# Lacunary hyperbolic groups

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# Outline

# ASYMPTOTIC CONES OF FINITELY GENERATED GROUPS

# Asymptotic cones of metric spaces

Let $X$ be a metric space. Fix an *observation point* $e = (e_n)$, $e_n \in X$, an increasing *scaling sequence* of positive integers $d = (d_n)$, and an ultrafilter $\omega$.

## Definition

Given two sequences $x = (x_n)$ and $y = (y_n)$ of elements of $X$, set

$$\mathrm{dist}(x, y) = \lim^{\omega} \frac{\mathrm{dist}(x_n, y_n)}{d_n}.$$

Further,

$$x \sim y \quad \Leftrightarrow \quad \mathrm{dist}(x, y) = 0.$$

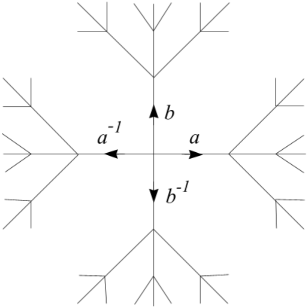
The *asymptotic cone of $X$ with respect to e, d, and $\omega$* is

$$\mathrm{Con}^{\omega}(X, d, e) = \{x = (x_n) \mid \mathrm{dist}(x, e) < \infty\}/\sim$$

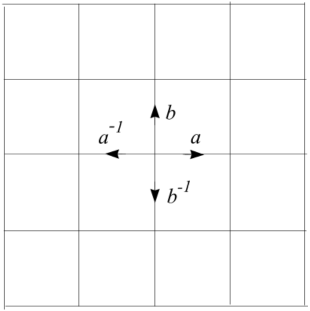
with the distance induced by $\mathrm{dist}$. If $X$ is homogeneous, the asymptotic cone in independent of $e$ and is denoted by $\mathrm{Con}^{\omega}(X, d)$.

# Cayley Graphs

Let $G = \langle S \rangle$. The set of vertices of the *Cayley graph* of $G$ is $G$ and vertices $g$ and $h$ are connected by an edge going from $g$ to $h$ and labelled by $s \in S$ whenever $h = gs$.




$F(a, b)$




$\langle a \rangle \oplus \langle b \rangle$

# Asymptotic cones of finitely generated groups

**Definition**

Asymptotic cone of a finitely generated group $G$, $\mathrm{Con}^\omega(G, d)$, is the asymptotic cone of its Cayley graph.

**Examples.**

1. If $X$ is a finite group, then $\mathrm{Con}^\omega(X, d)$ is a point $\forall\, d, \omega$.
2. $\forall\, d, \omega$, $\mathrm{Con}^\omega(\mathbb{Z}^n, d) = \mathbb{R}^n$.

   More generally, for any finitely generated nilpotent group $N$, $\mathrm{Con}^\omega(N, d)$ is homeomorphic to $\mathbb{R}^n$ $\forall\, d, \omega$, where $n$ is the Hirsch number of $N$.
3. $\forall\, d, \omega$, $\mathrm{Con}^\omega(F_n, d)$ is an $\mathbb{R}$-tree.

## Varying scaling sequences and ultrafilters

**Theorem (Thomas–Velickovic)**

*There exists a group $G$ and two ultrafilters $\omega_1, \omega_2$, such that $\mathrm{Con}^{\omega_1}(G,(n))$ is a real tree while $\mathrm{Con}^{\omega_2}(G,(n))$ is not simply connected.*

**Theorem (Drutu–Sapir)**

*There exists a finitely generated group with uncountably many non-homeomorphic asymptotic cones.*

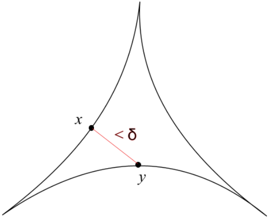
**Theorem (Olshanskii–Sapir)**

*There exists a finitely presented group with at least 2 non–homeomorphic asymptotic cones.*

# Hyperbolic Spaces

## Definition (Gromov)

A geodesic metric space $X$ is *hyperbolic*, if $\exists\ \delta \geq 0$ such that $\forall$ geodesic triangle $\Delta$ in $X$, each side of $\Delta$ belongs to the $\delta$–neighborhood of the union of the other two sides.




