Construction of certain boundedly simple groups and simple groups of infinite commutator width

Small-cancellation approach

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Abstract

1. There exist finitely generated infinite *boundedly* simple groups of arbitrarily large commutator width;
2. there exists a finitely generated (infinite) simple group of infinite commutator width;
3. such groups can be constructed with decidable word and conjugacy problems.

Outline

Definition of bounded simplicity

Notation: $x^y = yxy^{-1}$ (the conjugate of x by y).

Definition

A group G is *n-boundedly simple* if for every two elements $g, h \in G \setminus \{1\}$,

$$\big(\exists m \leq n\big)\,\big(\exists \sigma_1, \ldots, \sigma_m \in \{\pm 1\}\big)\,\big(\exists x_1, \ldots, x_m \in G\big)\,\big(g = (h^{\sigma_1})^{x_1} \ldots (h^{\sigma_m})^{x_m}\big).$$

A group G is *boundedly simple* if it is n-boundedly simple for some $n \in \mathbb{N}$.

Every boundedly simple group is simple, but the converse is not generally true (e.g., it is not true for an infinite alternating group).

Remark

The property of being n-boundedly simple is first-order, but the property of being simple is not.

Questions about bounded simplicity

Can a finitely generated boundedly simple group be infinite?

Can it have free non-cyclic subgroups?

Ivanov's and Osin's theorems

Theorem (S. Ivanov, 1989)

For every big enough prime p $(p > 10^{78})$, *there exists a* 2*-generated infinite group of exponent p*, *in which there are exactly p distinct conjugacy classes*, *and therefore*, *every subgroup of order p has elements from all of these classes*.

The group whose existence is stated in this theorem is $(p-1)$-boundedly simple.
Ivanov's proof uses techniques of graded diagrams.

Theorem (D. Osin, 2004)

Any countable group G can be embedded into a 2*-generated group C such that in C each two elements of the same order are conjugate*, *and every finite-order element of C is conjugate to an element of G*.

Osin's theorem shows that every countable torsion-free group can be embedded into a 2-generated 1-boundedly simple group.
(1-boundedly simple means all nontrivial elements are conjugate.)
Osin's proof uses theory of relatively hyperbolic groups.

Definition of commutator width

Notation: $[x,y] = xyx^{-1}y^{-1}$ (the commutator of x and y).

Definition

If G is a group, then the *commutator length* of an element $g \in [G,G]$, denoted $\mathrm{cl}_G(g)$, is the minimal n such that there exist elements $x_1,\dots,x_n,y_1,\dots,y_n \in G$ such that

$$g = [x_1,y_1]\dots[x_n,y_n].$$

The *commutator width* of a group G, denoted $\mathrm{cw}(G)$, is the maximum of the commutator lengths of elements of $[G,G]$.

Questions about commutator width of simple groups

1951 Oystein Ore conjectured that all elements in every non-abelian finite simple group are commutators. (In terms of commutator width: the commutator width of every finite simple group is 1.) This question still remains open.

1977 Martin Isaacs noted that no non-abelian simple group, finite or infinite, was known to contain a non-commutator (i.e., to have commutator width greater than 1).

1999 Valerij Bardakov posed the following question (Problem 14.13 in The Kourovka Notebook):

> *Does there exist a (finitely presented) simple group of infinite commutator width?*

Work of Barge, Gambaudo, and Ghys

Simple groups of infinite commutator width, realised as groups of certain surface diffeomorphisms, have been studied in:

- Jean Barge and Étienne Ghys, *Cocycles d'Euler et de Maslov*, Math. Ann. **294** (1992), no. 2, 235–265;
- Jean-Marc Gambaudo and Étienne Ghys, *Commutators and diffeomorphisms of surfaces*, Ergod. Th. & Dynam. Sys. **24** (2004), 1591–1617.

Those groups are not finitely generated, and the infinity of their commutator widths is established by constructing *non-trivial homogeneous quasi-morphisms*.

Boundedly simple groups

Theorem 1

For every $n \in \mathbb{N}$, there exists a torsion-free 2-generated simple group G with a rank-2 free subgroup H such that:

1. *for every $g \in G$ and every $x \in G \setminus \{1\}$, there exist $y_1, \dots, y_{2n+2}$ in G such that $g = x^{y_1} \dots x^{y_{2n+2}}$; and*
2. *for every $h \in H \setminus \{1\}$ and for every $m \geq 2n$, $\mathrm{cl}_G(h^m) > n$.*

In particular, G is $(2n+2)$-boundedly simple, and $n+1 \leq \mathrm{cw}(G) \leq 2n+2$. Moreover, there exists such a group G with decidable word and conjugacy problems.

Infinite commutator width

Theorem 2

There exists a torsion-free 2-generated simple group G with a rank-2 free subgroup H such that for every $h \in H \setminus \{1\}$,

$$\lim_{n\to+\infty} \mathrm{cl}_G(h^n) = +\infty.$$

In particular, G has infinite commutator width.
Moreover, there exists such a group G with decidable word and conjugacy problems.

Construction of a f.g. simple group of infinite c.w. (slide 1 of 3)

Let $\mathfrak{A} = \{a, b\}$.

Let $\{\lambda_n\}_{n=4,5,6,\dots}$ and $\{\mu_n\}_{n=4,5,6,\dots}$ be sequences of sufficiently small positive numbers tending to 0 sufficiently fast, e.g., $\lambda_n = \frac{1}{100n}$ and $\mu_n = \frac{1}{1000n^2}$.

Let $w_4, w_5, w_6, \dots$ be the list of all reduced group words over $\{a, b\}$ ordered deg-lex, so $0 = |w_4| < |w_5| \leq |w_6| \leq \dots$.

