Errata of the book

*Analysis and Geometry of Markov Diffusion Operators*
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- Page 30, line 8: instead of $\int (LP_t)^2 d\mu$ read $\int (LP_t f)^2 d\mu$, thanks to Kevin Tanguy.

- Page 35, line 10: instead of
  \[ \sum_{y \in E} \mu(x) L(x, y) = 0 \]
read
  \[ \sum_{x \in E} \mu(x) L(x, y) = 0, \]
thanks to Michal Strzelecki.

- Page 36, line 4: instead of $\hat{L}$ read $K$, thanks again to Michal Strzelecki.

- Page 39, line -1: instead of $F(x, T)$ read $F(T, x)$, thanks to Michal Strzelecki.

- Page 42, line -4: the matrix $g$ is supposed to be also definite-positive, thanks to Michal Strzelecki.

- Page 49, line -13: instead of $g = \psi'(f)$ read $g = \psi(f)$, thanks to Michal Strzelecki.

- Page 79, line -5: instead of
  \[ \sum_{i,j=1}^{n} (\partial_{ij}f)^2 \geq \frac{1}{n} (\sum_{i=1}^{n} \partial_{ii}f)^2 \]
read
  \[ \sum_{i,j=1}^{n} (\partial_{ij}^2 f)^2 \geq \frac{1}{n} (\sum_{i=1}^{n} \partial_{ii}^2 f)^2, \]
thanks to Michal Strzelecki.
Page 90, line 11: instead of
\[ k_3(t, d) = \frac{1}{4\pi t^{3/2}} \frac{d}{\sinh(d)} \exp \left( -t - \frac{d^2}{4t} \right) \]
read (add a )
\[ k_3(t, d) = \frac{1}{(4\pi t^{3/2})} \frac{d}{\sinh(d)} \exp \left( -t - \frac{d^2}{4t} \right) \]

Page 129, line -1: instead of
\[ \Gamma(f)(x) = \lim_{k \to \infty} \left( \frac{1}{2t_k} P_{t_k}(f^2)(x) - P_{t_k}(f)(x)^2 \right) \]
read
\[ \Gamma(f)(x) = \lim_{k \to \infty} \frac{1}{2t_k} \left( P_{t_k}(f^2)(x) - P_{t_k}(f)(x)^2 \right) \]
thanks to Michał Strzelecki.

Page 152, line -7: Item (iii) has to be understood as follows, for any functions \( f_1, \cdots, f_k \in A \) and \( \Psi : \mathbb{R}^k \to \mathbb{R} \) a smooth function \( (C^\infty) \), then \( \Psi(f_1, \cdots, f_k) \in A \).

Page 156, line 17: instead of \( f - g \) read \( f + g \) (three times), thanks to Michał Strzelecki.

Page 158, line 13: instead of
\[ H(f)(g, h) = \frac{1}{2} \left[ \Gamma(g, \Gamma(f, h)) + \Gamma(h, \Gamma(f, g)) - \Gamma(f, \Gamma(g, h)) \right] \]
read (add a )
\[ H(f)(g, h) = \frac{1}{2} \left[ \Gamma(g, \Gamma(f, h)) + \Gamma(h, \Gamma(f, g)) - \Gamma(f, \Gamma(g, h)) \right] \]

Page 170, line -6: remove \( L^*(f) \) at the beginning of the formula.

Page 200, line -8 to the end of the page. Replace the paragraph by the following:

The second set (ii) of inequalities, without any boundary condition, appears as a consequence of (iii) by symmetrization and periodization (for \( f : [0, 1] \to \mathbb{R} \) arbitrary, define \( g : [-1, +1] \to \mathbb{R} \) by \( g(x) = f(x) \) for \( x \in [0, +1] \), \( g(x) = f(-x) \) for \( x \in [-1, 0] \), and apply (iii) to \( g \) on the interval \([-1, +1]\) after re-scaling).

