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# Arithmetical birational invariants of linear algebraic groups over two-dimensional geometric fields ☆

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**Abstract**

Let $G$ be a connected linear algebraic group over a geometric field $k$ of cohomological dimension 2 of one of the types which were considered by Colliot-Thélène, Gille and Parimala. Basing on their results, we compute the group of classes of $R$-equivalence $G(k)/R$, the defect of weak approximation $A_\Sigma(G)$, the first Galois cohomology $H^1(k,G)$, and the Tate–Shafarevich kernel $Ш^1(k,G)$ (for suitable $k$) in terms of the algebraic fundamental group $\pi_1(G)$. We prove that the groups $G(k)/R$ and $A_\Sigma(G)$ and the set $Ш^1(k,G)$ are stably $k$-birational invariants of $G$.


*Keywords:* Two-dimensional geometric field; Linear algebraic group; Birational invariants; $R$-equivalence; Weak approximation; Tate–Shafarevich kernel

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## Introduction

*0.1.* Let $k$ be a field of one of the three types below, where $k_0$ is an algebraically closed field of characteristic 0:

(gl) *a function field $k$ in two variables over $k_0$, i.e., the function field of a smooth, projective, connected surface over $k_0$;*
(ll) *the field of fractions $k$ of a two-dimensional, excellent, henselian local domain $A$ with residue field $k_0$;*
(sl) *the Laurent series field $k = l((t))$ over a field $l$ of characteristic* 0 *and cohomological dimension* 1.

Let $G$ be a connected linear $k$-group. *In the case* (gl) *we always assume that $G$ has no factors of type $E_8$.*

*0.2.* In [11,12] the arithmetic of linear algebraic groups over such fields was investigated. In particular, when $G$ is semisimple simply connected, it was proved that $H^1(k, G) = 1$ and $G(k)/R = 1$ (where $G(k)/R$ denotes the group of classes of $R$-equivalence); in the cases (gl) or (ll) it was proved that the defect of weak approximation $A_\Sigma(G)$ equals 1 with respect to any finite set $\Sigma$ of associated discrete valuations, i.e., $G$ has weak approximation property with respect to $\Sigma$. It was proved that in the second non-abelian cohomology set $H^2(k, L)$ all the elements are neutral, if the $k$-kernel ($k$-band) $L = (\overline{G}, \kappa)$ is such that $\overline{G}$ is semisimple simply connected.

Assume that $G$ is a reductive $k$-group admitting a special covering, i.e., there exists an exact sequence

$$1 \to \mu \to G_0 \times N \to G \to 1,$$

where $G_0$ is a semisimple simply connected group, $N$ is a quasi-trivial torus and $\mu$ is a finite abelian $k$-group. For such groups $G$ the group of classes of $R$-equivalence $G(k)/R$ and the defect of weak approximation $A_\Sigma(G)$ were computed by Colliot-Thélène, Gille, and Parimala [12] in terms of $\mu$.

*0.3.* In the present paper we do not assume that $G$ admits a special covering. Basing on the fundamental results of [12], for a connected linear $k$-group $G$ we compute the group $G(k)/R$, the group $A_\Sigma(G)$ (in the cases (gl) and (ll)), the Galois cohomology set $H^1(k, G)$, and the Tate–Shafarevich set $Ш^1(k, G)$ (in the case (ll)) in terms of the algebraic fundamental group $\pi_1(G)$. We prove that the groups $G(k)/R$ and $A_\Sigma(G)$ and the set $Ш^1(k, G)$ are stably $k$-birational invariants of $G$. We also consider the case where $k$ is a number field.

*0.4.* We describe our results in more detail. First let $k$ be any field of characteristic 0. Let $\Gamma = \mathrm{Gal}(\bar{k}/k)$, where $\bar{k}$ is a fixed algebraic closure of $k$. For a reductive $k$-group $G$ let $\pi_1(G)$ denote the algebraic fundamental group of $G$ introduced in [6]. For any connected linear $k$-group $G$ let $G^{\mathrm{u}}$ denote its unipotent radical and let $G^{\mathrm{red}} = G/G^{\mathrm{u}}$; it is a reductive group. We set $\pi_1(G) := \pi_1(G^{\mathrm{red}})$; it is a finitely generated (over $\mathbf{Z}$) $\Gamma$-module. (For another definition of $\pi_1(G)$ see [33, Section 10].)

We consider an additive functor $\mathcal{H}$ from the category of $k$-tori to the category of abelian groups, with the following property: $\mathcal{H}(N) = 0$ for any quasi-trivial torus $N$. An example of such a functor is $T \mapsto H^1(k, T)$.

In Section 1 we consider a coflasque resolution of $\pi_1(G)$

$$0 \to Q \to P \to \pi_1(G) \to 0$$

(i.e., $P$ is a permutation $\Gamma$-module and $Q$ is a coflasque $\Gamma$-module). Let $F_G$ denote the flasque torus such that $\mathbf{X}_*(F_G) = Q$, where $\mathbf{X}_*$ denotes the cocharacter group. We show that $\mathcal{H}(F_G)$ is determined uniquely by $G$ up to canonical isomorphism, and we obtain a functor $G \mapsto \mathcal{H}(F_G)$.

In Section 2 we consider a smooth rational $k$-variety $X$. Let $V_X$ denote a smooth compactification of $X$. Write $\overline{V}_X = V_X \times_k \bar{k}$. Let $S_X$ be the Néron–Severi torus of $V_X$, i.e., the $k$-torus such that $\mathbf{X}^*(S_X) = \operatorname{Pic} \overline{V}_X$, where $\mathbf{X}^*$ denotes the character group. We show that $\mathcal{H}(S_X)$ is determined uniquely by $X$ up to canonical isomorphism, and we obtain a functor $X \mapsto \mathcal{H}(S_X)$. The group $\mathcal{H}(S_X)$ is a stably $k$-birational invariant of $X$.

In Section 3 we prove that for a connected $k$-group $G$, $\operatorname{Pic} \overline{V}_G$ is a flasque $\Gamma$-module (thus we generalize a theorem of Voskresenskiĭ on tori). Using this result we prove that $\mathcal{H}(F_G) \simeq \mathcal{H}(S_G)$ and that $F_G \times N_1 \simeq S_G \times N_2$ for some quasi-trivial $k$-tori $N_1$ and $N_2$.

In Section 4 we assume that $k$ is as in 0.1. We prove that $G(k)/R \simeq H^1(k, F_G)$. We take $\mathcal{H}(T) = H^1(k, T)$. Using the results of Sections 3 and 2, we obtain that $G(k)/R \simeq H^1(k, S_G)$ and therefore the group $G(k)/R$ is a stably $k$-birational invariant of $G$.

In Section 5 we consider weak approximation for $G$ with respect to a finite set $\Sigma$ of associated discrete valuations of $k$. We assume that $k$ is of type (gl) or (ll). For a $k$-torus $T$ set

$$\text{Ч}^1_\Sigma(k, T) = \operatorname{coker}\left[H^1(k, T) \to \prod_{v \in \Sigma} H^1(k_v, T)\right]$$

(Ч is pronounced “cheh”). We prove that $A_\Sigma(G) \simeq \text{Ч}^1_\Sigma(k, F_G)$. We take $\mathcal{H}(T) = \text{Ч}^1_\Sigma(k, T)$. Using the results of Sections 3 and 2, we obtain that $A_\Sigma(G) \simeq \text{Ч}^1_\Sigma(k, S_G)$ and therefore $A_\Sigma(G)$ is a stably $k$-birational invariant of $G$.

In Section 6 we consider $H^1(k, G)$. In [6] for any field $k$ of characteristic 0 the group of abelian Galois cohomology $H^1_{\mathrm{ab}}(k, G)$ was defined in terms of $\pi_1(G)$. A canonical abelianization map $\mathrm{ab}^1 : H^1(k, G) \to H^1_{\mathrm{ab}}(k, G)$ was defined. We prove here that if $k$ is as in 0.1, then $\mathrm{ab}^1$ is a bijection. Thus $H^1(k, G)$ has a canonical, functorial structure of an abelian group.

In Section 7 we consider the Hasse principle for $G$ when $k$ is of type (ll). Using the result of Section 6, we prove that there is a canonical bijection $\text{Ш}^1(k, G) \simeq \text{Ш}^2(k, F_G)$. We take $\mathcal{H}(T) = \text{Ш}^2(k, T)$. Using the results of Sections 3 and 2, we obtain that $\text{Ш}^1(k, G) \simeq \text{Ш}^2(k, S_G)$ and that the cardinality of the set $\text{Ш}^1(k, G)$ is a stably $k$-birational invariant of $G$. In particular, if $G$ is stably $k$-rational, then $\text{Ш}^1(k, G) = 1$.

The results of Sections 4–7 hold also when $k$ is a totally imaginary number field. In Section 8 we establish analogues of these results when $k$ is any number field, not necessarily totally imaginary.

The proof of our formula for $G(k)/R$ in Section 4 is based on the difficult Lemma 4.12. This lemma is proved in Appendix by P. Gille. Gille also proves a similar (and more difficult) result over a number field which we use in Section 8.

For a discussion of our results (with references) see the text of the paper below. Here we only note that we use the method of Kottwitz [27] in order to reduce the assertions to the known case of tori.

### *0.5. Notation and conventions*

$k$ is a field of characteristic 0, $\bar{k}$ is a fixed algebraic closure of $k$, $\Gamma = \mathrm{Gal}(\bar{k}/k)$. By a $\Gamma$-module we mean a finitely generated over $\mathbf{Z}$ discrete $\Gamma$-module.

Let $G$ be a connected linear algebraic group defined over $k$. We define $G^{\mathrm{u}}$ and $G^{\mathrm{red}}$ as in 0.4. Let $G^{\mathrm{ss}}$ denote the derived group of $G^{\mathrm{red}}$; it is semisimple. Set $G^{\mathrm{tor}} = G^{\mathrm{red}}/G^{\mathrm{ss}}$; it is a torus. Let $G^{\mathrm{sc}}$ denote the universal covering of $G^{\mathrm{ss}}$; it is simply connected.

