Orthogonal polynomials and diffusions

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(Joint work with S. Orevkov and M. Zani)

Motivations

- Describe natural bases in $\mathcal{L}^2(\mu)$ where computations are easy to made.
- Describe some measures hard to handle in high dimensions through formal manipulations : in particular compute moments.
- Describe examples of Markov diffusions where one may compute explicitly the spectral decomposition, and hence heat kernel measures, etc.
- Try to understand the underlying structure of sets on which such measure exist.
- Understand some specific properties of families of orthogonal polynomials : generating functions, associated Markov sequence problems, hypergroup properties, etc.

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Context

μ probability measure on $\mathbb{R}$ or $\mathbb{R}^d$ such that polynomials are dense in $\mathcal{L}^2(\mu)$.

Natural basis for $\mathcal{L}^2(\mu)$ given by orthogonal polynomials, obtained by orthonormalization of the sequence of monomials.

In dimension 1, orthonormalize the sequence $1, x, \ldots, x^n, \ldots$ to get a (unique up to the sign) sequence of polynomials P_n which are orthogonal and norm 1.

Not unique in higher dimension : for any k, a choice is made of a basis of the orthogonal complement of $\mathcal{P}_{k-1}$ in $\mathcal{P}_k$, where $\mathcal{P}_k$ is the space of polynomials with total degree $\leq k$.

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Context : symmetric diffusion generators L

Symmetry : $\int gL(f)d\mu = \int fL(g)d\mu$.

Diffusion : $L(\Phi(f_1, \cdots, f_k)) = \sum_i L(f_i)\partial_i\Phi + \sum_{ij}\Gamma(f_i, f_j)\partial^2_{ij}\Phi$, where

$$\Gamma(f_i, f_j) = \frac{1}{2}\Big(L(f_if_j) - f_iL(f_j) - f_jL(f_i)\Big).$$

Markov $\forall f,\ \Gamma(f, f) \geq 0$.

In particular $L(1) = 0$ and $\int L(f)d\mu = 0$ (invariance).

In $\mathbb{R}^n$, $\mu(dx) = \rho(x)dx$ then

$$L(f) = \frac{1}{\rho}\sum_{ij}\partial_i\Big(g^{ij}\rho\partial_j f\Big).$$

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Other formulations

If L self adjoint and has discrete spectrum : another natural basis for $\mathcal{L}^2(\mu)$ is given by the eigen vectors of L.
We are looking for the situation where those bases coincide.

In dimension 1 on an interval : eigen vectors of a Sturm Liouville operator with Neuman boundary conditions.

When does the algebra generated by a finite number of eigen vectors of L generates the full σ-algebra of measurable functions ?
In this case, we chose those finite numbers of eigenvectors as coordinates.
They may be algebraically correlated : in which case we face the same problem on an algebraic manifold.

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General remarks

$L(f) = \sum_{ij} g^{ij}(x)\partial^2_{ij} f + \sum_i b^i(x)\partial_i f$

$L(x^i) = b^i(x),\ \ g^{ij}(x) = \Gamma(x^i, x^j).$

$\mathcal{P}_n$:= polynomials with total degree less than n. If there is a basis of $\mathcal{P}_n$ formed with eigenvectors for L then

$$L : \mathcal{P}_n \mapsto \mathcal{P}_n.$$

$b^i(x)$ polynomial degree ≤ 1

$g^{ij}(x)$ polynomial degree ≤ 2.

$\int PL(Q)d\mu = \int QL(P)d\mu$ for any pair of polynomials.

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Dimension 1

Most famous examples

On $\mathbb{R}$: Hermite polynomials : $\mu(dx) = e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$.

On $[0,\infty)$: Laguerre polynomials : $\mu(dx) = C_a x^a e^{-x} dx$.

On $[-1,1]$: Jacobi polynomials $\mu(dx) = C_{a,b}(1-x)^a(1+x)^b dx$.