Finally (i) is a consequence of (iii) by anti-symmetrization and periodization. For \( f : [0, 1] \to \mathbb{R} \) such that \( f(0) = f(1) = 0 \), define \( g : [-1, +1] \to \mathbb{R} \) by \( g(x) = f(x) \) for \( x \in [0, +1] \), \( g(x) = -f(-x) \) for \( x \in [-1, 0] \). Then
\[ \int_{[0,1]} f^2 dx = \int_{[-1,1]} g^2 \frac{dx}{2} - \left( \int_{[-1,1]} g \frac{dx}{2} \right)^2 \leq \frac{1}{\pi^2} \int_{[-1,1]} g^2 \frac{dx}{2} = \frac{1}{\pi^2} \int_{[0,1]} f^2 dx, \]
where (iii) has been applied to the probability measure \( 1_{[-1,1]} \frac{dx}{2} \) with the optimal constant \( 1/\pi^2 \). The function \( f(x) = \sin(\pi x) \) is an optimal function from a direct computation.
• Page 201, line -7: instead of
\[ \int_K (f^2_\ell - \frac{1}{\mu(K)} \int_K f_\ell^2 d\mu)^2 d\mu, \]
read
\[ \int_K (f_\ell - \frac{1}{\mu(K)} \int_K f_\ell d\mu)^2 d\mu, \]
(thanks to Arnak Dalalyan).

• Page 205, Proposition 4.6.4: instead of
\[ C_{K \cup L} \leq \frac{\mu(K \cap L)}{\mu(K \cup L)} \max(C_K, C_L), \]
read
\[ C_{K \cup L} \leq 2\frac{\mu(K \cap L)}{\mu(K \cup L)} \max(C_K, C_L), \]
(thanks to Michał Strzelecki).

• Page 211, line 5: instead of \( \Gamma(P_t f) = O(t^{-1/2}) \) read \( \sqrt{\Gamma(P_t f)} = O(t^{-1/2}) \).

• Page 240, line -1: instead of \( \int_E f d\mu \) read \( s = f \), thanks to Michał Strzelecki.

• Page 249, Proposition 5.2.7: instead of \( E_1, E_2 \) read \( E_1, E_2 \).

• Page 251, formula (5.3.2), read \((q - 1)^k/2\) instead of \((q - 1)^k\), thanks to Max Fathi.

• Page 263, line 11: instead of \( \Lambda^{q-1}(s) \) in the LHS, read
\[ \frac{d^2}{dq^2} \Lambda^{q-1}(s) \Lambda'(s), \]
moreover the function \( q \) is decreasing, thanks to Michał Strzelecki.

• Page 267, line -7: instead of
\[ \frac{d(x, y)}{2t} \]
in the RHS, read
\[ \frac{d(x, y)}{2\sqrt{t}}, \]
thanks to Michał Strzelecki.

• Page 298, line -8: instead of
\[ P_t(f \log f) - P_t f \log P_t f \leq t \Delta P_t f + \frac{n}{2} \log(1 - \frac{2t}{n} \frac{P_t(f \Delta(\log f))}{P_t f}), \]
read
\[ P_t(f \log f) - P_t f \log P_t f \leq t \Delta P_t f + \frac{n}{2} P_t f \log(1 - \frac{2t}{n} \frac{P_t(f \Delta(\log f))}{P_t f}). \]
Page 298. The proof of Theorem 6.7.3 can be simplified as follows. Let $f$ be a nonnegative function and let, as usual, for $s \in [0, t]$

$$\Lambda(s) = P_s(P_{t-s}f \log P_{t-s}f).$$

As already observed,

$$\Lambda'(s) = P_s(P_{t-s}f \Gamma(\log P_{t-s}f)),
\Lambda''(s) = 2P_s(P_{t-s}f \Gamma_2(\log P_{t-s}f))$$

and the $CD(0, n)$ condition yields the inequality (6.7.6) page 300,

$$\Lambda''(s) \geq 2nP_t f [LP_t f - \Lambda'(s)]^2.$$ 

Now, letting $\varphi(s) = \Lambda(s) - sLP_t f$, the previous inequality can be reformulated as,

$$\varphi''(s) \geq 2 \frac{nP_t f}{\varphi'(s)} \varphi'(s)^2,$$ 

$s \in [0, t]$.

In other words, the map

$$[0, t] \ni s \mapsto \exp\left(-\frac{2}{nP_t f} \varphi(s)\right)$$

is concave.