**Examples**

1. Any bounded space $X$ is hyperbolic with $\delta = \operatorname{diam} X$.
2. Any tree is hyperbolic with $\delta = 0$.
3. $\mathbb{H}^n$, and, more generally, any $CAT(\kappa)$–space for $\kappa < 0$.
4. $\mathbb{R}^2$ is not hyperbolic.

# Hyperbolic Groups

**Definition (Gromov)**

A finitely generated group is *hyperbolic* if its Cayley graph is hyperbolic.

## Examples

- Finite groups are hyperbolic.
- Finitely generated free groups are hyperbolic.
- Fundamental groups of closed hyperbolic manifolds are hyperbolic.
- $\mathbb{Z} \oplus \mathbb{Z}$ is not hyperbolic.

# Asymptotic cones of hyperbolic groups

**Theorem (Gromov)**

*A finitely generated group $G$ is hyperbolic iff all asymptotic cones of $G$ are $\mathbb{R}$-trees.*

**Theorem (M. Kapovich–Kleiner)**

*If $G$ is a finitely presented group and at least one asymptotic cone of $G$ is an $\mathbb{R}$-tree, then $G$ is hyperbolic.*

**Definition**

A finitely generated group $G$ is *lacunary hyperbolic* if at least one asymptotic cone of $G$ is a real tree.

# LACUNARY HYPERBOLIC GROUPS: A CHARACTERIZATION AND EXAMPLES

# Equivalent definitions of lacunary hyperbolic groups

Given a homomorphism $\alpha\colon G \to H$ and a generating set $S$ of $G$, we define the *injectivity radius* $IR_S(\alpha)$ of $\alpha$ with respect to $S$ to be the radius of the largest ball in $G$ on which $\alpha$ is injective.

## Theorem (Olshanskii–Osin–Sapir)

*Let $G$ be a finitely generated group. Then the following conditions are equivalent.*

1. *$G$ is lacunary hyperbolic.*
2. *There exists a scaling sequence $d = (d_n)$ such that $\mathrm{Con}^\omega(G, d)$ is an $\mathbb{R}$–tree for any ultrafilter $\omega$.*
3. *$G$ is the direct limit of a sequence of hyperbolic groups $G_i$ generated by finite sets $S_i$ and epimorphisms*

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \dots,$$

*where $\alpha_i(S_i) = S_{i+1}$, $G_i$ is $\delta_i$–hyperbolic with respect to $S_i$, and $\delta_i = o(IR_{S_i}(\alpha_i))$.*

# Limits of hyperbolic groups that are not lacunary hyperbolic

**Theorem (Drutu–Sapir)**

*Let $G$ be a non-elementary finitely generated group. If $\mathrm{Con}^{\omega}(G, d)$ has a cut point, then $\prod^{\omega} G$ contains a non–abelian free subgroup.*

**Corollary**

*Non-elementary groups satisfying a law are not lacunary hyperbolic.*

**Example.** The wreath product $(\mathbb{Z}/2\mathbb{Z}) \,\mathrm{wr}\, \mathbb{Z}$ and the free Burnside group $B(m, n)$ are directed limits of hyperbolic groups, but are not lacunary hyperbolic.

# Lacunary hyperbolic small cancellation groups

We say that a subset $L \subset \mathbb{N}$ is *sparse*, if for any $\varepsilon > 0$, there exists a segment $I = [a, b] \subset (1, +\infty)$ such that $I \cap L = \emptyset$ and $a/b < \varepsilon$.

**Proposition (Olshanskii–Osin–Sapir)**

*Let $G = \langle X \mid \mathcal{R} \rangle$ be a group presentation, where $X$ is finite and $\mathcal{R}$ satisfies the $C'(\lambda)$ small cancellation condition for some $\lambda < 1/6$. Then $G$ is lacunary hyperbolic if and only if the set $\{|R| \mid R \in \mathcal{R}\}$ is sparse.*

**Corollary**

*There are lacunary hyperbolic groups $H_1, H_2$ such that $H_1 * H_2$ is not lacunary hyperbolic.*

*Idea of the proof:* The union of two sparse sets is not necessarily sparse.

# CONSTRICTED GROUPS

# Constricted groups

**Definition (Drutu-Sapir)**

A group $G$ is *constricted* if all asymptotic cones of $G$ have cut points.

