On equations in tree-free groups

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Summary

- Equations in groups

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- Tree-free groups

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- Tree-free groups
- Lyndon's equation in tree-free groups

Equations in groups

- G - a group
- $X = \{X_1, \ldots, X_n\}$ - a set of variables.

An *equation* in variables $X_1, \ldots, X_n$ with coefficients g_j in G is a formal expression of the form

$$g_1 X_{i_1}^{\epsilon_1} g_2 X_{i_2}^{\epsilon_2} \ldots X_{i_m}^{\epsilon_m} g_{m+1} = 1$$

where $\epsilon_j \in \{1, -1\}$.

Equations in functional notation:

$$f(X_1, \ldots, X_n, g_1, \ldots, g_{m+1}) = 1 \qquad (1)$$

A tuple $(h_1, \ldots, h_n)$ of elements from G is a *solution* of the equation (1) if

$$f(h_1, \ldots, h_n, g_1, \ldots, g_{m+1}) = 1.$$

Questions

Diophantine Problem (DP)

Does there exist an algorithm which for any equation $f = 1$ with coefficients in G determines whether $f = 1$ has a solution or not?

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Search Diophantine Problem (SDP)

Does there exist an algorithm that finds a solution (all solutions) for any solvable equation $f = 1$ in G?

Example

- F - free group on a and b, X and Y variables.

$$XYX^{-1}Y^{-1} = aba^{-1}b^{-1}$$

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$$X = a,\ Y = ba^m$$

Solving equations and the geometry of the groups

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- Free groups (Makanin (1982) and Razborov (1985))
- Torsion-free hyperbolic groups (Rips, Sela)
- Some relatively hyperbolic groups (Dahmani, Groves)
- Various free constructions (Diekert and others)

Connections

Tarski's Conjecture (Sela, Kharlampovich and Myasnikov)
The elementary theories of non-abelian free groups with different number of generators coincide.

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Geometry (Nielsen, Edmunds-Comerford, Culler, etc.)
Quadratic equations (every variable appears twice) : well understood.

Tree-free groups

Λ-spaces

- An *ordered abelian group* is an abelian group Λ, together with a total ordering $\leq$ on Λ, such that for all a, b and $c \in \Lambda$, $a \leq b$ implies $a + c \leq b + c$.

Λ-spaces

- An *ordered abelian group* is an abelian group Λ, together with a total ordering $\leq$ on Λ, such that for all a, b and $c \in \Lambda$, $a \leq b$ implies $a + c \leq b + c$.

- A Λ-*metric space* (X, d) can be defined in the same way as a conventional metric space.

  That is, $d : X \times X \to \Lambda$ is symmetric, satisfies the triangle inequality and satisfies $d(x, y) = 0$ if and only if $x = y$.

Λ-trees

Definition. A Λ-tree is a geodesic Λ-metric space (X, d) such that:

(a) if two segments of (X, d) intersect in a single point, which is an endpoint of both, then their union is a segment;

(b) the intersection of two segments with a common endpoint is also a segment.

Groups acting on Λ-trees

• Let G be a group that acts on X via isometries. Isometries of Λ-trees are analogous to those of ordinary trees in that we can classify them as $inversions$, $elliptic$ and $hyperbolic$.

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• If g is hyperbolic, let A_g be the maximal g-invariant linear subtree of X on which g acts by translation.

• $Translation\ length\ function$ of any non-inversion g given by $\|g\|$.

Tree-free groups

• We consider only *free* actions, that is, actions without inversions in which no non-trivial element of the group fixes a point in the tree. Thus all non-trivial isometries are hyperbolic.

• tree-free group $=$ a group acting freely on a Λ-tree for some Λ.

Properties of Λ-free groups

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More on Λ-free groups

Archimeadean actions: well understood

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Non - Archimeadean actions

- Structure theory for $\mathbb{Z}^n$-free groups (Bass, Martino - O Rourke) and $\mathbb{R}^n$-free groups (Guirardel).
- All limit groups are $\mathbb{R}^m$-free groups (Sela, Kharlampovich - Myasnikov, Guirardel).

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Definition. A group G is fully residually free if for every finite set $S \subset G$ of non-trivial elements there exists a homomorphism $\phi : G \longrightarrow F$ into a free group F such that $\phi(g) \neq 1$ for every $g \in S$.

Lyndon's Equation

Theorem (Lyndon, Schützenberger, Baumslag, . . .)