Then the two inequalities hold true:

$$-\frac{2}{nP_t f} \varphi'(t) \exp\left(-\frac{2}{nP_t f} \varphi(t)\right) \leq \frac{\exp\left(-\frac{2}{nP_t f} \varphi(t)\right) - \exp\left(-\frac{2}{nP_t f} \varphi(0)\right)}{t} \leq -\frac{2}{nP_t f} \varphi'(0) \exp\left(-\frac{2}{nP_t f} \varphi(0)\right).$$

The first inequality can be written as

$$P_t \left(\frac{\Gamma(f)}{f}\right) - LP_t f + \frac{n}{2t} P_t f \geq \frac{n}{2t} P_t f \exp\left(-\frac{2}{nP_t f} (\varphi(0) - \varphi(t))\right),$$

which is a reformulation of inequality (6.7.4), and the second one can be written as

$$-\frac{\Gamma(P_t f)}{P_t f} + LP_t f + \frac{n}{2t} P_t f \geq \frac{n}{2t} P_t f \exp\left(-\frac{2}{nP_t f} (\varphi(t) - \varphi(0))\right),$$

which is a reformulation of inequality (6.7.5). We recover the Li-Yau inequality since the exponential is positive.

Page 301, line 11: instead of

$$\Lambda''(s) \geq \frac{2[LP_t f - \Lambda'(s)]^2}{nP_t f} + \rho \Lambda'(s),$$

read

$$\Lambda''(s) \geq \frac{2[LP_t f - \Lambda'(s)]^2}{nP_t f} + 2\rho \Lambda'(s).$$
• Page 308, additional information on Theorem 6.8.3. For all the computations explained on page 309, the extremal function $f$ has to satisfy some properties.

First, from the identity

$$\int (f^{q-1} - (1 + \epsilon)f)u d\mu = CE(f, u),$$

we get

$$\int f^{q-1}u d\mu = C \int f \left( \frac{1}{C} u - Lu \right) d\mu.$$

That is, if $R_\lambda(u) = g$ with $\lambda = \frac{1 + \epsilon}{C}$, the equality becomes

$$\int (R_\lambda(f^{q-1}) - Cf)g d\mu = 0.$$

This equation implies back that

$$f = \frac{1}{C}R_\lambda(f^{q-1})$$

and then, $f \in D(L)$.

It is proved that $f$ is bounded from above and below (by a strictly positive constant). From the equation satisfied by $f$, we know that $Lf$ is also bounded. To apply the various integration by parts formula, we need to prove that for any constant $a \in \mathbb{R}$, $f^a \in D(L)$. One way to prove it is to show that $\Gamma(f)$ is a bounded function.

From the first formula page 312, we have

$$f = \frac{1}{C}R_\lambda(f^{q-1}),$$

which implies that

$$\sqrt{\Gamma(f)} \leq \frac{1}{C} \int_0^\infty e^{-\lambda t} \sqrt{\Gamma(P_t(f^{q-1}))} dt.$$

Now, since the model satisfies the $CD(0, \infty)$ condition and $f^{q-1}$ is a bounded function, Inequality 4.7.7 page 211 implies that

$$\sqrt{\Gamma(P_t(f^{q-1}))} \leq \frac{||f^{q-1}||_{\infty}}{\sqrt{t}}, \quad t > 0.$$

The two previous inequalities imply that $\Gamma(f)$ is a bounded function.

• Page 315, formula (6.9.2): instead of $\hat{L}$, read $\hat{L}(f)$.
• Page 317, line 12: instead of $\nabla W(f)$, read $\Gamma(W, f)$.
• Page 318, line 13: instead of $\mu$, read $\mu_\gamma$.
• Page 321, Proposition 6.9.6. The proposition and the proof have to be replaced by the following (see also [1] for a more developed proof).
Proposition 6.9.6 Let $d\mu = e^{-W}d\mu_\#$ and $\alpha \in \mathbb{R}$, then

$$S_\alpha(\mu, \Gamma) = \gamma_n(\alpha)[sc_\# - \alpha \Delta_\# W + \beta_n(\alpha)\Gamma(W)]$$