**Examples.**

1. (Osin–Sapir) Relatively hyperbolic groups (fundamental groups of finite-volume complete hyperbolic manyfolds, groups acting freely on $\mathbb{R}^n$-trees, free products, etc.)
2. Mapping class groups (Behrstock).
3. Groups acting $k$–acylindrically on trees (Drutu–Mozes–Sapir).

Under some mild assumptions, an action of a group on a homogeneous space with cut points leads to an action on an $\mathbb{R}$-tree. This allows to apply the Rips theory to study constricted groups.

# Some questions about constricted groups

The following natural questions were open until now:

1. Does every non–elementary constricted group contain a free non–abelian subgroup?
2. Is every infinite constricted group non-simple?
3. Can a constricted group be periodic?
4. Suppose a finitely generated group has cut points in some asymptotic cone. Is it constricted?

# Strongly lacunary hyperbolic groups

## Definition

A geodesic metric space $X$ is *tree-graded* with respect to a collection of connected subsets $\mathcal{P}$ (called *pieces*) if:

1. Any two distinct pieces intersect by at most one point.
2. Every non-trivial simple geodesic triangle in $X$ is contained in a single piece.

If a geodesic space $X$ is tree-graded with respect to a collection of circles whose diameters are uniformly bounded from above and from below, we call $X$ a *cicrcle-tree*.

## Theorem (Olshanskii–Osin–Sapir)

1. *There exist infinite periodic strongly lacunary hyperbolic groups.*
2. *There exist strongly lacunary hyperbolic Tarskii Monsters (i.e., non-elementary finitely generated groups all of whose proper subgroups are cyclic). In particular these groups are simple.*

# AMENABLE LACUNARY HYPERBOLIC GROUPS

# Amenable groups

**Definition**

A group $G$ is *amenable*, if there exists a finitely additive measure on the set of all subsets of $G$ which is invariant under the action of $G$ on itself by left multiplications.

**Theorem (von Neumann)**

*The class of amenable groups contains all abelian and finite groups and is closed under taking subgroups, quotients, extensions, and directed limits.*

**Example.** Non-abelian free groups are non amenable.

**Problem (Kleiner)**

*Suppose that a group G is finitely generated, amenable, and not virtually cyclic. Can it have cut points in at least one asymptotic cone?*

# Amenable lacunary hyperbolic group

**Theorem (Olshanskii–Osin–Sapir)**

*There exists a finitely generated group $G$ satisfying the following properties.*

1. *$G$ is not virtually cyclic.*
2. *$G$ is lacunary hyperbolic.*
3. *$G$ splits as $1 \to L \to G \to \mathbb{Z} \to 1$, where $L$ is locally finite. In particular, $G$ is elementary amenable.*

Pick a prime $p$ and a non-decreasing sequence of positive integers $c_1 \le c_2 \le \dots$ with fast growth.

$$G = \left\langle t, a \;\middle|\; \begin{array}{c} [...[a, t^{-i_1} a t^{i_1}], ..., t^{-i_{c_n}} a t^{i_{c_n}}] = 1, \;\; n \in \mathbb{Z}, \; -n \le i_1, \dots, i_{c_n} \le n \\ a^p = 1 \end{array} \right\rangle$$

# CENTRAL EXTENSIONS OF LACUNARY HYPERBOLIC GROUPS

# Bounded cohomology and asymptotic cones

**Theorem (Mineyev)**

*Let $G$ be a hyperbolic group. Then the natural map $H_b^n(G, \mathbb{Z}) \to H^n(G, \mathbb{Z})$ is surjective for all $n \geq 2$.*

$H_b^2(G, \mathbb{Z}) \to H^2(G, \mathbb{Z})$ is surjective.

$\Downarrow$ (Gersten)

Any central extension $1 \to \mathbb{Z} \to H \to G \to 1$ is quasi-isometric to $G \times \mathbb{Z}$.

$\Downarrow$

For any $d = (d_n)$ and $\omega$, $\mathrm{Con}^{\omega}(H, d)$ is bi-Lipschitz equivalent to $\mathrm{Con}^{\omega}(G, d) \times \mathbb{R}$.