In those three examples, the associated polynomials are also eigenvectors of Diffusion Operators, that is second order elliptic differential operators.

- Hermite case : $L(f) = f'' - xf'$, $LP_n = -nP_n$.
- Laguerre case : $L(f) = xf'' - (a+1-x)f'$, $L(P_n) = -nP_n$
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How to use it ? Moments

Computation of $\int x^n \, d\mu$ for the Gaussian measure :

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Recurrence formula for the moments.

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Recurrence formula for the moments.

Computation of $\int x^n \, d\mu$ for the Gaussian measure :

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Complex representation for Hermite Polynomials

On $\mathbb{R}^2$, $L = L_x + L_y$ symmetric wrt $d\mu(x)d\mu(y)$.

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Geometric interpretation for Jacobi

Laplace operator on spheres $S^{n-1} \subset \mathbb{R}^n$; x_i coordinates in R^n

$L(x_i) = -(n-1)x_i$.

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$X := 2(x_1^2 + \cdots x_p^2) - 1$, $1 \leq p < n$.

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Geometric interpretation for Hermite and Laguerre

Jacobi to Hermite scale Jacobi on $(-\sqrt{n}, \sqrt{n})$, $a = b = n$, $n \to \infty$

Jacobi to Laguerre move and scale Jacobi on $(0, \sqrt{n})$, limit $a = n \to \infty$, b fixed.

Hermite to Laguerre Hermite on R^d, applied on $f(X)$ with $X = x_1^2 + \cdots x_d^2$: Laguerre with parameter $a = d/2 + 1$.

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Higher dimensional models

Few examples

- Dirichlet measures on the simplex $x_i \geq 0, \sum_i x_i \leq 1$.
- On the unit ball $\sum_i x_i^2 \leq 1$. $\mu(dx) = (1 - \|x\|^2)^a dx$.
- Law of the spectrum of random matrices : GOE, GUE, $SO(n)$, $SU(n)$, $Sp(n)$, and many other on matrices. The variables are then the elementary symmetric functions of the eigenvalues.
- Root systems, Affine Hecke algebras (McDonald polynomials).
- etc.

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Lie group actions

G compact group of matrices acting on $\mathbb{R}^d$ or a linear space (may be a space of matrices).

Examples : $g \mapsto Mg$, $g \mapsto g^* Mg$, etc.

$A \in \mathcal{L}(G)$ $(e^{tA} \in G)$ $X_A(F)(x) = \lim_{t\to 0} \frac{F(e^{tA}x)-F(x)}{t}$.

Then $X_A(F)(x) = \sum_{ijk} A_{ik} x_k \partial_{x_j} F$.
X_A preserves the polynomials of degree $\leq k$ in the variables (x_i).
$L = \sum X_{A_i}^2$ maps polynomials in the entries into polynomials in the entries, and is symmetric with respect to the image of the Haar measure.

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Example Laplace operators (Casimir operators) on compact groups . In general non elliptic in $\mathbb{R}^d$: the associated process lives on an orbit of the group.
Still true for functions which are invariant under actions of subgroups : main source of natural elliptic examples for models.
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General Problem

Find

- all regular open sets $\Omega \subset R^n$, (piecewise smooth boundary)
- all probability measures μ on Ω (with dense polynomials),
- all symmetric diffusion operators L on Ω,

such that $\mathcal{L}^2(\mu)$ has a orthonormal basis formed of eigenvectors for L which are polynomials.

We shall restrict to the elliptic case : $(g^{ij})(x)$ everywhere positive definite on Ω. In this case, the inverse matrix $g_{ij}(x)$ defines a Riemanian metric on Ω.

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General formulation of the problem

$L(f) = \sum_{ij} g^{ij}\partial^2_{ij} f + b^i(x)\partial_i f$

b^i polynomials of degree ≤ 1 and g^{ij} polynomials of degree ≤ 2.