Let F be a free group, and let X, Y and Z be elements in the free group. If

$$X^p Y^q Z^r = 1$$

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$$X^p Y^q Z^r = 1$$

where $p, q, r \geq 2$, then X, Y and Z commute.

Lyndon's equation in tree-free groups

Theorem (N. Brady, C., A. Martino, S. O Rourke)

Let G be a tree-free group and let x, y, z be elements in G. If $x^p y^q = z^r$ with $p,\ q,\ r \geq 4$, then $x,\ y$ and z commute.

Observation

The equation $x^2y^2 = z^2$ implies that x, y and z commute in free groups, while in Λ-free groups this is not true, since the exceptional surface group

$$\langle x, y, z \mid x^2y^2z^2 = 1 \rangle$$

acts freely on a $\mathbb{Z}^2$-tree (Gaglione, Spellman).

Consequences

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- G_{pqr} do not act freely on any Λ-tree, for $p, q, r \geq 4$.
- G_{pqr} are all small-cancellation groups. $(C(6) - T(4))$

Corollary. *The groups G_{pqr} form a sequence of word hyperbolic groups which cannot act freely, and without inversions, by isometries on any Λ-tree.*

Actions on negatively-curved spaces

Proposition. The groups G_{pqr} admit CAT(-1) structures corresponding to each isometry class of triangles in the hyperbolic plane.

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• G_{pqr} are fundamental groups of graphs of groups. with underlying graph a tripod, edge groups all infinite cyclic, valence 1 vertex groups all infinite cyclic, and valence 3 vertex group being free of rank 2.

Observation

The groups

$$\langle x_1, x_2, \ldots, x_n \,|\, x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} = 1 \rangle$$

are $\mathbb{Z}^2$-free for $n \geq 4$, provided at least four α_i are non-zero (Bass, Martino - O Rourke).

Proof

A_x, A_y and A_z: axes of translation of x, y and z.

Assumption: $\|x^p\| \geq \|y^q\| \geq \|z^r\|$.

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A_x, A_y and A_z: axes of translation of x, y and z.

Assumption: $\|x^p\| \geq \|y^q\| \geq \|z^r\|$.

- A_x and A_y do not intersect;
- A_x and A_y do intersect.

