Small-cancellation approach to constructing boundedly generated groups

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Abstract

1. for every $n \geq 27$, there exists an infinite simple group G and its elements $a_1, \ldots, a_n$ such that every element of G can be presented in the form $a_1^{k_1} \ldots a_n^{k_n}$, $k_i \in \mathbb{Z}$;
2. for every $n \geq 63$, there exists a torsion-free group G and its elements $a_1, \ldots, a_n$ such that:
    a) every element of G has a *unique* presentation in the form $a_1^{k_1} \ldots a_n^{k_n}$, $k_i \in \mathbb{Z}$, and
    b) every pair of elements from among $a_1, \ldots, a_n$ freely generate a (rank-2) free subgroup of G.

Outline

Bounded generation

Definition

A group G is *n-boundedly generated* iff it has a system of generators $a_1, \dots, a_n$ such that

$$G = \{a_1^{k_1} \dots a_n^{k_n} | k_1, \dots, k_n \in \mathbb{Z}\}.$$

David Carter and Gordon Keller, 1983:
any matrix in $SL_n(\mathcal{O})$, where $n \geq 3$ and $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}$, is the product of a bounded number of elementary matrices (bounded depending only on n and $\mathcal{O}$).

Bounded generation was studied in connection with the
Congruence Subgroup Property (V. Platonov, A. Rapinchuk, 1992),
Kazhdan's Property (T) (Y. Shalom, 1999).

Questions about bounded generation

- Is every boundedly generated group residually finite?
- Problem 13.11 by Vasiliy Bludov in the Kourovka Notebook (1995):

  If a torsion-free group G has a finite system of generators $a_1, \dots, a_n$ such that every element of G has a unique presentation in the form $a_1^{k_1} \dots a_n^{k_n}$ where $k_i \in \mathbb{Z}$, is it true then that G is virtually polycyclic?

Definition

A group G is *polycyclic* if there are subgroups $G_0, \dots, G_n$ such that

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = \{1\},$$

and each of the quotients G_{i-1}/G_i is cyclic.
A group is *virtually polycyclic* if it has a polycyclic subgroup of finite index.

Infinite simple boundedly generated group

Theorem 1

For every $n \geq 27$*, there exists an infinite simple* 2*-generated group* G *and elements* $a_1, \dots, a_n \in G$ *such that* $G = \{\, a_1^k \dots a_n^k \mid k \in \mathbb{N} \,\}$.
Such a group is in particular n*-boundedly generated.*

Torsion-free group with finite regular file basis but not virtually polycyclic

Theorem 2

For every $n \geq 63$, there exists a group G and pairwise distinct elements $a_1, \ldots, a_n$ in G such that:

0. *G is generated by $\{a_1, \ldots, a_n\}$*;
1. *every $n - 21$ elements out of $\{a_1, \ldots, a_n\}$ freely generate a free subgroup F such that every two elements of this subgroup F are conjugate in G only if they are conjugate in F itself* (*in particular*, *G is not virtually polycyclic*);
2. *for every element g of G, there is a unique n-tuple $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ such that $g = a_1^{k_1} \ldots a_n^{k_n}$*;
3. *G is torsion-free*;
4. *G is the direct limit of a sequence of hyperbolic groups with respect to a family of surjective homomorphisms*;
5. *G is recursively presented and has solvable word and conjugacy problems*.

Presentations by generators and defining relations

Every group can be presented by a set of generators and a set of defining relations among them.

Example

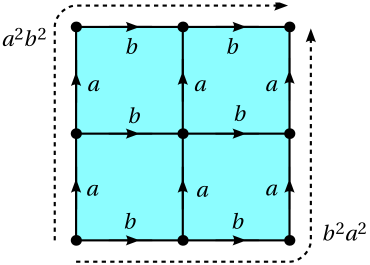
- Infinite cyclic group: $\mathbb{Z} \cong \langle\, a \,\|\, \rangle$.
- Free abelian group of rank 2: $\mathbb{Z}^2 \cong \langle\, a, b \,\|\, [a, b] = 1 \,\rangle$.
- Dihedral group of order $2n$: $D_n \cong \langle\, r, s \,\|\, r^n = 1, s^2 = 1, srs = r^{-1} \,\rangle$

Van Kampen diagrams

Claim

In the group $\langle\, a, b \,\|\, aba^{-1}b^{-1} = 1 \,\rangle$*,*
a^2b^2 *equals* b^2a^2*.*

Proof.

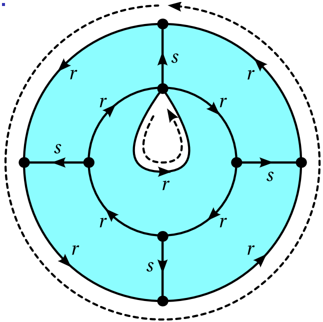


□

Claim

In $\langle\, r, s \,\|\, r^5 = 1, srs^{-1} = r^{-1} \,\rangle$*,*
r^4 *and* r *are conjugate.*

Proof.