In Sections 1–3, $\mathcal{H}$ is a covariant functor from the category of $k$-tori to the category of abelian groups satisfying the following conditions:

(1) let $f_1, f_2 : T' \to T''$ be two homomorphisms of $k$-tori, then $\mathcal{H}(f_1 + f_2) = \mathcal{H}(f_1) + \mathcal{H}(f_2)$;
(2) $\mathcal{H}(T_1 \times T_2) \simeq \mathcal{H}(T_1) \oplus \mathcal{H}(T_2)$ for any two $k$-tori $T_1$ and $T_2$;
(3) $\mathcal{H}(N) = 0$ for any quasi-trivial $k$-torus $N$.

A functor satisfying (1) is called additive, and (2) follows from (1), cf. [30, Chapter VIII.2, Proposition 4 on p. 193]. From (2) and (3) follows the following property:

(4) if $p_T : T \times N \to T$ is the projection, where $N$ is a quasi-trivial torus, then $p_{T*} : \mathcal{H}(T \times N) \to \mathcal{H}(T)$ is an isomorphism.

An example of such a functor is $\mathcal{H}(T) = H^1(k, T)$. Another example is $\mathcal{H}(T) = Ш^2(k, T)$ when $k$ is a number field.

## 1. Functor $\mathcal{H}(F_G)$

Let $k$ be a field of characteristic 0. In this section we construct a functor $G \mapsto \mathcal{H}(F_G)$ from the category of connected linear algebraic $k$-groups to abelian groups. Here $F_G$ is the flasque torus coming from a coflasque resolution of $\pi_1(G)$.

*1.1.* A $\Gamma$-module $P$ is called a permutation module if it is torsion-free and has a $\Gamma$-invariant basis. A $\Gamma$-module is called coflasque if it is torsion-free and $H^1(\Gamma', Q) = 0$ for every open subgroup $\Gamma' \subset \Gamma$. Any permutation module is coflasque.

A coflasque resolution of a $\Gamma$-module $A$ is an exact sequence of $\Gamma$-modules

$$0 \to Q \to P \xrightarrow{\alpha} A \to 0, \tag{R}$$

where $P$ is a permutation module and $Q$ is a coflasque module.

**Lemma 1.2** [16, Lemme 0.6]. *Every $\Gamma$-module $A$ admits a coflasque resolution. Moreover if $\overline{\Gamma}$ is the image of $\Gamma$ in* Aut $A$, *then there exists a coflasque resolution* (R) *of $A$ such that $\Gamma$ acts on $P$ and $Q$ through $\overline{\Gamma}$.*

*1.3.* A $k$-torus $F$ is called flasque if its cocharacter group $\mathbf{X}_*(F)$ is a coflasque $\Gamma$-module. A $k$-torus $N$ is called quasi-trivial if it is isomorphic to the product $\prod_i R_{K_i/k}\mathbf{G}_{m,K_i}$, where $K_i/k$ are finite extensions. In other words, $N$ is quasi-trivial if and only if $\mathbf{X}_*(N)$ is a permutation $\Gamma$-module.

Let (R) be a coflasque resolution of a $\Gamma$-module $A$. Let $F_{(\mathrm{R})}$ denote the flasque torus such that $\mathbf{X}_*(F_{(\mathrm{R})}) = Q$. Set $\mathcal{F}(\mathrm{R}) = \mathcal{H}(F_{(\mathrm{R})})$, where $\mathcal{H}$ is a functor as in Section 0.5. We shall prove that $\mathcal{F}(\mathrm{R})$ depends only on $A$ and is functorial in $A$.

Note that for two coflasque resolutions

$$0 \to Q_i \to P_i \to A \to 0 \quad (i = 1, 2) \tag{$\mathrm{R}'_i$}$$

of a $\Gamma$-module $A$, we have $Q_1 \oplus P'_1 \simeq Q_2 \oplus P'_2$ for some permutation modules $P'_1$ and $P'_2$, cf. [16, Lemme 0.6]. Thus $\mathcal{H}(F_{(\mathrm{R}'_1)}) \simeq \mathcal{H}(F_{(\mathrm{R}'_2)})$ by property (4) of $\mathcal{H}$, see Section 0.5. We prove below that there exists a *canonical* isomorphism, permitting to identify $\mathcal{H}(F_{(\mathrm{R}'_1)})$ and $\mathcal{H}(F_{(\mathrm{R}'_2)})$.

*1.4.* Let

$$0 \to Q_i \to P_i \xrightarrow{\alpha_i} A_i \to 0 \quad (i = 1, 2) \tag{$\mathrm{R}_i$}$$

be coflasque resolutions. We always regard $Q_i$ as a subgroup of $P_i$. A *morphism* $(\mathrm{R}_1) \to (\mathrm{R}_2)$ is a pair of homomorphisms of $\Gamma$-modules $f : A_1 \to A_2$, $\psi : P_1 \to P_2$ such that the following diagram is commutative:

$$\begin{array}{ccc} P_1 & \xrightarrow{\alpha_1} & A_1 \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle f} \\ P_2 & \xrightarrow{\alpha_2} & A_2 \end{array}$$

Then $\psi$ defines a homomorphism $Q_1 \to Q_2$ (as the restriction of $\psi$ to $Q_1$). Thus a pair $(f, \psi)$ gives rise to a homomorphism $\mathcal{F}(f, \psi) : \mathcal{F}(\mathrm{R}_1) \to \mathcal{F}(\mathrm{R}_2)$.

**Lemma 1.5.** *Let*

$$0 \to Q' \to P' \xrightarrow{\alpha'} A' \to 0, \tag{$\mathrm{R}'$}$$

$$0 \to Q'' \to P'' \xrightarrow{\alpha''} A'' \to 0 \qquad (\mathrm{R}'')$$

*be coflasque resolutions. Let* $f : A' \to A''$ *be a homomorphism of* $\Gamma$*-modules. Then* $f$ *extends to a morphism* $(f, \psi) : (\mathrm{R}') \to (\mathrm{R}'')$.

**Proof.** Set $P = P' \times_{A''} P'' = \{(x', x'') \in P' \times P'' \mid f(\alpha'(x')) = \alpha''(x'')\}$. Let $p' : P \to P'$ denote the projection defined by $p'(x', x'') = x'$. Clearly $\ker p' \simeq Q''$. We obtain an exact sequence

$$0 \to Q'' \to P \xrightarrow{p'} P' \to 0.$$

Since $P'$ is a permutation module and $Q''$ is a coflasque module, we have $\mathrm{Ext}^1(P', Q'') = 0$ (cf. [29, Proposition 1.2]). Thus there exists a splitting $\beta : P' \to P$ such that $p' \circ \beta = \mathrm{id}_{P'}$. Write $\beta(x') = (x', \psi(x'))$, where $\psi(x') \in P''$, $\alpha''(\psi(x')) = f(\alpha'(x'))$. Clearly $(f, \psi)$ is a morphism $(\mathrm{R}') \to (\mathrm{R}'')$ extending $f$. □

**Lemma 1.6.** *Let* $(\mathrm{R}')$, $(\mathrm{R}'')$ *be as in Lemma* 1.5. *Let* $(f, \psi) : (\mathrm{R}') \to (\mathrm{R}'')$ *be any morphism of coflasque resolutions. Then the homomorphism* $\mathcal{F}(f, \psi) : \mathcal{F}(\mathrm{R}_1) \to \mathcal{F}(\mathrm{R}_2)$ *does not depend on* $\psi$.

**Proof.** Let $\psi_1, \psi_2 : P' \to P''$ be two homomorphisms of $\Gamma$-modules compatible with $f : A' \to A''$. Let $\chi = \psi_1 - \psi_2 : P' \to P''$. Then clearly $\mathrm{im}\,\chi \in \ker \alpha'' = Q''$. We may and shall regard $\chi$ as a homomorphism $\chi : P' \to Q''$.

Let $\theta_i : Q' \to Q''$ be the homomorphisms induced by $\psi_i$ $(i = 1, 2)$, where we regard $Q'$, $Q''$ as submodules of $P'$, $P''$, respectively. Then $\theta_i(x') = \psi_i(x')$ for any $x' \in Q'$. We see that $\theta_2 - \theta_1 = \chi|_{Q'}$. But $\chi|_{Q'} : Q' \to Q''$ factors through $P'$. It follows that

$$\mathcal{F}(f, \psi_2) - \mathcal{F}(f, \psi_1) : \mathcal{H}(F_{(\mathrm{R}')}) \to \mathcal{H}(F_{(\mathrm{R}'')})$$

factors through $\mathcal{H}(N')$, where $N'$ is the $k$-torus such that $\mathbf{X}_*(N') = P'$. Since $N'$ is a quasi-trivial torus, we have $\mathcal{H}(N') = 0$ and $\mathcal{F}(f, \psi_2) - \mathcal{F}(f, \psi_1) = 0$. Thus $\mathcal{F}(f, \psi_1) = \mathcal{F}(f, \psi_2)$. □

We shall write $\mathcal{F}(f)_{(\mathrm{R}', \mathrm{R}'')}$ instead of $\mathcal{F}(f, \psi)$.

*1.7.* Now using Lemmas 1.2, 1.5 and 1.6, we shall prove by a categoric argument that the correspondence $A \mapsto \mathcal{F}(\mathrm{R})$ defines a functor $A \mapsto \mathcal{F}(A)$ from the category of $\Gamma$-modules to the category of abelian groups.

(i) Assume we have three coflasque resolutions $(\mathrm{R}')$, $(\mathrm{R}'')$, $(\mathrm{R}''')$ as above. Let $f : A' \to A''$ and $g : A'' \to A'''$ be homomorphisms of $\Gamma$-modules. Then it is easy to see that

$$\mathcal{F}(g \circ f)_{(\mathrm{R}', \mathrm{R}''')} = \mathcal{F}(g)_{(\mathrm{R}'', \mathrm{R}''')} \circ \mathcal{F}(f)_{(\mathrm{R}', \mathrm{R}'')}.$$

(ii) Consider the case when we have one $\Gamma$-module $A$ and one coflasque resolution

$$0 \to Q \to P \to A \to 0. \qquad (\mathrm{R})$$

Then $\mathcal{F}(\mathrm{id}_A)_{(\mathrm{R},\mathrm{R})} : \mathcal{F}(\mathrm{R}) \to \mathcal{F}(\mathrm{R})$ equals $\mathrm{id}_{\mathcal{F}(\mathrm{R})}$.