Let $\Delta(x, y)$ be the intersection of the two axes. Then

$$|\Delta(x, y)| < \|x\| + \|y\|.$$

A_x and A_y intersect

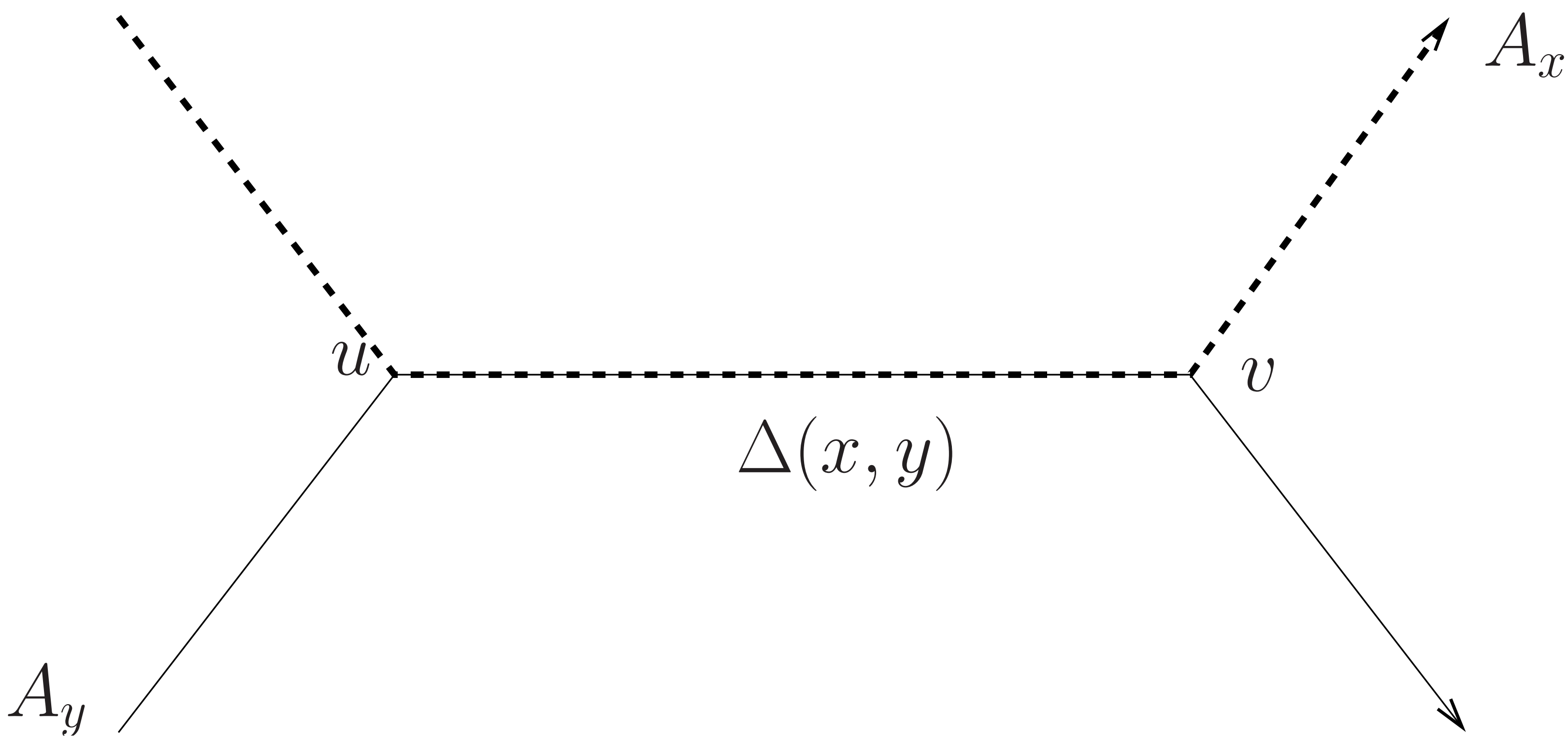

Figure 1: Coherent axes

A_x and A_y intersect incoherently

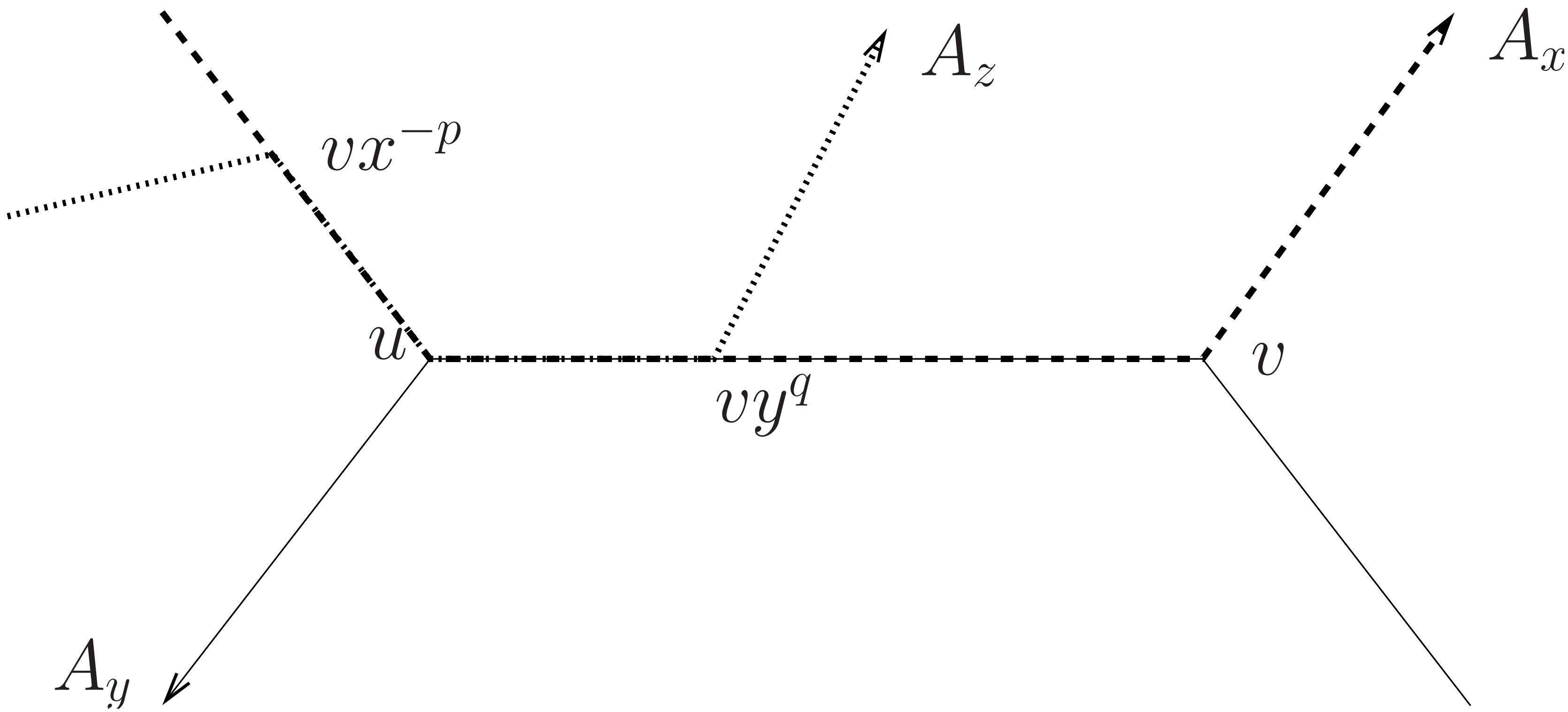

Figure 2: Large Intersection

A_x and A_y intersect incoherently

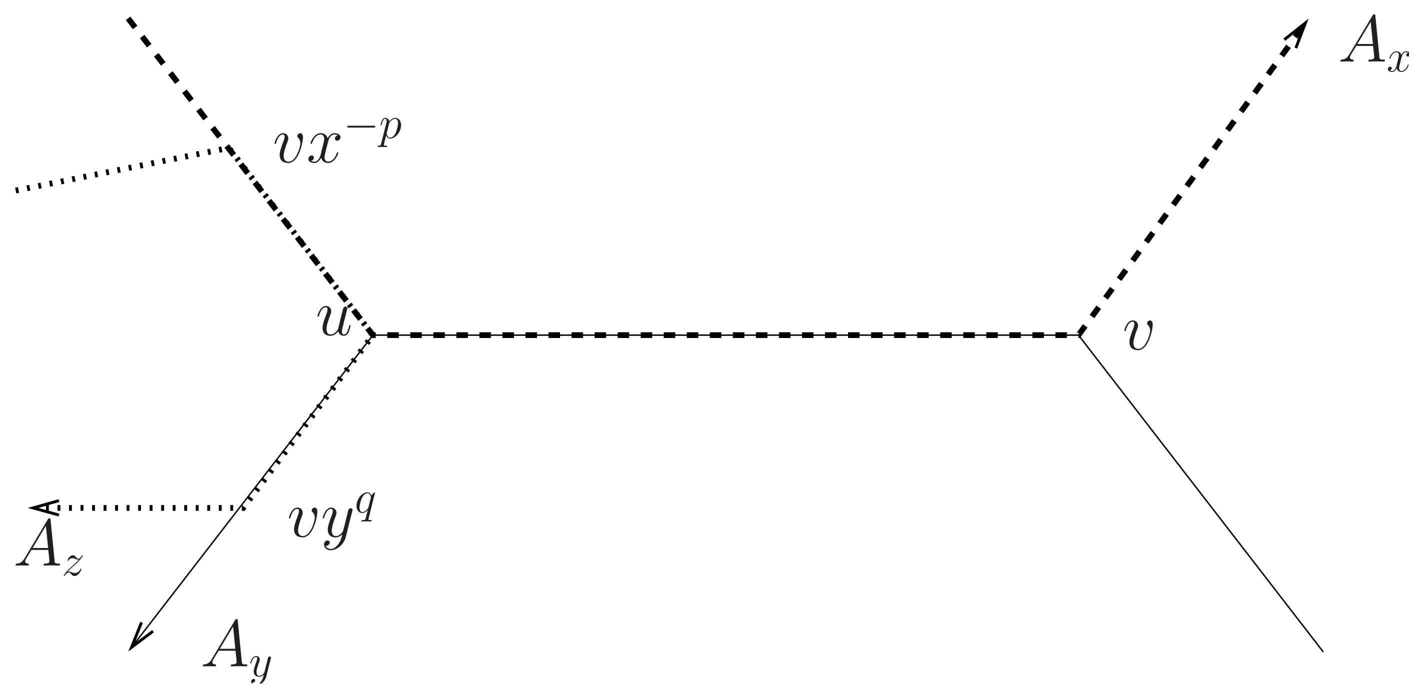

Figure 3: Small intersection

A_x and A_y intersect incoherently