is $n$-conformal invariant where

$$\beta_n(\alpha) = \frac{\alpha(n - 2n_0 + 2) - 2(n_0 - 1)}{2(n - n_0)}$$

and

$$\gamma_n(\alpha) = \frac{n - 2}{4(n_0 - 1) - 2\alpha(n - n_0)}.$$ 

Proof

It is enough to check that $S_\alpha(\mu, \Gamma)$ satisfies the condition (6.9.1). The measure $\mu$ is transformed to $\hat{\mu} = c^{-n}\mu$, and $\Gamma$ to $\hat{\Gamma} = c^2\Gamma$. From the previous computations, $sc_\#$ becomes

$$\hat{sc}_\# = c^2[sc_\# + (n_0 - 1)(2\Delta_\# \tau - (n_0 - 2)\Gamma(\tau))],$$

$W = -\log \frac{dw}{d\mu_\#}$ becomes

$$\hat{W} = -\log \frac{d\hat{\mu}}{d\mu_\#} = -\log \frac{c^{-n}d\mu}{c^{-n_0}d\mu_\#} = -\log c^{n_0-n} \frac{d\mu}{d\mu_\#} = W + (n - n_0)\tau,$$

and finally, $\Delta_\#$ becomes

$$\hat{\Delta}_\# = c^2[\Delta_\# - (n_0 - 2)\Gamma(\tau, \cdot)].$$

So,

$$S_\alpha(c^{-n}\mu, c^2\Gamma) = c^2\gamma_n(\alpha)\left[sc_\# + [2(n_0 - 1) - \alpha(n - n_0)]\Delta_\#(\tau)
+ [\beta_n(\alpha)(n - n_0)^2 - (n_0 - 1)(n_0 - 2) + \alpha(n_0 - 2)(n - n_0)]\Gamma(\tau)
- \alpha\Delta_\#(W) + [\alpha(n_0 - 2) + 2\beta_n(\alpha)(n - n_0)]\Gamma(\tau, W) + \beta_n(\alpha)\Gamma(W)\right].$$

It has to be equal to

$$c^2\left[\gamma_n(\alpha)[sc_\# - \alpha\Delta_\#(W) + \beta_n(\alpha)\Gamma(W)] + \frac{n - 2}{2}\left(\Delta_\#(\tau) - \Gamma(W, \tau) - \frac{n - 2}{2}\Gamma(\tau)\right)\right].$$

On can check the values of $\gamma_n(\alpha)$ and $\alpha_n(\alpha)$ proposed do the job.

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- Page 338, line -1: instead of $I(u)$, read $I_{\mu,F}(u)$.
- Page 364, line -6: The sentence starting by In the finite measure case... is not correct. It has to be replaced by the following one: In the finite measure case, the tight Nash inequality (3.2.3), p. 281, corresponds to a function $\Phi$ which is the inverse function of $(1, +\infty) \ni x \mapsto (x^{1+2/n} - x)/C.$
• Page 372, line -5: instead of $e^{-C/t}$, read $e^{-t/C}$.

• Page 373, line -13: instead of $w(x) = p(x)^{1/2}(1+x^2)^{-\beta}$, read $w(x) = p(x)^{-1/2}(1+x^2)^{-\beta}$ (thanks to Persi Diaconis).

• Page 425, Theorem 8.6.3: the set $A_{d_t}$ should be here the $d_t$-closed neighborhood of $A$ instead of the open one ($A_{d_t} = \{x \in E; d(x,A) \leq d_t\}$ instead of $A_{d_t} = \{x \in E; d(x,A) < d_t\}$).

• Page 448, line -7: (the line before formula (9.3.5)) the integration is w.r.t. the measure $u^{1-1/n}dx$ instead of $udx$ (thanks to Emanuel Milman).

• Page 464, formula (9.7.4) should be

$$W_2^2(P_tf\mu, P_tg\mu) \leq W_2^2(f\mu, g\mu) + 2n(\sqrt{t} - \sqrt{s})^2,$$

thanks to Luigia Ripani.

• Page 516, in the formula (C.6.5) the last term should be

$$H(f_i)(f_j, f_l)$$

instead of

$$H(f_i)(f_i, f_l)$$

thanks to François Bolley.

• Page 514, line -2: instead of wrapped product, read, of course, warped products!

References