(iii) Consider the case of two coflasque resolutions of the same $\Gamma$-module $A$:

$$0 \to Q_i \to P_i \to A \to 0 \quad (i = 1, 2). \tag{$\mathrm{R}_i$}$$

Set $\varphi_{12} = \mathcal{F}(\mathrm{id}_A)_{(\mathrm{R}_1,\mathrm{R}_2)} : \mathcal{F}(\mathrm{R}_1) \to \mathcal{F}(\mathrm{R}_2)$. One can easily prove that $\varphi_{12}$ is an isomorphism. We have constructed a canonical isomorphism $\varphi_{12} : \mathcal{F}(\mathrm{R}_1) \to \mathcal{F}(\mathrm{R}_2)$.

(iv) Now consider three coflasque resolutions of one $\Gamma$-module $A$:

$$0 \to Q_i \to P_i \to A \to 0 \quad (i = 1, 2, 3). \tag{$\mathrm{R}_i$}$$

By (iii) we have canonical isomorphisms $\varphi_{ij} : \mathcal{F}(\mathrm{R}_i) \to \mathcal{F}(\mathrm{R}_j)$. By (i) we have $\varphi_{23} \circ \varphi_{12} = \varphi_{13}$.

(v) Let $A$ be a $\Gamma$-module. For any two coflasque resolutions $(\mathrm{R}_1)$, $(\mathrm{R}_2)$ of $A$ we identify $\mathcal{F}(\mathrm{R}_1)$ with $\mathcal{F}(\mathrm{R}_2)$ using the canonical isomorphism $\varphi_{12}$. We thus obtain an abelian group which we denote by $\mathcal{F}(A)$. Note that the group $\mathcal{F}(A)$ is well defined because of (iv).

(vi) Let $f : A' \to A''$ be a homomorphism of $\Gamma$-modules, and let $(\mathrm{R}'_i) \to (\mathrm{R}''_i)$ $(i = 1, 2)$ be two morphisms of coflasque resolutions extending $f$. Then it is easy to see that the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{F}(\mathrm{R}'_1) & \xrightarrow{\mathcal{F}(f)_{(\mathrm{R}'_1,\mathrm{R}''_1)}} & \mathcal{F}(\mathrm{R}''_1) \\
\Big\downarrow{\scriptstyle \varphi'_{12}} & & \Big\downarrow{\scriptstyle \varphi''_{12}} \\
\mathcal{F}(\mathrm{R}'_2) & \xrightarrow{\mathcal{F}(f)_{(\mathrm{R}'_2,\mathrm{R}''_2)}} & \mathcal{F}(\mathrm{R}''_2)
\end{array}$$

(vii) Let $f : A' \to A''$ be a homomorphism of $\Gamma$-modules. Choose coflasque resolutions $(\mathrm{R}')$ and $(\mathrm{R}'')$ of $A'$ and $A''$, respectively. We define $\mathcal{F}(f) : \mathcal{F}(A') \to \mathcal{F}(A'')$ to be $\mathcal{F}(f)_{(\mathrm{R}',\mathrm{R}'')} : \mathcal{F}(\mathrm{R}') \to \mathcal{F}(\mathrm{R}'')$. By (vi) this homomorphism is well defined (does not depend on the choice of resolutions).

Thus we have defined a functor $A \mapsto \mathcal{F}(A)$ from $\Gamma$-modules to abelian groups. We shall denote $\mathcal{F}(A)$ by $\mathcal{H}(F_A)$.

*1.8.* We recall the definition of algebraic fundamental group $\pi_1(G)$ of a connected linear algebraic group $G$ from [6].

First assume that $G$ is reductive. Consider the composition

$$\rho : G^{\mathrm{sc}} \to G^{\mathrm{ss}} \hookrightarrow G.$$

In general the homomorphism $\rho$ is neither surjective nor injective. Let $T \subset G$ be a maximal torus (defined over $k$). Set $T^{\mathrm{sc}} = \rho^{-1}(T) \subset G^{\mathrm{sc}}$, it is a maximal torus in $G^{\mathrm{sc}}$. The homomorphism $\rho : T^{\mathrm{sc}} \to T$ induces a homomorphism of $\Gamma$-modules $\rho_* : \mathbf{X}_*(T^{\mathrm{sc}}) \to \mathbf{X}_*(T)$, where $\mathbf{X}_*$ denotes the cocharacter group. Set $\pi_1(G) = \mathbf{X}_*(T)/\rho_*\mathbf{X}_*(T^{\mathrm{sc}})$. It is shown in [6] that the $\Gamma$-module $\pi_1(G)$ is well defined, i.e., does not depend on the choice of a maximal torus $T \subset G$. To a homomorphism $f: G_1 \to G_2$ there corresponds a homomorphism of $\Gamma$-modules $f_*: \pi_1(G_1) \to \pi_1(G_2)$.

For an arbitrary connected linear algebraic $k$-group $G$ (not necessarily reductive) we set $\pi_1(G) := \pi_1(G^{\mathrm{red}})$. Then $\pi_1$ is a functor from the category of connected linear algebraic $k$-groups to the category of $\Gamma$-modules.

*1.9.* Consider the functor $\mathcal{F} \circ \pi_1 : G \mapsto \mathcal{H}(F_{\pi_1(G)})$ from the category of connected linear algebraic $k$-groups to the category of abelian groups. We shall write $\mathcal{H}(F_G)$ for $\mathcal{H}(F_{\pi_1(G)})$.

Recall that a finite group is called metacyclic if all its Sylow subgroups are cyclic.

**Proposition 1.10.** *Assume that the image $\overline{\Gamma}$ of $\Gamma$ in* $\operatorname{Aut}\pi_1(G)$ *is a metacyclic group. Then* $\mathcal{H}(F_G) = 0$.

**Proof.** By definition $\mathcal{H}(F_G) = \mathcal{H}(F)$ for a flasque torus $F$ coming from a coflasque resolution of $\pi_1(G)$. By Lemma 1.2 we may assume that $F$ splits over a metacyclic extension. By a theorem of Endo and Miyata (cf. [15, Proposition 2, p. 184]) there exists a $k$-torus $T$ such that the torus $F \times T$ is quasi-trivial. We obtain

$$\mathcal{H}(F) \oplus \mathcal{H}(T) = \mathcal{H}(F \times T) = 0,$$

hence $\mathcal{H}(F_G) = \mathcal{H}(F) = 0$. □

## 2. Functor $\mathcal{H}(S_X)$

*2.1.* Let $k$ be a field of characteristic 0. Let $X$ be a smooth rational $k$-variety (i.e., $X \times_k \bar{k}$ is birational to an affine space). Let $V_X$ be a smooth $k$-compactification of $X$. We consider the $\Gamma$-module $\operatorname{Pic} \overline{V}_X$, where $\overline{V}_X = V_X \times_k \bar{k}$. It is a torsion-free group of finite $\mathbf{Z}$-rank (cf., e.g., [44, 4.5]). Let $S_X$ denote the Néron–Severi torus of $V_X$, i.e., the $k$-torus with character group $\mathbf{X}^*(S_X) = \operatorname{Pic} \overline{V}_X$. We shall show in this section that $\mathcal{H}(S_X)$ does not depend on the choice of $V_X$, and that the correspondence $X \mapsto \mathcal{H}(S_X)$ extends to a functor from the category of smooth rational $k$-varieties to the category of abelian groups. (The similar assertion about the correspondence $X \mapsto H^1(k, \operatorname{Pic} \overline{V}_X)$ is known to experts, cf. [39, 9.0], but we could not find a reference where it was written in detail.) Moreover we shall prove that $\mathcal{H}(S_X)$ is a stably $k$-birational invariant of $X$.

*2.2.* Let $X$ be a smooth geometrically integral $k$-variety. A smooth compactification $V$ of $X$ is a pair $(V, \nu : X \hookrightarrow V)$, where $V$ is a smooth complete $k$-variety, and $\nu$ is an embedding of $X$ into $V$ as a dense open subset. We often write just $V$ instead of $(V, \nu)$. We say that a smooth compactification $(V', \nu')$ dominates $(V, \nu)$ if there exists a $k$-morphism $\lambda : V' \to V$ such that $\nu = \lambda \circ \nu'$. Then such $\lambda$ is unique (because $\nu'(X)$ is dense in $V'$).

We need three propositions on smooth compactifications.

**Proposition 2.3** [24]. *For any smooth geometrically integral $k$-variety $X$ there exists a smooth compactification $(V, \nu)$ of $X$.*

**Proposition 2.4.** *For any two smooth compactifications $V_1$, $V_2$ of a smooth geometrically integral $k$-variety $X$, there exists a smooth compactification $V_3$ of $X$ dominating both $V_1$ and $V_2$.*

**Proof.** The proposition is a special case of Proposition 2.6 below. □

*2.5.* Let $f : X' \to X''$ be a morphism of smooth $k$-varieties. Let $(V', \nu')$, $(V'', \nu'')$ be smooth compactifications of $X'$, $X''$, respectively. We say that a morphism $\psi : V' \to V''$ is *compatible* with $f$ if the following diagram commutes:

$$\begin{array}{ccc} V' & \xrightarrow{\psi} & V'' \\ {\scriptstyle \nu'}\uparrow & & \uparrow{\scriptstyle \nu''} \\ X' & \xrightarrow{f} & X'' \end{array}$$

**Proposition 2.6.** *Let $f : X' \to X''$ be a morphism of smooth geometrically integral varieties, and let $V'$, $V''$ be smooth compactifications of $X'$, $X''$, respectively. Then there exist a smooth compactification $V_1'$ of $X'$ dominating $V'$ and a morphism $\psi : V_1' \to V''$ compatible with $f$.*

**Proof.** See [7, 1.2.2]. This proof was communicated to us by J.-L. Colliot-Thélène. □

*2.7.* From now on to the end of the section we assume that $X$ is a smooth *rational* variety (i.e., $\bar{k}$-rational). Let $V_X$ be a smooth compactification of $X$. We define $S_X$ as in 2.1.