$b^i(x) = \sum_j \partial_j g^{ij}(x) + \sum_j g^{ij}\partial_j \log(\rho)$.

$\partial_i \log(\rho) = \sum_j (g^{-1})^{ij}\hat{L}_j$, where $\hat{L}_i$ is affine function.

In addition : symmetry holds for every pair of polynomials

$\forall i,\ \sum_j g^{ij} n_j \rho(x) = 0$ on $\partial\Omega$, with n_j = normal vector on $\partial\Omega$.

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General Formulation continued

$\partial\Omega$ is in some algebraic surface with degree $\leq 2n$.

With $\{D = 0\}$ the irreducible equation of $\partial\Omega$
$\forall i,\ \sum_j g^{ij}\partial_j D = 0$ on $\{D = 0\}$.

$\forall i,\ \sum_j g^{ij}\partial_j D = L_i D$ for some first order polynomials L_i.

The admissible domains are exactly those for which the above equation admits a non trivial solution.

In general, D divides the determinant of the metric.

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Measures

Given D (boundary equation), with $D = D_1 \cdots D_k$ (irreducible factors)

Equation on ρ

$$\partial_i(\log(\rho)) = (g^{ij})^{-1} L_j,$$

with L_j degree 1.

Compare with

$$g^{ij}\partial D_p = L_i D_p.$$

For any $n_1, \cdots, n_p$, $\rho = D_1^{n_1} \cdots D_k^{n_k}$ is an admissible solution.

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$$\Delta\big(= \det(g)\big) = D_1^{m_1} \cdots D_p^{m_p}.$$

(real decomposition in irreducible factors)
For every irreducible real factor D_j which may be factorized in $\mathcal{C}[X, Y]$, set

$$D_j = (\mathcal{R}_j + i\mathcal{I}_j)(\mathcal{R}_j - i\mathcal{I}_j).$$

There exist real constants (α_i, β_j), and some polynomial Q with $\deg(Q) \leq 2n - \deg(\Delta)$, such that

$$\rho = \prod_i |\Delta_i|^{\alpha_i} \exp\Big(\frac{Q}{\Delta_1^{m_1-1} \cdots \Delta_p^{m_p-1}} + \sum_j \beta_j \arctan \frac{\mathcal{I}_j}{\mathcal{R}_j}\Big).$$

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- $\Omega = (0, \infty)$:

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No other examples (up to affine transformations) (Mazet, '97)

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The 11 compact models in dimension 2 : triangle

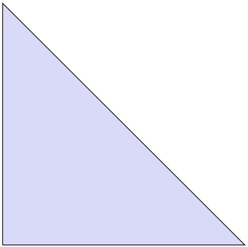

FIGURE: 1 Triangle. Curv= ?

Equation : $xy(1-x-y)=0$.
Measure $\rho(x)=x^a y^b(1-x-y)^c$.

The 11 compact models in dimension 2 : circle

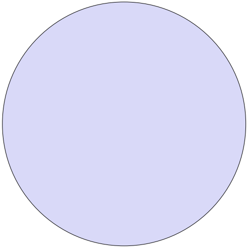

FIGURE: 2 Circle. Curv = ?

Equation : $(1 - x^2 - y^2) = 0$.
Measure $\rho(x) = (1 - x^2 - y^2)^a$.

The 11 compact models in dimension 2 : square

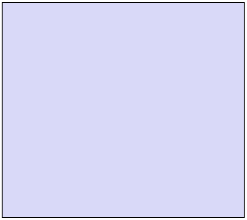

FIGURE: 3 Square (root system $A_1 \times A_1$). Curv =0

Equation : $(1 - x)(1 + x)(1 - y)(1 + y) = 0$.
Measure $\rho(x) = (1 - x)^a (1 + x)^b (1 - y)^c (1 + y)^d$.