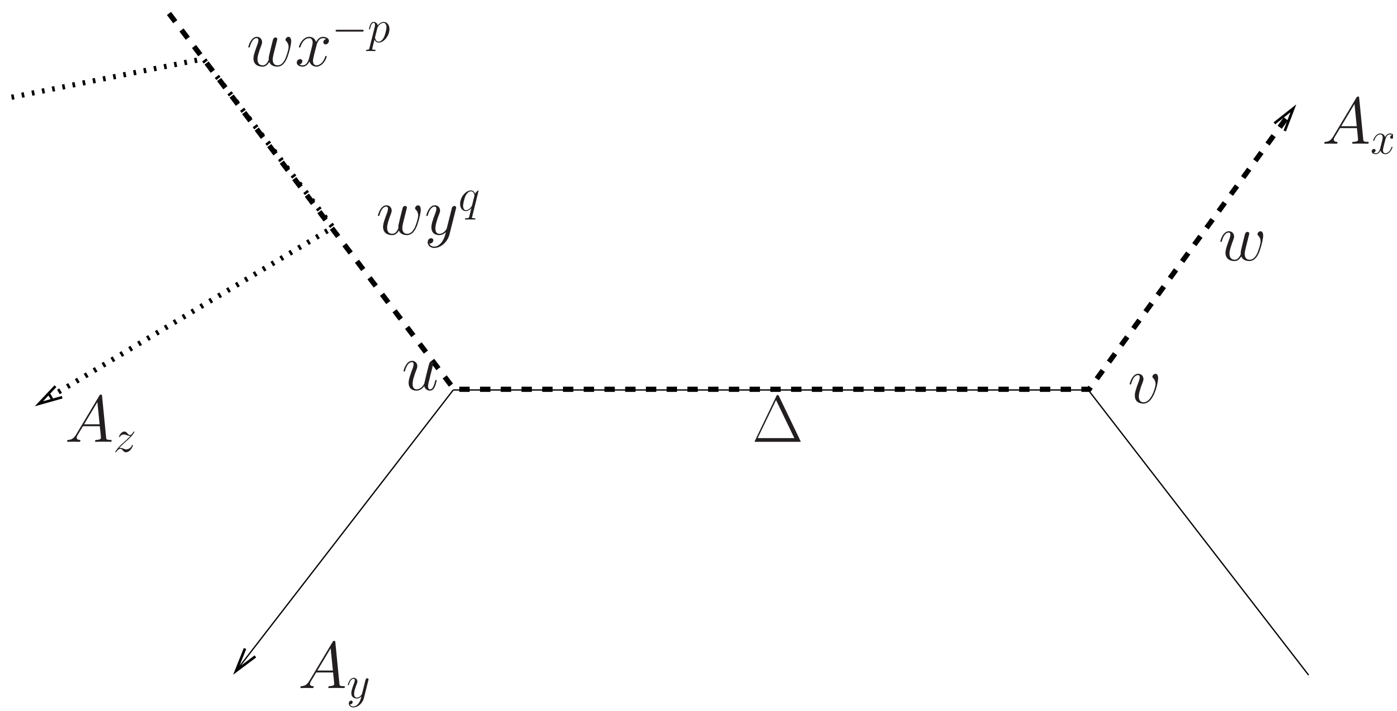

Figure 4: Exact intersection

More equations?

The equation $[X, Y] = Z^n$ has no non-trivial solutions in a free groups, where $n \geq 2$. What about tree-free groups?

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Theorem.(Martino-O Rourke) The group

$$G = \langle x, y, x_1, \ldots, x_n \mid [x, y] = w(x_1, \ldots, x_n) \rangle$$

acts freely on a $\mathbb{Z} \times \mathbb{Z}$-tree, where $w(x_1, \ldots, x_n)$ is a word in $\{x_1, \ldots, x_n\}$.

- In a free group, the equation

$$[X_1, Y_1][X_2, Y_2] = Z^m$$

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(Comerford - Comerford - Edmunds)

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- The group

$$G = \langle x, y, z, t \mid [x, y]x^m = [z, t] \rangle$$

acts freely on a $\mathbb{Z}^n$-tree. (Martino-O Rourke)