*2.8.* Let $V_1$, $V_2$ be two smooth compactifications of $X$, and let $S_1$, $S_2$ be the corresponding Néron–Severi tori (i.e., $\mathbf{X}^*(S_i) = \operatorname{Pic} \overline{V}_i$, $i = 1, 2$). We wish to construct an isomorphism $\varphi_{12} : \mathcal{H}(S_1) \to \mathcal{H}(S_2)$. By Proposition 2.4, there exists a smooth compactification $V$ of $X$ dominating both $V_1$ and $V_2$. Let $S$ denote the corresponding Néron–Severi torus. The domination morphism $\lambda_1 : V \to V_1$ induces a homomorphism $\lambda_{1*} : S \to S_1$, and there exists an isomorphism $S \simeq S_1 \times N_1$, where $N_1$ is a quasi-trivial $k$-torus and $\lambda_{1*}$ corresponds to the projection of $S_1 \times N_1$ onto $S_1$ (cf. [44, 4.4]). We thus obtain an isomorphism $\varphi_1 : \mathcal{H}(S) \to \mathcal{H}(S_1)$ by property (4) of $\mathcal{H}$, see Introduction. Similarly, the domination morphism $\lambda_2 : V \to V_2$ induces an isomorphism $\varphi_2 : \mathcal{H}(S) \to \mathcal{H}(S_2)$. We obtain an isomorphism $\varphi_{12} = \varphi_2 \circ \varphi_1^{-1} : \mathcal{H}(S_1) \to \mathcal{H}(S_2)$.

*2.9.* If $V'$ is another smooth compactification of $X$ dominating both $V_1$ and $V_2$, we obtain another isomorphism $\varphi_{12}' : \mathcal{H}(S_1) \to \mathcal{H}(S_2)$. However there exists a smooth compactification $V''$ of $X$ dominating both $V$ and $V'$, and using this fact one can easily show that $\varphi_{12}' = \varphi_{12}$. Thus we have constructed a canonical isomorphism $\varphi_{12} : \mathcal{H}(S_1) \to \mathcal{H}(S_2)$.

*2.10.* Now let $V_1$, $V_2$, $V_3$ be three smooth compactifications of $X$, let $S_1$, $S_2$, $S_3$ denote the corresponding Néron–Severi tori, and let $\varphi_{ij} : \mathcal{H}(S_i) \to \mathcal{H}(S_j)$ be the canonical isomorphisms. Let $V_{12}$ (respectively $V_{23}$) be a smooth compactification of $X$ dominating $V_1$ and $V_2$ (respectively $V_2$ and $V_3$). Let $V$ be a smooth compactification of $X$ dominating $V_{12}$ and $V_{23}$. Clearly $V$ dominates $V_1$, $V_2$, and $V_3$, and using this fact, one can easily show that $\varphi_{13} = \varphi_{23} \circ \varphi_{12}$.

*2.11.* Let $V_1$, $V_2$, $S_1$, $S_2$ be as in 2.8. We can now identify $\mathcal{H}(S_1)$ with $\mathcal{H}(S_2)$ using the canonical isomorphism $\varphi_{12}$, for all pairs $(V_1, V_2)$. We denote the obtained group by $\mathcal{H}(S_X)$. The group $\mathcal{H}(S_X)$ is well defined because of the equality $\varphi_{13} = \varphi_{23} \circ \varphi_{12}$ of 2.10.

*2.12.* Let $f : X' \to X''$ be a morphism of smooth rational varieties. By Proposition 2.6 there exists a morphism of smooth compactifications $\psi : V' \to V''$; here $(V', \nu')$ and $(V'', \nu'')$ are smooth compactifications of $X'$ and $X''$, respectively, and the diagram

$$\begin{array}{ccc} V' & \xrightarrow{\psi} & V'' \\ {\scriptstyle \nu'}\uparrow & & \uparrow{\scriptstyle \nu''} \\ X' & \xrightarrow{f} & X'' \end{array}$$

commutes. We obtain a homomorphism $\psi_* : \mathcal{H}(S_{X_1}) \to \mathcal{H}(S_{X_2})$.

Let now $\psi_1 : V_1' \to V_1''$ and $\psi_2 : V_2' \to V_2''$ be two morphisms of smooth compactifications extending a $k$-morphism $f : X_1 \to X_2$. Then using Propositions 2.4 and 2.6, we can construct a morphism of smooth compactifications $\psi_3 : V_3' \to V_3''$ dominating both $\psi_1$ and $\psi_2$ (in the obvious sense). Using this fact one can easily show that the diagram

$$\begin{array}{ccc} \mathcal{H}(S_1') & \xrightarrow{\psi_{1*}} & \mathcal{H}(S_1'') \\ {\scriptstyle \varphi_{12}'}\downarrow & & \downarrow{\scriptstyle \varphi_{12}''} \\ \mathcal{H}(S_2') & \xrightarrow{\psi_{2*}} & \mathcal{H}(S_2'') \end{array}$$

commutes. (Here $S_1'$ is the Néron–Severi torus of $V_1'$, and so on.) Thus we have constructed a canonical homomorphism $f_* : \mathcal{H}(S_{X'}) \to \mathcal{H}(S_{X''})$.

If $f : X' \to X''$, $g : X'' \to X'''$ are $k$-morphisms of smooth rational varieties, then using Proposition 2.6 one can construct a commutative diagram

$$\begin{array}{ccccc} V' & \longrightarrow & V'' & \longrightarrow & V''' \\ \uparrow & & \uparrow & & \uparrow \\ X' & \xrightarrow{f} & X'' & \xrightarrow{g} & X''' \end{array}$$

where $V'$, $V''$, $V'''$ are smooth compactifications of $X'$, $X''$, $X'''$, respectively. It follows that $(g \circ f)_* = g_* \circ f_*$. We see that we have constructed a functor $X \mapsto \mathcal{H}(S_X)$ from the category of smooth rational $k$-varieties to the category of abelian groups.

*2.13.* Let $f : X_1 \to X_2$ be a rational map of smooth rational varieties defined over $k$. In other words, let $U_1 \subset X_1$ and $U_2 \subset X_2$ be open subvarieties and $f' : U_1 \to U_2$ a regular map, all defined over $k$. We may take $V_{U_\nu} = V_{X_\nu}$ $(\nu = 1, 2)$, thus we can identify $\mathcal{H}(S_{U_\nu})$ with $\mathcal{H}(S_{X_\nu})$. The regular map $f' : U_1 \to U_2$ induces a homomorphism of abelian groups $f'_* : \mathcal{H}(S_{U_1}) \to \mathcal{H}(S_{U_2})$, see 2.12. Thus we obtain a homomorphism $f_* : \mathcal{H}(S_{X_1}) \to \mathcal{H}(S_{X_2})$ which does not depend on the choice of $U_1$ and $U_2$. Clearly if $f : X_1 \to X_2$ is a birational isomorphism, then $f_*$ is an isomorphism.

Recall that two $k$-varieties $X_1$, $X_2$ are called stably $k$-birationally equivalent, if $X_1 \times \mathbf{P}_k^{n_1}$ and $X_2 \times \mathbf{P}_k^{n_2}$ are $k$-birationally equivalent for some $n_1$, $n_2$ (here $\mathbf{P}_k^{n_1}$ and $\mathbf{P}_k^{n_2}$ are projective spaces).

**Proposition 2.14.** *$\mathcal{H}(S_X)$ is a stably $k$-birational invariant of $X$.*

**Proof.** Let $X_1$ and $X_2$ be two stably $k$-birationally equivalent varieties. Then $X_1 \times \mathbf{P}_k^{n_1}$ and $X_2 \times \mathbf{P}_k^{n_2}$ are $k$-birationally equivalent for some $n_1$, $n_2$. Set $Y_\nu = X_\nu \times \mathbf{P}_k^{n_\nu}$ $(\nu = 1, 2)$, then there is a $k$-birational isomorphism $f : Y_1 \to Y_2$. The birational isomorphism $f$ induces an isomorphism $f_* : \mathcal{H}(S_{Y_1}) \to \mathcal{H}(S_{Y_2})$, see 2.13. The projections $\psi_\nu : Y_\nu \to X_\nu$ induce isomorphisms $\psi_{\nu *} : \mathcal{H}(S_{Y_\nu}) \to \mathcal{H}(S_{X_\nu})$ $(\nu = 1, 2)$. We obtain an isomorphism $\psi_{2*} \circ f_* \circ \psi_{1*}^{-1} : \mathcal{H}(S_{X_1}) \to \mathcal{H}(S_{X_2})$. □

## 3. Isomorphism $\mathcal{H}(F_G) \simeq \mathcal{H}(S_G)$

Let $k$ be a field of characteristic 0. In this section we construct an isomorphism of functors $G \mapsto \mathcal{H}(S_G)$ and $G \mapsto \mathcal{H}(F_G)$ on the category of connected linear algebraic $k$-groups. But first we need to generalize a result of Voskresenskiĭ.

**Proposition 3.1** ([43, 4.8], [44, 4.6]). *For any $k$-torus $T$ the $\Gamma$-module* $\operatorname{Pic} \overline{V}_T$ *is flasque.*

Here a $\Gamma$-module $M$ is called flasque if the dual module $M^\vee := \operatorname{Hom}(M, \mathbf{Z})$ is coflasque.

We prove the following theorem.

**Theorem 3.2.** *Let $G$ be a connected linear algebraic $k$-group. Then* $\operatorname{Pic} \overline{V}_G$ *is a flasque module.*

**Proof.** (i) First, we reduce the assertion to the case of a reductive group. A Levi decomposition gives an isomorphism of $k$-varieties $G \simeq G^{\mathrm{red}} \times G^{\mathrm{u}}$, where $G^{\mathrm{u}}$ is a $k$-rational variety. We may take $V_G = V_{G^{\mathrm{red}}} \times V_{G^{\mathrm{u}}}$, then $\operatorname{Pic} \overline{V}_G = \operatorname{Pic} \overline{V}_{G^{\mathrm{red}}} \oplus P$, where $P$ is a permutation module. Thus if $\operatorname{Pic} \overline{V}_{G^{\mathrm{red}}}$ is flasque, then $\operatorname{Pic} \overline{V}_G$ is also flasque. So we may and shall assume that $G$ is reductive.