Construction of a f.g. simple group of infinite c.w. (slide 2 of 3)

Let for every $n = 4, 5, 6, \ldots$,

$$r_{a,n} = w_n^{u_{n,1}} \ldots w_n^{u_{n,n}} a^{-1}, \qquad r_{b,n} = w_n^{u_{n,n+1}} \ldots w_n^{u_{n,2n}} b^{-1}$$

where $\{r_{x,n}\}$ and $\{u_{n,i}\}$ are families of group words over $\{a, b\}$ such that:

1. for every $n = 4, 5, 6, \ldots$,
   1.1 $|u_{n,1}| = |u_{n,2}| = \cdots = |u_{n,2n}|$, and hence $|r_{a,n}| = |r_{b,n}|$,
   1.2 $1 + n|w_n| \leq \lambda_n |r_{a,n}|$,
   1.3 $|u_{n,1}| \leq |u_{n+1,1}|$ and $\mu_n |r_{a,n}| \leq \mu_{n+1} |r_{a,n+1}|$;
2. the family $\{u_{ni}\}_{n=4,5,6,\ldots;i=1,\ldots,2n}$ satisfies the following small-cancellation condition: if $u_{n_1 i_1}^{\sigma_1} = p_1 s q_1$ and $u_{n_2 i_2}^{\sigma_2} = p_2 s q_2$ ($\sigma_1, \sigma_2 \in \{\pm 1\}$), then either

   $$(n_1, i_1, \sigma_1, p_1, q_1) = (n_2, i_2, \sigma_2, p_2, q_2),$$

   or

   $$\mu_{n_1} |r_{a,n_1}| \geq |s| \leq \mu_{n_2} |r_{a,n_2}|;$$

3. if s is a common subword of $u_{n,i}$ and of the concatenation of several copies of $a^{\pm 2}$ and $b^{\pm 2}$, then

   $$|s| \leq \mu_n |r_{a,n}| = \mu_n |r_{b,n}|.$$

Construction of a f.g. simple group of infinite c.w. (slide 3 of 3)

Inductively construct a presentation $\langle a, b \,\|\, \mathscr{R} \rangle$ as follows:

1. $\mathscr{R}_0 = \mathscr{R}_1 = \mathscr{R}_2 = \mathscr{R}_3 = \varnothing$,
2. For $n = 4, 5, \dots$, if the relation '$w_n = 1$' is a consequence of the relations '$r = 1$', $r \in \mathscr{R}_{n-1}$, then define $\mathscr{R}_n = \mathscr{R}_{n-1}$;
   otherwise, define $\mathscr{R}_n = \mathscr{R}_{n-1} \cup \{r_{a,n}, r_{b,n}\}$.

Let $\mathscr{R} = \bigcup_{n=4}^{+\infty} \mathscr{R}_n$.

Let G be the group presented by $\langle a, b \,\|\, \mathscr{R} \rangle$, and H be the subgroup generated by $[a^2]_G$ and $[b^2]_G$. Then:

0. G is generated by $[a]_G$ and $[b]_G$,
1. G is torsion-free,
2. H is a free (sub)group freely generated by $[a]_G^2$ and $[b]_G^2$,
3. for every $h \in H \setminus \{1\}$, $\lim_{n \to +\infty} \mathrm{cl}_G(h^n) = +\infty$,
4. G has decidable word and conjugacy problems if the family of group words $\{r_{x,n}\}$ is recursive.

Van Kampen diagrams of spheres with handles

If w is a group word and $[w]_G \in [G, G]$, then to show that $\mathrm{cl}_G([w]_G) \geq n$, it is enough to show that there is no van Kampen diagram Δ on a sphere with less than n handles and a hole such that the label of $\partial\Delta$ is w.

If “cancellations are small,” this can be proved by contradiction using Euler characteristic and Hall’s Lemma.

Hall's Lemma

If X is a set, then $\|X\|$ denotes that cardinality of X, and $\mathscr{P}(X)$ denotes the set of all subsets of X (the power set).

Lemma (Philip Hall, 1935)

Let A and B be finite sets. Let $f\colon A \to \mathscr{P}(B)$. Let $F\colon \mathscr{P}(A) \to \mathscr{P}(B)$ be defined by $F(X) = \bigcup_{x\in X} f(x)$ for all $X \subset A$. Then the following are equivalent:

(I) *There exists an injection $h\colon A \to B$ such that for each $x \in A$, $h(x) \in f(x)$.*

(II) *For each $X \subset A$, $\|X\| \le \|F(X)\|$.*

Corollary of Hall's Lemma

Corollary

Let A and B be two finite sets. Let $f\colon A \to \mathscr{P}(B)$ and $c\colon B \to \mathbb{N}$. Let $F\colon \mathscr{P}(A) \to \mathscr{P}(B)$ be defined by $F(X) = \bigcup_{x \in X} f(x)$ for all $X \subset A$. Then the following are equivalent:

(I) *There exists a function $h\colon A \to B$ such that:*

1. *for each $x \in A$, $h(x) \in f(x)$, and*
2. *for each $y \in B$, $\|h^{-1}(y)\| \leq c(y)$.*

(II) *For each $X \subset A$, $\|X\| \leq \sum_{y \in F(X)} c(y)$.*

(III) *For each $Y \subset B$, $\|\{\, x \mid f(x) \subset Y \,\}\| \leq \sum_{y \in Y} c(y)$.*

One particular result

Theorem

There exists a 4-boundedly simple group generated by two non-commutators.

C'est tout, merci de votre attention.