Given a product $X \times Y$ of metric spaces $X$ and $Y$, we define a metric on $X \times Y$ by the rule

$$\mathrm{dist}_{X\times Y}((x_1, y_1), (x_2, y_2)) = \mathrm{dist}_X(x_1, x_2) + \mathrm{dist}_Y(y_1, y_2).$$

We write $X \sim_{Lip} Y$ if metric spaces $X$ and $Y$ are bi-Lipschitz equivalent.

**Theorem**

*Let $N$ be a central subgroup of a finitely generated group $G$. Suppose that $\mathrm{Con}^\omega(G/N, d)$ is an $\mathbb{R}$–tree for some $d = (d_n)$ and $\omega$. Then*

$$\mathrm{Con}^\omega(G, d) \sim_{Lip} \mathrm{Con}^\omega(N, d) \times \mathrm{Con}^\omega(G/N, d),$$

*where $\mathrm{Con}^\omega(N, d)$ is taken with respect to the metric on $N$ induced from $G$.*

# Main example

Fix an infinite presentation

$$H = \langle a, b \mid R_1, R_2, \ldots \rangle$$

such that:

(a) The set of relations satisfies $C'(1/24)$.

(b) Lengths $r_i = |R_i|$ grow sufficiently fast. In particular, $H$ is lacunary hyperbolic.

Given a sequence of integers $k = (k_n)$, $k_n \geq 2$, consider the central extension of $H$ defined by

$$G(k) = \left\langle a,\ b \;\middle|\; [R_n, a] = 1,\ [R_n, b] = 1,\ R_n^{k_n} = 1,\ n = 1, 2, \ldots \right\rangle \qquad (1)$$

# Cut points in asymptotic cones

## Problem (Drutu–Sapir)

*Suppose an asymptotic cone of a finitely generated group G has cut points. Does every asymptotic cone of G have cut points?*

By a *connectedness degree* $c(X) \in \{0, 1, \ldots, \infty\}$ of a metric space $X$ we mean the minimal number of points whose removal disconnects $X$.

The negative answer to the above question is provided by

## Theorem (Olshanskii–Osin–Sapir)

*Let $G(k)$ be the group corresponding to the sequence $k_n = m \geq 2$. Then for any ultrafilter $\omega$ and any scaling sequence $d = (d_n)$, exactly one of the following possibilities occurs and both of them can be realized for suitable $\omega$ and $d$.*

1. $c(\mathrm{Con}^{\omega}(G(k), d)) = m$.
2. $\mathrm{Con}^{\omega}(G(k), d))$ *is an* $\mathbb{R}$*–tree.*

# Fundamental groups of asymptotic cones

**Theorem (Erschler–Osin)**

*Any countable group can be realized as a subgroup of $\mathrm{Con}^\omega(G, d)$ for some $G$, $d$, and $\omega$.*

**Theorem (Drutu-Sapir)**

*For any countable groups $Q$, there exist $G$, $d$, $\omega$ such that $\pi_1(\mathrm{Con}^\omega(G, d))$ is the free product of uncountably many copies of $Q$.*

**Problem (Gromov)**

*Can the fundamental group of an asymptotic cone of a finitely generated group be countable and non-trivial?*

The main difficulty comes from the fact that the (uncountable) group $\prod^\omega G$ acts on $\mathrm{Con}^\omega(G, d)$ transitively.

# Asymptotic cones with cyclic fundamental groups

Let $G = G(k)$ be the group corresponding to a sequence $k = (k_n)$ such that

$$k_n \to \infty \quad \text{and} \quad k_n|R_n| = o(|R_{n+1}|).$$

Let $N = \langle R_1, R_2, \ldots \rangle$. Clearly $N$ is central in $G$.

**Theorem (Olshanskii–Osin–Sapir)**

*There exists a scaling sequence $d = (d_n)$ such that for any ultrafilter $\omega$ the following conditions hold:*

1. $\mathrm{Con}^\omega(G/N, d)$ *is an* $\mathbb{R}$*–tree.*
2. $\mathrm{Con}^\omega(N, d)$ *is isometric to* $\mathbb{S}^1$*.*

*In particular,*

$$\mathrm{Con}^\omega(G, d) \sim_{Lip} \mathbb{S}^1 \times (\mathbb{R}-\text{tree})$$

*and* $\pi_1(\mathrm{Con}^\omega(G, d)) = \mathbb{Z}$*.*