(ii) Let us now prove the assertion of the theorem in the case where $G$ is quasi-split, i.e., has a Borel subgroup $B$ defined over $k$. Then it follows from the Bruhat decomposition that there exists an open subset in $G$ isomorphic to $U^- \times T \times U$, where $T$ is a maximal torus of $G$, $U$ is the unipotent radical of $B$, and $U^-$ is the opposite unipotent subgroup of $G$. Here $U$ and $U^-$ are $k$-rational varieties. It follows that $\operatorname{Pic}\overline{V}_G \simeq \operatorname{Pic}\overline{V}_T \oplus P$, where $P$ is a permutation module. Since $\operatorname{Pic}\overline{V}_T$ is flasque by Proposition 3.1, we conclude that $\operatorname{Pic}\overline{V}_G$ is flasque.

(iii) The general case can be reduced to the quasi-split case by the device of passage to the variety of Borel subgroups. The following argument mimics [17, Theorem 2.B.1] (see also [13, Theorem 4.2] and [7, Theorem 2.4]).

Let $G$ be any connected reductive $k$-group (not necessarily quasi-split). Let $Y$ denote the variety of Borel subgroups of $G$ (see [40, t. III, Exp. XXII, 5.8.3] for the definition). It is a geometrically integral smooth $k$-variety, because $Y_{\bar{k}} \simeq G_{\bar{k}}/\overline{B}$, where $\overline{B} \subset G_{\bar{k}}$ is a Borel subgroup. The variety $Y$ has the following property: if $Y(k') \neq \emptyset$ for a field extension $k'/k$, then $G_{k'}$ is quasi-split, and then by (ii) the assertion of the theorem holds for such $G_{k'}$.

Let $\overline{k(Y)}$ be an algebraic closure of $k(Y)$ containing $\bar{k}(Y)$. Since $Y$ is geometrically integral, we see that $k$ is algebraically closed in $k(Y)$, and therefore $\operatorname{Gal}(\bar{k}(Y)/k(Y)) \simeq \operatorname{Gal}(\bar{k}/k)$. The variety $Y$ has a $k(Y)$-point (the generic point of $Y$), hence $G_{k(Y)}$ is quasi-split. It follows that $\operatorname{Pic}\overline{V}_{G_{k(Y)}}$ is a flasque module.

(iv) We can now finish the proof of the theorem. Let $V = V_G$ be a smooth compactification of $G$. Since $G$ is $\bar{k}$-rational, it follows from [44, 4.4] that $\operatorname{Pic}(V \times_k \bar{k})$ is a torsion-free abelian group of finite rank, and that $\operatorname{Pic}(V \times_k \bar{k}) = \operatorname{Pic}(V \times_k \bar{k}(Y)) = \operatorname{Pic}(V \times_k \overline{k(Y)})$. We denote this group by $\operatorname{Pic}\overline{V}$. Let $Q = \operatorname{Hom}(\operatorname{Pic}\overline{V}, \mathbf{Z})$. We wish to prove that $\operatorname{Pic}\overline{V}$ is a flasque $\Gamma$-module, i.e., that $Q$ is a coflasque $\Gamma$-module. We know that $Q$ is a coflasque $\operatorname{Gal}(\overline{k(Y)}/k(Y))$-module because $\operatorname{Pic}\overline{V}$ is a flasque $\operatorname{Gal}(\overline{k(Y)}/k(Y))$-module.

Let $k'/k$ be a finite field extension in $\bar{k}$. Set $\Gamma' = \operatorname{Gal}(\bar{k}(Y)/k'(Y))$, $\mathfrak{g}' = \operatorname{Gal}(\overline{k(Y)}/k'(Y))$, $\mathfrak{h} = \operatorname{Gal}(\overline{k(Y)}/\bar{k}(Y))$. Then $\mathfrak{h}$ acts trivially on $\operatorname{Pic}\overline{V}$ and hence on $Q$. We have an isomorphism $\Gamma' \simeq \mathfrak{g}'/\mathfrak{h}$.

We have an inflation-restriction exact sequence

$$0 \to H^1\big(\Gamma', Q^{\mathfrak{h}}\big) \to H^1\big(\mathfrak{g}', Q\big) \to H^1(\mathfrak{h}, Q),$$

cf. [1, Chapter IV, Proposition 5.1]. We have $Q^{\mathfrak{h}} = Q$. Since $Q$ is a coflasque $\operatorname{Gal}(\overline{k(Y)}/k(Y))$-module, we have $H^1(\mathfrak{g}', Q) = 0$. Hence $H^1(\Gamma', Q) = 0$. We have proved that $Q$ is a coflasque $\Gamma$-module. Thus $\operatorname{Pic}\overline{V}$ is a flasque $\Gamma$-module. □

**Lemma 3.3.** *Let $L$ be a flasque $\Gamma$-module. Then $H^1(\gamma, L) = 0$ for any closed procyclic subgroup $\gamma \subset \Gamma$.*

**Proof.** Let $Q = L^\vee$, then $Q$ is a coflasque module. Let $\bar{\gamma}$ denote the image of $\gamma$ in $\operatorname{Aut} L$, it is a finite cyclic group. Since $Q$ is coflasque, $H^1(\bar{\gamma}, Q) = 0$. By duality for Tate cohomology with coefficients in a torsion-free module (cf. [9, Chapter VI, §7, Exercise 3]), we have $H^{-1}(\bar{\gamma}, L) = 0$. By periodicity for Tate cohomology of finite cyclic groups (cf. [1, Chapter IV, Theorem 8.1]) we have $H^1(\bar{\gamma}, L) = 0$. Thus $H^1(\gamma, L) = 0$. □

**Corollary 3.4** [13, Proposition 3.2]. *Let $G$ be a connected linear algebraic $k$-group, then $H^1(\gamma, \operatorname{Pic}\overline{V}_G) = 0$ for any closed procyclic subgroup $\gamma \subset \operatorname{Gal}(\bar{k}/k)$.*

**Proof.** The corollary follows from Theorem 3.2 and Lemma 3.3. □

**Theorem 3.5.** *There exists a canonical isomorphism of functors $G \mapsto \mathcal{H}(F_G)$ and $G \mapsto \mathcal{H}(S_G)$ from the category of connected linear algebraic $k$-groups to the category of abelian groups.*

**Corollary 3.6.** *$\mathcal{H}(F_G)$ is a stably $k$-birational invariant of $G$.*

**Proof.** The corollary follows from Theorem 3.5 and Proposition 2.14. □

In the proof of Theorem 3.5 we use the method of Kottwitz [27].

We need the following lemma which was stated in [6] without proof.

**Lemma 3.7.** *Let $1 \to G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \to 1$ be an exact sequence of connected reductive $k$-groups. Then the sequence*

$$1 \to \pi_1(G_1) \to \pi_1(G_2) \to \pi_1(G_3) \to 1$$

*is exact.*

**Proof.** Let $T_2 \subset G_2$ be a maximal torus, $T_3 = \beta(T_2) \subset G_3$, $T_1 = \alpha^{-1}(T_2) \subset G_1$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{X}_*\big(T_1^{\mathrm{sc}}\big) & \longrightarrow & \mathbf{X}_*\big(T_2^{\mathrm{sc}}\big) & \longrightarrow & \mathbf{X}_*\big(T_3^{\mathrm{sc}}\big) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{X}_*(T_1) & \longrightarrow & \mathbf{X}_*(T_2) & \longrightarrow & \mathbf{X}_*(T_3) & \longrightarrow & 0 \end{array}$$

where the vertical arrows are injective and $\operatorname{coker}[\mathbf{X}_*(T_i^{\mathrm{sc}}) \to \mathbf{X}_*(T_i)] = \pi_1(G_i)$ for $i = 1, 2, 3$. Now our lemma follows from the snake lemma. □

**Corollary 3.8.** *If $G$ is a reductive $k$-group and $G^{\mathrm{ss}}$ is simply connected, then the map $t: G \to G^{\mathrm{tor}}$ induces a canonical isomorphism $t_*: \pi_1(G) \xrightarrow{\sim} \mathbf{X}_*(G^{\mathrm{tor}})$.*

**Proof.** We have an exact sequence $1 \to G^{\mathrm{ss}} \to G \to G^{\mathrm{tor}} \to 1$, where $\pi_1(G^{\mathrm{ss}}) = 1$ and $\pi_1(G^{\mathrm{tor}}) = \mathbf{X}_*(G^{\mathrm{tor}})$. □

*3.9.* We now construct an isomorphism of functors $\xi_G: \mathcal{H}(F_G) \to \mathcal{H}(S_G)$ for reductive groups $G$ such that $G^{\mathrm{ss}}$ is simply connected.

Choose a smooth compactification $V_G$ of $G$. Consider the exact sequence of Voskresenskiĭ ([41,42], [44, 4.5])

$$0 \to \mathbf{X}^*(G) \to P \to \operatorname{Pic} \overline{V}_G \to \operatorname{Pic} \overline{G} \to 0,$$

where $P$ is a permutation module. We have $\mathbf{X}^*(G) = \mathbf{X}^*(G^{\mathrm{tor}})$. Since $G^{\mathrm{ss}}$ is simply connected, we have $\operatorname{Pic} \overline{G} = 0$ (cf. [39, 6.9, 6.11]). We thus obtain an exact sequence of torsion-free $\Gamma$-modules

$$0 \to \mathbf{X}^*\big(G^{\mathrm{tor}}\big) \to P \to \operatorname{Pic} \overline{V}_G \to 0.$$

The dual exact sequence is

$$0 \to \mathbf{X}_*(S_G) \to P' \to \mathbf{X}_*\big(G^{\mathrm{tor}}\big) \to 0, \tag{3.1}$$

where $P'$ is a permutation module. By Theorem 3.2, $\operatorname{Pic} \overline{V}_G$ is a flasque module, hence $\mathbf{X}_*(S_G)$ is a coflasque module. By Corollary 3.8, $\mathbf{X}_*(G^{\mathrm{tor}}) = \pi_1(G)$. We see that (3.1) is a coflasque resolution of $\pi_1(G)$. Thus we may take $F_G = S_G$. We obtain an isomorphism $\xi_G : \mathcal{H}(F_G) \xrightarrow{\sim} \mathcal{H}(S_G)$.

*3.10.* We show that $\xi_G$ does not depend on the choice of a smooth compactification $V_G$ of $G$.

