty)$ criterion.

(ii) For all $t \geq 0$,

$$\mathrm{Ent}_{P_t}^{\Phi_p}(f) \leq \frac{1-\mathrm{e}^{-2\rho t}}{2\rho} P_t(f^{p-1}\Gamma(f))\xi\left(\frac{1-\mathrm{e}^{-2\rho t}}{2\rho}K_1\right), \tag{3.7}$$

where

$$K_1 = \big|(2-p)(1-p)\big|(\beta-1)\left(\frac{P_t\big(f^{p-1}\Gamma(f)^{\frac{b}{2\alpha}}\big)}{P_t\big(f^{p-2}\Gamma(f)\big)}\right),$$

with $b = 2\,\frac{\beta-2}{\beta-1}$, $\xi(x) = \frac{1-\beta}{2-\beta}\frac{(1+x)^{\frac{1-\beta}{2-\beta}}-1}{x}$ and

$$(p,\alpha,\beta) \in \Delta = \begin{cases} \alpha \in (0,p),\ \beta \in [0,1],\ p \in (0,1), \\ \alpha = 1,\ \beta > \dfrac{4-p}{2-p},\ p \in (1,2), \\ \alpha = 1,\ \beta = 2,\ p = 2,\ (\equiv \text{ Poincaré inequality}), \\ \alpha = 1,\ \beta \in \left[\max\left\{\dfrac{p-4}{p-2}, 0\right\}, 1\right),\ p > 2. \end{cases}$$

Remark 3.1.1. If $\alpha = 1,\ \beta > \frac{4-p}{2-p},\ p \in (1,2)$, we are in the admissible case. In such a case $\xi(x) < 1$, so we can get a better inequality. Instead the case $\alpha \in (0,p)$ is much worse since $\xi(x) > 1$ and so the inequality provides us with less information.

Remark 3.1.2 (Open question)**.** The case $p < 0$ is still open.

Corollary 3.1.1. If $\rho > 0$ and $(P_t)_{t\geq 0}$ is μ-ergodic, you can get the related inequality (3.7), simply by replacing P_t by μ.

Remark 3.1.3. Another interesting case to study is $\widetilde{\Phi}_p(x) = \frac{x^p-1}{p(p-1)}$. In this case

$$\lim_{p\to 0^+} \widetilde{\Phi}_p(x) = -\log x.$$

Hence, one can do similar computations and get the same results that we have showed for Φ_p.
Moreover, passing then to the limit as $p \to 0^+$, one can deduce

$$\log \int f d\mu - \int \log f d\mu = \mathrm{Ent}_{\mu}^{\widetilde{\Phi}_0}(f) \leq \frac{1}{2\rho}\int \frac{\Gamma(f)}{f^2}d\mu + \frac{1}{2\rho^2}\left\|\frac{\Gamma(f)^2}{f^4}\right\|_{\infty}.$$

We refer to [6] for more detail on this case.

Sketch of proof of Theorem 3.1.1. The method of the proof is similar to the one used in the previous proofs. We define $\Psi(s) = P_s(\Phi_p(g))$ and $g = P_{t-s}f$ and we calculate

$$\Psi''(s) = 2P_s\left(\frac{1}{\Phi''(g)}\Gamma_2(\Phi'(g))\right) + CP_s\left(\frac{1}{g^p}\big(g^{p-2}\Gamma(g)\big)^2\right).$$

Then by the $\mathcal{CD}(\rho,\infty)$ criterion we deduce

$$\begin{aligned}\Psi''(s) \geq 2\rho\Psi'(s)+CP_s\big(g^{p-4}\Gamma(g)^2\big) &= 2\rho\Psi'(s)+CP_s\Big(\big(g^{p-2}\Gamma(g)\big)^\beta\big(g^{p-b}\Gamma(g)^{\frac{b}{2}}\big)^{1-\beta}\Big)\\ &\geq 2\rho\Psi'(s) + P_s\big(g^{p-1}\Gamma(g)\big)^\beta P_s\big(g^{p-b}\Gamma(g)^{\frac{\beta}{2}}\big)^{1-\beta}.\end{aligned}$$

Using again $\mathcal{CD}(\rho,\infty)$ criterion, we have $\Gamma(P_tf) \leq \mathrm{e}^{-2\rho t}P_t\left(\sqrt{\Gamma(f)}\right)^2$. We omit the calculations necessary to conclude the proof since they are very similar to the ones given in the case $p \in (1,2)$. □

To conclude this note, we would like to quote an important result proved by F.Y. Wang in the case of the Ornstein-Uhlenbeck semigroup.

Theorem 3.1.2 (F.Y. Wang)**.** Let $Lf = \Delta f - \langle\nabla\Psi,\nabla f\rangle$ with $D = Id$ and $a = \nabla\Psi$ and

$$dX_t = \sqrt{2}dB_t - \nabla\Psi(X_t)dt.$$

Assuming $\rho\in\mathbb{R}$, the following properties are equivalent:

(i) L satisfies $\mathcal{CD}(\rho,\infty)$, i.e. $\mathrm{Hess}(f) \geq \rho Id$.

(ii) The Harnack inequality holds, i.e. $\forall\, f\geq 0$, $\forall x,y\in\mathbb{R}^n$, $\forall p>1$,

$$\big(P_tf\big)^p(x) \leq P_t\big(f^p\big)(y)\exp\left(\frac{p}{p-1}\left\|x-y\right\|^2\frac{2\rho}{\mathrm{e}^{2\rho t}-1}\right).$$

(iii) Let su consider the Wasserstein distance

$$W_2^2(\mu,\nu) = \inf_\pi\int\left\|x-y\right\|^2 d\pi(x,y),$$

where π is any map on $\mathbb{R}^n\times\mathbb{R}^n$ having the measures μ and ν as marginals, then

$$W_2^2(P_t^*\mu,P_t^*\nu)\leq \mathrm{e}^{-2\rho t}W_2^2(\mu,\nu),$$

where $P_t^*\mu = \mathcal{L}(X_t)$ with $X_0=\mu$.

(iv) For all $t\geq 0$

$$W_2^2(P_t^*\delta_x,P_t^*\delta_y)\leq \mathrm{e}^{-2\rho t}W_2^2(\delta_x,\delta_y) = \mathrm{e}^{-2\rho t}\left\|x-y\right\|^2.$$

An important consequence of Wang's Theorem is that by property *(iii)* or property *(iv)* it is possible to deduce the concentration property and this implies the Logarithm-Sobolev inequality.

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