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Diagrams may look strange:

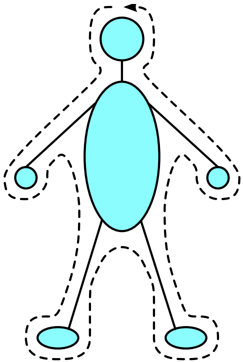


Figure: A disc diagram

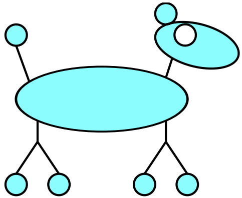


Figure: An annular diagram

Condition $C'(\lambda)$

Definition

A set S of group words is *symmetrised* if all elements of S are cyclically reduced, and for every $w \in S$, all cyclic shifts of w and w^{-1} are also in S.
A symmetrized set of group words S satisfies the *small cancellation condition* $C'(\lambda)$, $\lambda > 0$, if for any two its distinct elements w_1 and w_2 with a common prefix s, $|s| < \lambda \min\{|w_1|, |w_2|\}$.

Grindlinger's Lemma (simplified version)

Let $\langle A \,\|\, R \rangle$ be an arbitrary group presentation such that R is symmetrised and satisfies $C'(1/6)$. Let w be an arbitrary non-trivial cyclically reduced group word over A that represents the identity 1 in the group defined by $\langle A \,\|\, R \rangle$.
Then there exist a subword s of w and a relator $r \in R$ such that
s is a subword of r and $|s| > (1/2)|r|$.

Condition $\mathscr{B}(\lambda,\mu)$

Consider a group presentation $\langle A \,\|\, v_i = u_i, i \in I \rangle$.
It satisfies the *condition* $\mathscr{B}(\lambda,\mu)$, $\lambda \geq 0$, $\mu \geq 0$, if:

1. for every i, u_i is reduced;
2. for every i, $|v_i| \leq \lambda |u_i|$;
3. for every i_1, i_2, σ_1, σ_2 ($\sigma_1, \sigma_2 \in \{\pm 1\}$), p_1, p_2, q_1, q_2, s, if $u_{i_1}^{\sigma_1} = p_1 s q_1$ and $u_{i_2}^{\sigma_2} = p_2 s q_2$, then either $(i_1, \sigma_1, p_1, q_1) = (i_2, \sigma_2, p_2, q_2)$, or

$$\mu |u_{i_1}| \geq |s| \leq \mu |u_{i_2}|.$$

Lemma about condition $\mathscr{B}(\lambda,\mu)$

Lemma

Let $\langle A \,\|\, v_i = u_i, i \in I\rangle$ be a presentation which satisfies $\mathscr{B}(\lambda,\mu)$, and let G be the group defined by this presentation. Let $r_i = u_i v_i^{-1}$ for every $i \in I$.
Let $\gamma = 2\lambda + 13\mu$.
Let w be an arbitrary nontrivial cyclically reduced group word over A such that $w \overset{G}{=} 1$.
Then in the free group F on A, w can be written as

$$w \overset{F}{=} z_1 r_{i_1}^{\sigma_1} z_1^{-1} \dots z_n r_{i_n}^{\sigma_n} z_n^{-1}$$

(here $\sigma_1, \dots, \sigma_n \in \{\pm 1\}$) so that:

1. $|w| \geq (1-\gamma)(|r_{i_1}| + \cdots + |r_{i_n}|)$,
2. *for some k, there is a common subword s of w and $u_{i_k}^{\sigma_k}$ such that* $|s| \geq \dfrac{1-\gamma}{2} |r_{i_k}^{\sigma_k}|$.

Inductive construction of presentations of b.g. groups

Let $A = \{x_1, \dots, x_n\}$ be a big finite alphabet.
Let $w_1, w_2, w_3, \dots$ be the list of all nontrivial reduced group words over A in the *deg-lex* order.

Start with the free group $\langle A \| \varnothing \rangle$ and "carefully" impose relations of the form

$$w_i = x_1^{m_i} \dots x_n^{m_i},$$

where $m_i \gg |w_i|$.

For an infinite boundedly generated simple group, these relations should be chosen so that to "kill" all finite quotients,
and the quotient over a maximal proper normal subgroup should be taken as a desired simple group.

If the group has to be 2-generated, this can be achieved by imposing $n-2$ additional relations.

Most edges on the boundary

Any reduced disc or annular diagram over any of the constructed presentations has most of its edges on the boundary (because of small cancellatons).

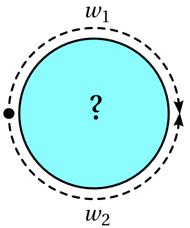
This is shown by distributing all internal edges among faces so that the number of edges associated to each face is very small compared to the contour length of the face.

Minimal diagram

Hardest part of Theorem 2:

> *for every element g of G, there is a unique n-tuple $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ such that $g = a_1^{k_1} \ldots a_n^{k_n}$.*

Proved by contradiction: assume there is no uniqueness, and consider a minimal diagram of the form ....



where w_1 and w_2 are distinct words of the form $a_1^{k_1} \ldots a_n^{k_n}$.

C'est tout.