Let $V_1$ and $V_2$ be two smooth compactifications of $G$. Proposition 2.4 shows that it suffices to consider the case when $V_1$ dominates $V_2$. Let $\lambda : V_1 \to V_2$ denote the domination morphism. Let $S_1$ and $S_2$ be the Néron–Severi tori of $V_1$ and $V_2$, respectively. Then $\lambda$ induces a homomorphism $\lambda_* : S_1 \to S_2$ and an isomorphism $\varphi_{12} = \lambda_* : \mathcal{H}(S_1) \to \mathcal{H}(S_2)$, where $\varphi_{12}$ is the canonical isomorphism defined in 2.8. Since Voskresenskiĭ's exact sequence is functorial in $(G, V_G)$, the morphism $\lambda : (G, V_1) \to (G, V_2)$ induces a morphism of coflasque resolutions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{X}_*(S_1) & \longrightarrow & P_1 & \longrightarrow & \pi_1(G) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda_*} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbf{X}_*(S_2) & \longrightarrow & P_2 & \longrightarrow & \pi_1(G) & \longrightarrow & 0
\end{array}$$

where $P_1$ and $P_2$ are permutation modules. Thus $\lambda_* : S_1 \to S_2$ is the morphism of flasque tori corresponding to a morphism of coflasque resolutions of the $\Gamma$-module $\pi_1(G)$. In other words, if we set $F_1 = S_1$ and $F_2 = S_2$, then we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{H}(S_1) & = & \mathcal{H}(F_1) \\
{\scriptstyle \varphi_{12}}\downarrow & & \downarrow{\scriptstyle \varphi_{12}} \\
\mathcal{H}(S_2) & = & \mathcal{H}(F_2)
\end{array}$$

where the left vertical arrow is defined in 2.8, while the right vertical arrow is defined in 1.7(iii). Thus the isomorphism $\xi_G$ is well defined (does not depend on the choice of a smooth compactification of $G$).

One can easily show that $\xi_G$ is functorial in $G$ (using Proposition 2.6 and the fact that Voskresenskiĭ's exact sequence is functorial in $(G, V_G)$).

*3.11.* The next step is to extend $\xi_G$ to all connected reductive $k$-groups. We use the method of $z$-extensions.

A $z$-extension of a reductive $k$-group $G$ is an exact sequence of connected reductive $k$-groups

$$1 \to Z \to H \xrightarrow{\beta} G \to 1$$

such that $H^{\mathrm{ss}}$ is simply connected and $Z$ is a quasi-trivial $k$-torus. By a lemma of Langlands, cf. [34, Proposition 3.1], every reductive $k$-group admits a $z$-extension.

We need two lemmas.

**Lemma 3.12.** *Let*

$$1 \to G_1 \to G_2 \xrightarrow{\beta} G_3 \to 1$$

*be an exact sequence of connected linear $k$-groups. Assume that $G_1$ is $k$-rational and that $H^1(K, G_1) = 1$ for any field extension $K/k$. Then $\beta_*: \mathcal{H}(S_{G_2}) \to \mathcal{H}(S_{G_3})$ is an isomorphism of abelian groups.*

**Proof.** Since $H^1(K, G_1) = 0$ for any field extension $K/k$, in particular for $K = k(G_3)$, the epimorphism $\beta$ admits a rational section $s: U_3 \to U_2$, where $U_3$ is an open subset in $G_3$ and $U_2 = \beta^{-1}(U_3)$. Let $\beta': U_2 \to U_3$ be the map induced by $\beta$, then $\beta' \circ s = \mathrm{id}_{U_3}$. We define an isomorphism of $k$-varieties

$$\lambda: U_3 \times G_1 \to U_2, \quad (g_3, g_1) \mapsto s(g_3)g_1$$

(we assume that $G_1 \subset G_2$). By [15, Lemme 11] we have $\mathrm{Pic}(\overline{V}_{U_3} \times \overline{V}_{G_1}) = \mathrm{Pic}\,\overline{V}_{U_3} \oplus \mathrm{Pic}\,\overline{V}_{G_1}$, hence $\lambda$ induces an isomorphism $\mathcal{H}(S_{U_3}) \times \mathcal{H}(S_{G_1}) \to \mathcal{H}(S_{U_2})$. Since $G_1$ is a $k$-rational variety, $S_G$ is a quasi-trivial torus, and $\mathcal{H}(S_{G_1}) = 0$. We see that $s$ induces an isomorphism $s_*: \mathcal{H}(S_{U_3}) \to \mathcal{H}(S_{U_2})$. Since $\beta' \circ s = \mathrm{id}_{U_3}$, we have $\beta'_* \circ s_* = \mathrm{id}$. We see that $\beta'_*: \mathcal{H}(S_{U_2}) \to \mathcal{H}(S_{U_3})$ is an isomorphism.

Consider the commutative diagrams

$$\begin{array}{ccc} U_2 & \xrightarrow{i_2} & G_2 \\ {\scriptstyle \beta'}\downarrow & & \downarrow{\scriptstyle \beta} \\ U_3 & \xrightarrow{i_3} & G_3 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{H}(S_{U_2}) & \xrightarrow{i_{2*}} & \mathcal{H}(S_{G_2}) \\ {\scriptstyle \beta'_*}\downarrow & & \downarrow{\scriptstyle \beta_*} \\ \mathcal{H}(S_{U_3}) & \xrightarrow{i_{3*}} & \mathcal{H}(S_{G_3}) \end{array}$$

where $i_2$ and $i_3$ are the inclusions. Clearly $i_{2*}$ and $i_{3*}$ in the right diagram are isomorphisms (we may take $V_{U_2} = V_{G_2}$ and $V_{U_3} = V_{G_3}$). We have proved that $\beta'_*$ is an isomorphism, hence $\beta_*$ is an isomorphism. □

**Corollary 3.13.** *Let $H \xrightarrow{\beta} G$ be a $z$-extension with kernel $Z$. Then $\beta_*: \mathcal{H}(S_H) \to \mathcal{H}(S_G)$ is an isomorphism of abelian groups.*

**Corollary 3.14.** *Let $G$ be a connected $k$-group, $r : G \to G^{\mathrm{red}}$ the canonical epimorphism. Then the induced homomorphism $r_* : \mathcal{H}(S_G) \to \mathcal{H}(S_{G^{\mathrm{red}}})$ is an isomorphism.*

**Lemma 3.15.** *Let $H \xrightarrow{\beta} G$ be a $z$-extension with kernel $Z$. Then $\beta_* : \mathcal{H}(F_H) \to \mathcal{H}(F_G)$ is an isomorphism of abelian groups.*

**Proof.** By Lemma 3.7 we have an exact sequence

$$0 \to \mathbf{X}_*(Z) \to \pi_1(H) \xrightarrow{\beta_*} \pi_1(G) \to 0.$$

Let

$$0 \to Q_G \to P_G \to \pi_1(G) \to 0$$

be a coflasque resolution of $\pi_1(G)$. Set $P = P_G \times_{\pi_1(G)} \pi_1(H)$. We have exact sequences

$$0 \to \mathbf{X}_*(Z) \to P \xrightarrow{p_G} P_G \to 0, \tag{3.2}$$

$$0 \to Q_G \to P \xrightarrow{p_H} \pi_1(H) \to 0, \tag{3.3}$$

where $p_G$ and $p_H$ are the projections. Since $\mathbf{X}_*(Z)$ and $P_G$ are permutation modules, the sequence (3.2) splits. Therefore $P$ is a permutation module, and (3.3) is a coflasque resolution of $\pi_1(H)$.

Consider the morphism of resolutions $(\beta_*, p_G)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_G & \longrightarrow & P & \xrightarrow{p_H} & \pi_1(H) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle p_G} & & \downarrow{\scriptstyle \beta_*} & & \\ 0 & \longrightarrow & Q_G & \longrightarrow & P_G & \longrightarrow & \pi_1(G) & \longrightarrow & 0 \end{array}$$

Clearly $p_G|_{Q_G} : Q_G \to Q_G$ is the identity map. Thus the induced homomorphism $\mathcal{H}(F_H) \to \mathcal{H}(F_G)$ is an isomorphism. □

We shall use the following lemma.

**Lemma 3.16** [27, Lemma 2.4.4]. *Let $G_1 \to G_2$ be a homomorphism of connected reductive $k$-groups, and let $H_i \to G_i$ $(i = 1, 2)$ be $z$-extensions. Then there exists a commutative diagram*

$$\begin{array}{ccccc} H_1 & \longleftarrow & H_3 & \longrightarrow & H_2 \\ \downarrow & & \downarrow & & \downarrow \\ G_1 & \xleftarrow{\mathrm{id}} & G_1 & \longrightarrow & G_2 \end{array}$$

*in which the homomorphisms $H_3 \to H_1$ and $H_3 \to H_2$ are surjective, and $H_3 \to G_1$ is a $z$-extension.*

*3.17.* We can now extend the isomorphism $\xi_G : \mathcal{H}(F_G) \to \mathcal{H}(S_G)$ to all connected reductive $k$-groups. Let $G$ be a reductive group. Choose a $z$-extension $H \xrightarrow{\beta} G$. The isomorphism $\xi_H$ is already defined because $H^{ss}$ is simply connected. We must define $\xi_G$ so that the following diagram of isomorphisms is commutative:

$$\begin{array}{ccc} \mathcal{H}(F_H) & \xrightarrow{\xi_H} & \mathcal{H}(S_H) \\ \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_*} \\ \mathcal{H}(F_G) & \xrightarrow{\xi_G} & \mathcal{H}(S_G) \end{array}$$

By Corollary 3.13 and Lemma 3.15, the vertical arrows are isomorphisms, and $\xi_G$ is thus defined. Using Lemma 3.16, one can easily check that our $\xi_G$ does not depend on the choice of a $z$-extension $H \xrightarrow{\beta} G$ and is functorial in $G$.

To extend $\xi_G$ to all connected $k$-groups $G$, we need a lemma.

**Lemma 3.18.** *Let $G$ be a connected $k$-group, $r : G \to G^{red}$ the canonical epimorphism. Then the induced homomorphism $r_* : \mathcal{H}(F_G) \to \mathcal{H}(F_{G^{red}})$ is an isomorphism.*

**Proof.** By definition $\pi_1(G) = \pi_1(G^{red})$, and therefore $\mathcal{H}(F_G) = \mathcal{H}(F_{G^{red}})$. □

*3.19.* We can now extend $\xi_G$ to the category of all connected $k$-groups $G$. We must define $\xi_G$ so that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{H}(F_G) & \xrightarrow{\xi_G} & \mathcal{H}(S_G) \\ \downarrow{\scriptstyle r_*} & & \downarrow{\scriptstyle r_*} \\ \mathcal{H}(F_{G^{red}}) & \xrightarrow{\xi_{G^{red}}} & \mathcal{H}(S_{G^{red}}) \end{array}$$

By Corollary 3.14 and Lemma 3.18, the vertical arrows are isomorphisms, and $\xi_G$ is thus defined. This isomorphism $\xi_G$ is functorial in $G$.