The 11 compact models in dimension 2 : double parabola

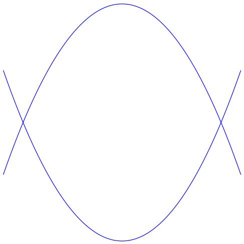

FIGURE: 4 Coaxial Parabolas. Curv= 1

Equation : $(y - x^2 + 1)(y - 1 + \alpha x^2) = 0$.
Measure $\rho(x) = (y - x^2 + 1)^a(y - 1 + \alpha x^2)^b$.

The 11 compact models in dimension 2 : Parabola with two lines 1

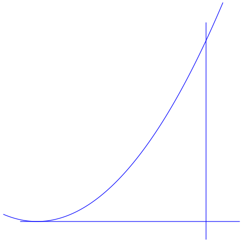

FIGURE: 5 Parabola with two lines 1. Curv= 1

Equation : $(y - x^2)y(1 - x) = 0$.
Measure $\rho(x) = (y - x^2)^a y^b (1 - x)^c$.

The 11 compact models in dimension 2 : Parabola with two lines 2

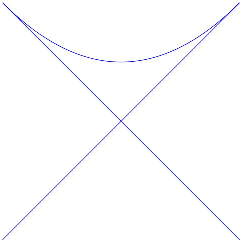

FIGURE: 6 Parabola with two lines 2 (root system B_2). Curv= 0

Equation : $(y - x^2)(y + 2x + 1)(y - 2x + 1) = 0$.
Measure $\rho(x) = (y - x^2)^a(y + 2x + 1)^b(y - 2x + 1)^c$.

The 11 compact models in dimension 2 : Cuspidal Cubic 1

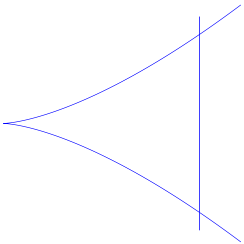

FIGURE: 7 Cuspidal cubic 1. Curv =1

Equation : $(y^2 - x^3)(1 - x) = 0$.
Measure $\rho(x) = (y^2 - x^3)^a(1 - x)^b$.

The 11 compact models in dimension 2 : Cuspidal Cubic 2

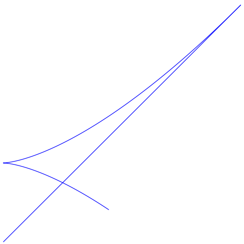

FIGURE: 8 Cuspidal cubic 2. Curv = 1

Equation : $(y^2 - x^3)(2y - 3x + 2) = 0$.
Measure $\rho(x) = (y^2 - x^3)^a(2y - 3x + 2)^b$.

The 11 compact models in dimension 2 : Nodal Cubic

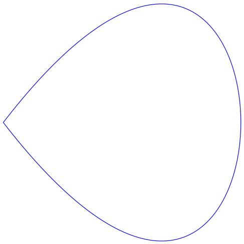

FIGURE: 9 Nodal Cubic. Curv = ?

Equation : $y^2 - x^2(1-x) = 0$.
Measure $\rho(x) = (y^2 - x^2(1-x))^a$.

The 11 compact models in dimension 2 : Swallow Tail

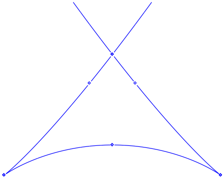

FIGURE: 10 Swallow Tail. Curv = 1

Equation :$4\,x^2 - 27\,x^4 + 16\,y - 128\,y^2 - 144\,x^2 y + 256\,y^3 = 0$
Measure
$\rho(x) = (4\,x^2 - 27\,x^4 + 16\,y - 128\,y^2 - 144\,x^2 y + 256\,y^3)^a$.