This completes the proof of Theorem 3.5.

**Remark 3.20.** (i) Theorem 3.2 generalizes [13, Proposition 3.2]. It was an observation by V. Chernousov that the device of passage to the variety of Borel subgroups can simplify the proof of that proposition. This observation, along with discussions with P. Gille, led us to Theorem 3.2. P. Gille suggested another proof of Theorem 3.2.

(ii) A particular case of Theorem 3.5 (for semisimple groups over number fields) was proved in [21] in the course of the proof of Theorem III.4.3. Discussions with P. Gille around this result led us to Theorem 3.5. P. Gille suggested another proof.

**Theorem 3.21.** *Set $Q_G = \mathbf{X}_*(F_G)$, then the $\Gamma$-modules $Q_G$ and $(\operatorname{Pic} \overline{V}_G)^\vee$ are similar, i.e., $Q_G \oplus P_1 \simeq (\operatorname{Pic} \overline{V}_G)^\vee \oplus P_2$, where $P_1$ and $P_2$ are some permutation modules (recall that $(\ )^\vee$ denotes the dual module).*

**Proof.** We have actually proved this while proving Theorem 3.5. Indeed, in 3.9 we proved that for a reductive group $G$ such that $G^{\mathrm{ss}}$ is simply connected, we may take $Q_G = (\operatorname{Pic} \overline{V}_G)^\vee$. In the proofs of Corollaries 3.13, 3.14 and Lemmas 3.15 and 3.18 we proved that if $G$ is any connected $k$-group, then $Q_G$ is similar to $Q_H$ and $(\operatorname{Pic} \overline{V}_G)^\vee$ is similar to $(\operatorname{Pic} \overline{V}_H)^\vee$ for some reductive group $H$ such that $H^{\mathrm{ss}}$ is simply connected. □

**Remark 3.22.** In Sections 1–3 we assumed that $\mathcal{H}$ is a *covariant* functor only for simplicity. All the results (with evident changes) also hold for an additive *contravariant* functor $\mathcal{H}$ such that $\mathcal{H}(N) = 0$ for any quasi-trivial $k$-torus $N$.

**Theorem 3.23.** *Let $G$ be a connected linear $k$-group. Then there is a canonical functorial isomorphism*

$$H^1\big(k, \operatorname{Pic} \overline{V}_G\big) \simeq H^1\big(k, Q_G^\vee\big),$$

*where $Q_G^\vee$ is the dual module to $Q_G$, and $Q_G$ comes from a coflasque resolution*

$$0 \to Q_G \to P \to \pi_1(G) \to 0.$$

**Proof.** Since $\operatorname{Pic} \overline{V}_G = \mathbf{X}^*(S_G)$ and $Q_G^\vee = \mathbf{X}^*(F_G)$, the theorem follows from Theorem 3.5 applied to the contravariant functor $T \mapsto \mathcal{H}(T) = \mathbf{X}^*(T)$. □

**Corollary 3.24.** *Let $E$ be a principal homogeneous space of a connected linear $k$-group $G$. Then there is a canonical isomorphism*

$$H^1\big(k, \operatorname{Pic} \overline{V}_E\big) \simeq H^1\big(k, Q_G^\vee\big).$$

**Proof.** The functor $X \mapsto \mathcal{F}(X) = H^1(k, \operatorname{Pic} \overline{V}_X)$ on the category of rational $k$-varieties is additive, i.e., $\mathcal{F}(X_1 \times X_2) = \mathcal{F}(X_1) \oplus \mathcal{F}(X_2)$, cf. [15, Lemme 11, p. 188]. By [39, Lemme 6.4] applied to the functor $\mathcal{F}$, there is a canonical isomorphism $H^1(k, \operatorname{Pic} \overline{V}_E) \simeq H^1(k, \operatorname{Pic} \overline{V}_G)$, and the corollary follows from Theorem 3.23. □

**Remark 3.25.** By Corollary 3.4 and Theorem 3.21 we can write Corollary 3.24 as follows:

$$H^1\big(k, \operatorname{Pic} \overline{V}_E\big) \simeq \ker\Big[H^1\big(k, Q_G^\vee\big) \to \prod_\gamma H^1\big(\gamma, Q_G^\vee\big)\Big],$$

where $\gamma$ runs over closed procyclic subgroups of $\operatorname{Gal}(\bar{k}/k)$. From this formula one can deduce the formula of [7, Theorem 2.4].

## 4. *R*-equivalence

In this section for $k$ as in 0.1 we construct an isomorphism of functors $G(k)/R \to H^1(k, F_G)$. (Clearly the functor $T \mapsto H^1(k,T)$ on the category of $k$-tori satisfies conditions (1)–(3) of Introduction, so we have functors $G \mapsto H^1(k, F_G)$ and $X \mapsto H^1(k, S_X)$ as in Sections 1 and 2.) We start with stating the results of [15] on $R$-equivalence on tori and the results of [12,22] on $R$-equivalence on reductive groups admitting special coverings. We derive some corollaries which will be used below.

*4.1.* The notion of $R$-equivalence was introduced by Manin [31]. Let $X$ be an algebraic variety over a field $k$. We say that two points $x, y \in X(k)$ are elementarily related if there exists a rational map $f$ of the projective line $\mathbf{P}^1$ to $X$ such that $f$ is defined in 0 and 1 and $f(0) = x$, $f(1) = y$. Two points $x$, $y$ are called $R$-equivalent if there exists a finite sequence of points $x_0 = x, x_1, \ldots, x_n = y$ such that $x_i$ is elementarily related to $x_{i-1}$ for $i = 1, \ldots, n$. We denote by $X(k)/R$ the set of equivalence classes in $X(k)$. If $G$ is a connected linear algebraic group over $k$, then the set $G(k)/R$ has a natural group structure.

*4.2.* Let $T$ be a $k$-torus. Let

$$0 \to Q \to P \to \mathbf{X}_*(T) \to 0 \tag{R}$$

be a coflasque resolution. Let

$$1 \to F_T \to N \to T \to 1$$

be the corresponding exact sequence of tori, where $\mathbf{X}_*(N) = P$ and $\mathbf{X}_*(F_T) = Q$. Consider the exact sequence

$$N(k) \to T(k) \xrightarrow{\delta_T} H^1(k, F_T) \to H^1(k, N) = 0.$$

**Theorem 4.3** [15, Theorem 2, p. 199]. *The map $\delta_T$ induces an isomorphism $\delta_{T*}: T(k)/R \xrightarrow{\sim} H^1(k, F_T)$.*

**Corollary 4.4.** *The collection of isomorphisms $\delta_{T*}: T(k)/R \xrightarrow{\sim} H^1(k, F_T)$ is an isomorphism of functors (from $k$-tori to abelian groups).*

**Proof.** Easy diagram chasing. □

Let now $k$ be as in 0.1. Let $G$ be a connected linear $k$-group. In the case (gl) we always assume that $G$ has no factors of type $E_8$.

We say that a connected $k$-group $G$ admits a special covering if $G$ is reductive and there is an exact sequence

$$1 \to \mu \to G' \to G \to 1$$

with $\mu$ finite and $G'$ the product of a semisimple simply connected group and a quasi-trivial torus.

**Theorem 4.5** ([22], [12, Theorem 4.12]). *Let $k$ be as in* 0.1. *Let $G$ be a connected reductive $k$-group admitting a special covering. In the* (gl) *case, assume that $G$ contains no factor of type $E_8$. Let*

$$1 \to \mu \to F \to N \to 1$$

*be a flasque resolution of $\mu$ (i.e., $F$ is a flasque torus and $N$ is a quasi-trivial torus). Then the Galois cohomology sequences induce an isomorphism of groups $G(k)/R \simeq H^1(k, F)$.*

**Corollary 4.6.** *Under the hypotheses of Theorem* 4.5*, suppose that $f : G_1 \to G_2$ is a homomorphism of $k$-groups admitting special coverings, and assume that $f$ extends to a morphism of coverings*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_1 & \longrightarrow & G'_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle f} & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & G'_2 & \longrightarrow & G_2 & \longrightarrow & 1 \end{array}$$

*where $\varphi : \mu_1 \to \mu_2$ is an isomorphism. Then the induced homomorphism $f_* : G_1(k)/R \to G_2(k)/R$ is an isomorphism.*

**Proof** (*idea*). We can choose flasque resolutions

$$1 \to \mu_i \to F_i \to N_i \to 1 \quad (i = 1, 2)$$

so that $\varphi$ extends to an isomorphism of resolutions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_1 & \longrightarrow & F_1 & \longrightarrow & N_1 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & F_2 & \longrightarrow & N_2 & \longrightarrow & 1 \end{array}$$

(i.e., $\varphi$, $\alpha$ and $\beta$ are isomorphisms). □

*4.7.* We can now state and prove our main result on $R$-equivalence on groups over a field $k$ as in 0.1.

Let $k$ be a field of characteristic 0. Consider two functors from the category of connected linear algebraic $k$-groups to the category of abelian groups: $G \mapsto G(k)/R$ and $G \mapsto H^1(k, F_G)$ (the latter functor was introduced in 1.9). The collection of maps $\delta_{T*}$ of Theorem 4.3 is an isomorphism of these functors on the category of $k$-tori.