The 11 compact models in dimension 2 : Deltoid

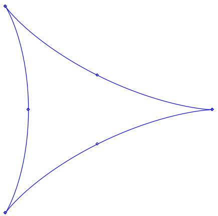

FIGURE: 11 Deltoid (root system A_2). Curv = 0

Equation : $(x^2+y^2)^2+18(x^2+y^2)-8x^3+24xy^2-27=0.$
Measure
$\rho(x)=\Big((x^2+y^2)^2+18(x^2+y^2)-8x^3+24xy^2-27\Big)^a.$

Comments

- Boundaries of Ω have degrees $2, 3$ or 4.
- When the boundary is degree 4 (all except triangle, circle and nodal cubic) , the associated metric has constant curvature.
- Curvature is 0 for square, parabola with two tangents, and deltoid.
- Curvature is constant positive in every other case.
- In circle and triangle case, the metric $g^{ij}(x)$ is not unique.
- Unique in every other case
- For nodal cubic, the metric is unique, the curvature is not constant, but when $a = -1/2$, it has a natural interpretation coming from the 4-d sphere (twisted Hopf fibration).

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$\forall i,\ \sum_j g^{ij}\partial_j D = L_i D$ for some degree 1polynomials L_i and degree 2 g^{ij}.

Implies that $\{D = 0\}$ has no flex points and no flat points (in the complex projective 2-plane) (except at infinity). Moreover, studying the valuations along analytic branches leads to further restrictions on singular points.

Implies that the dual curve has no singular points of some type, hence the curve itself have singular points (use Plucker formulas and and the genus formula).

Leads through the above classification through the inspection of singular points of $\{D = 0\}$

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Example : Cuspidal cubic with 1 tangent : 1 cusp, one tangent and 1 secant.
1 angle $\pi/2$, one $\pi/3$, one $\pi/4$.

Curvature 1 For the associated Laplacian : cut the sphere in 48

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The nodal cubic

Degree 3 boundary.

Laplace operator is not a solution.

If P is the equation of the boundary, only admissible ρ's are P^a.

For $a = -1/2$, comes from Laplace on the 3-d sphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = |z_1|^2 + |z_2|^2 = 1.$$

Functions invariant under $z_1 \mapsto e^{i\theta} z_1$, $z_2 \mapsto e^{2i\theta} z_2$. (Not the Hopf fibration)

Coded with X degree 2 polynomial and Y degree 3 polynomial in x_1, x_2, x_3, x_4.

$$X = x_1^2 + x_2^2, \;\; Y = (x_1^2 - x_2^2)x_3 + 2x_1x_2x_4.$$

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Functions invariant under $z_1 \mapsto e^{i\theta} z_1$, $z_2 \mapsto e^{2i\theta} z_2$. (Not the Hopf fibration)
Coded with X degree 2 polynomial and Y degree 3 polynomial in x_1, x_2, x_3, x_4.

$$X = x_1^2 + x_2^2, \ \ Y = (x_1^2 - x_2^2)x_3 + 2x_1 x_2 x_4.$$

Non compact cases

Any kind of products of intervals (5 different models)

In addition : above a parabola or to the right of the cuspidal cubic

Measures : same as before with some exponential factors (similar to the Laguerre case)

When no boundaries : only the Gaussian measures (indeed the hardest case)

Always appear as limits of compact cases.

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Larger dimension

Genus method not available in higher dimension.

Not even able to prove that in the maximal degree case, the curvature is constant. Is that even true ?

Easy to construct models in 3-d from models in 2-d by double cover (pass from equation $P(x, y) = 0$ to equation $z^2 - P(x, y) = 0$) : works as soon as no cusp ad no double tangents. (Many metrics in this case).

Not able to show that every model should come from Lie group representation. Not even proved in the above double covers.

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Open questions

Understand the geometric interpretations in $2D$.

Find generic classes in higher dimension.

Is the curvature always constant when the Laplace operator is a solution ?

Find good generic formulae for the associated orthogonal polynomials (Rodrigues Formulae).
In dimension 1 : $P_n(x) = \frac{1}{\rho}\partial^n(\rho g^n)$.
Curvature creation when changing the parameters of the measure ?
Discrete case (replace differentiation by finite differences or q-differences).
etc.

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Thank You For Your Attention

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