**Theorem 4.8.** *Assume that $k$ is as in* 0.1. *In the* (gl) *case assume that $G$ has no $E_8$-factors. Then the isomorphism of functors $\delta_{T*}$ extends uniquely to an isomorphism of functors $\theta_G : G(k)/R \to H^1(k, F_G)$.*

**Corollary 4.9.** *For $k$ and $G$ as in Theorem* 4.8, *if the image of* $\mathrm{Gal}(\bar{k}/k)$ *in* $\mathrm{Aut}\,\pi_1(G)$ *is a metacyclic group, then $G(k)/R = 1$.*

**Proof.** The corollary follows from Theorem 4.8 and Proposition 1.10. □

**Corollary 4.10.** *Let $k$ and $G$ be as in Theorem* 4.8, *then*

(i) *there is a canonical isomorphism $G(k)/R \simeq H^1(k, S_G)$;*
(ii) *the group $G(k)/R$ is a stably $k$-birational invariant of $G$.*

**Proof.** The corollary follows from Theorem 4.8, Theorem 3.5, and Proposition 2.14. □

**Remark 4.11.** (i) It is clear that the *set* $G(k)/R$ is a stably $k$-birational invariant of $G$, but it is not clear *a priori* that the *group* $G(k)/R$ is a stably $k$-birational invariant of $G$, cf. [15, p. 201].

(ii) Let $k, G_1, G_2$ be as in Theorem 4.8, and let $f : G_1 \to G_2$ be a rational map defined over $k$. In other words, we are given open subvarieties $U_\nu \subset G_\nu$ $(\nu = 1, 2)$ and a regular map $f' : U_1 \to U_2$, all defined over $k$. The map $f'$ induces a map $f'_* : U_1(k)/R \to U_2(k)/R$. Let $i_\nu : U_\nu \to G_\nu$ denote the inclusions, then $i_{\nu*} : U_\nu(k)/R \to G_\nu(k)/R$ are bijections, cf. [15, Proposition 11]. We identify $U_\nu(k)/R$ with $G_\nu(k)/R$ using $i_{\nu*}$ $(\nu = 1, 2)$. Then we obtain a map $f_* : G_1(k)/R \to G_2(k)/R$. On the other hand, in 2.13 we constructed the induced homomorphism $f_* : H^1(k, S_{G_1}) \to H^1(k, S_{G_2})$. By Corollary 4.10(i) we have canonical isomorphisms $G_\nu(k)/R \to H^1(k, S_{G_\nu})$ $(\nu = 1, 2)$. However in general the diagram

$$\begin{array}{ccc} G_1(k)/R & \xrightarrow{f_*} & G_2(k)/R \\ \downarrow & & \downarrow \\ H^1(k, S_{G_1}) & \xrightarrow{f_*} & H^1(k, S_{G_2}) \end{array}$$

is *not* commutative! For example take $G_1 = G_2 = G$, and let $f$ be a left translation, i.e., $f(g) = ag$ $(g \in G)$ for a fixed element $a \in G(k)$. Then $f_* : G(k)/R \to G(k)/R$ may take the identity element to another element, while $f_* : H^1(k, S_G) \to H^1(k, S_G)$ is an isomorphism of abelian groups.

(iii) Corollary 4.9 and the similar corollaries below (Corollaries 5.11 and 7.8) generalize results of [12] (Corollary 4.11(iv), Corollary 4.14(iv), and Theorem 5.2(b)(i)) proved for *semisimple* groups splitting over a metacyclic extension.

To prove Theorem 4.8 we need some lemmas.

**Lemma 4.12.** *Let $k$ be as in* 0.1. *Let $G$ be a reductive group such that $G^{\mathrm{ss}}$ is simply connected. Then the map $G(k) \to G^{\mathrm{tor}}(k)$ induces an isomorphism $G(k)/R \to G^{\mathrm{tor}}(k)/R$.*

**Proof.** We give two proofs.

(1) See Appendix by P. Gille, Theorem 1(b).

(2) (with the help of J.-L. Colliot-Thélène) (i) First assume that $G$ admits a special covering

$$1 \to \mu \to G_0 \times N \to G \to 1,$$

where $G_0$ is a simply connected group and $N$ is a quasi-trivial torus. Since $G^{\mathrm{ss}}$ is simply connected, we see that $\mu \cap G_0 = 1$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu & \longrightarrow & G_0 \times N & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \mu^t & \longrightarrow & N & \longrightarrow & G^{\mathrm{tor}} & \longrightarrow & 1 \end{array}$$

where $\mu \to \mu^t$ is an isomorphism. Hence by Corollary 4.6 we get an isomorphism $G(k)/R \to G^{\mathrm{tor}}(k)/R$.

(ii) Let now $G$ be any reductive $k$-group such that $G^{\mathrm{ss}}$ is simply connected. By [39, Lemme 1.10] there exist a natural number $m$ and a quasi-trivial torus $T$ such that $G^m \times T$ admits a special covering. Clearly $(G^m \times T)^{\mathrm{ss}}$ is simply connected. By (i),

$$\bigl(G^m \times T\bigr)(k)/R \to \bigl(\bigl(G^m\bigr)^{\mathrm{tor}} \times T\bigr)(k)/R$$

is an isomorphism. Thus $(G(k)/R)^m \to (G^{\mathrm{tor}}(k)/R)^m$ is an isomorphism, and $G(k)/R \to G^{\mathrm{tor}}(k)/R$ is an isomorphism. □

**Lemma 4.13.** *Let $k$ be a field of characteristic* 0 *and let $G$ be a connected reductive $k$-group such that $G^{\mathrm{ss}}$ is simply connected. Then the map $t: G \to G^{\mathrm{tor}}$ induces an isomorphism $t_*: H^1(k, F_G) \to H^1(k, F_{G^{\mathrm{tor}}})$.*

**Proof.** By Lemma 3.7, $\pi_1(G) \simeq \pi_1(G^{\mathrm{tor}})$, and the lemma follows. □

*4.14.* We can now extend the isomorphism $\theta_G : G(k)/R \to H^1(k, F_G)$ from the category of $k$-tori to the category of reductive $k$-groups $G$ such that $G^{\mathrm{ss}}$ is simply connected. Namely, we must define an isomorphism $\theta_G : G(k)/R \to H^1(k, F_G)$ so that the following diagram is commutative:

$$\begin{array}{ccc} G(k)/R & \xrightarrow{\theta_G} & H^1(k, F_G) \\ {\scriptstyle t_*}\downarrow & & \downarrow{\scriptstyle t_*} \\ G^{\mathrm{tor}}(k)/R & \xrightarrow{\theta_{G^{\mathrm{tor}}}} & H^1(k, F_{G^{\mathrm{tor}}}) \end{array}$$

where $t : G \to G^{\mathrm{tor}}$ is the canonical epimorphism. Here the left vertical arrow is an isomorphism by Lemma 4.12, and the right vertical arrow is an isomorphism by Lemma 4.13. Thus $\theta_G$ is defined.

The next step is to extend $\theta_G$ to all connected reductive $k$-groups. We use the method of $z$-extensions. We need a lemma.

**Lemma 4.15.** *Let $k$ be a field of characteristic* 0 *and let*

$$1 \to G_1 \to G_2 \xrightarrow{\beta} G_3 \to 1$$

*be an exact sequence of connected linear $k$-groups. Assume that $G_1(k)/R = 1$ and that $H^1(K, G_1) = 1$ for any field extension $K/k$. Then $\beta_* : G_2(k)/R \to G_3(k)/R$ is a group isomorphism.*

**Proof.** The map $\beta_*$ is clearly a group homomorphism. We wish to prove that $\beta_*$ is bijective.

Since $H^1(K, G_1) = 0$ for any field extension $K/k$, the epimorphism $\beta$ admits a rational section $s : U_3 \to U_2$, where $U_3$ is an open subset in $G_3$ and $U_2 = \beta^{-1}(U_3)$. Let $\beta' : U_2 \to U_3$ be the map induced by $\beta$, then $\beta' \circ s = \mathrm{id}_{U_3}$. As in the proof of Lemma 3.12, we define an isomorphism of $k$-varieties

$$\lambda : U_3 \times G_1 \to U_2, \quad (g_3, g_1) \mapsto s(g_3)g_1$$

(we assume that $G_1 \subset G_2$). Then $\lambda$ induces a bijection $U_3(k)/R \times G_1(k)/R \to U_2(k)/R$. By assumption $G_1(k)/R = 1$. We see that $s$ induces a bijection $s_* : U_3(k)/R \to U_2(k)/R$. Since $\beta' \circ s = \mathrm{id}_{U_3}$, we have $\beta'_* \circ s_* = \mathrm{id}$. We see that $\beta'_* : U_2(k)/R \to U_3(k)/R$ is a bijection.

Consider the commutative diagrams

$$\begin{array}{ccc} U_2 & \xrightarrow{i_2} & G_2 \\ {\scriptstyle \beta'}\downarrow & & \downarrow{\scriptstyle \beta} \\ U_3 & \xrightarrow{i_3} & G_3 \end{array} \quad \text{and} \quad \begin{array}{ccc} U_2(k)/R & \xrightarrow{i_{2*}} & G_2(k)/R \\ {\scriptstyle \beta'_*}\downarrow & & \downarrow{\scriptstyle \beta_*} \\ U_3(k)/R & \xrightarrow{i_{3*}} & G_3(k)/R \end{array}$$

where $i_2$ and $i_3$ are the inclusions. By [15, Proposition 11], $i_{2*}$ and $i_{3*}$ in the right diagram are bijections. We have proved that $\beta'_*$ is a bijection, hence $\beta_*$ is a bijection. We conclude that $\beta_*$ is a group isomorphism. □

**Corollary 4.16.** *Let $H \xrightarrow{\beta} G$ be a $z$-extension with kernel $Z$. Then $\beta_* : H(k)/R \to G(k)/R$ is an isomorphism of groups.*

**Corollary 4.17.** *Let $G$ be a connected $k$-group. Then the homomorphism $r_* : G(k)/R \to G^{\mathrm{red}}(k)/R$ is an isomorphism.*

*4.18.* We can now extend the isomorphism $\theta_G : G(k)/R \to H^1(k, F_G)$ to all reductive $k$-groups. The construction is similar to that of 3.17. We choose a $z$-extension $H \xrightarrow{\beta} G$. We must define $\theta_G$ so that the following diagram is commutative:

$$\begin{array}{ccc} H(k)/R & \xrightarrow{\theta_H} & H^1(k, F_H) \\ \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_*} \\ G(k)/R & \xrightarrow{\xi_G} & H^1(k, F_G) \end{array}$$

Here the left vertical arrow is an isomorphism by Corollary 4.16, and the right vertical arrow is an isomorphism by Lemma 3.15. As in 3.17, using Lemma 3.16, one can easily check that our $\theta_G$ does not depend on the choice of the $z$-extension $H \xrightarrow{\beta} G$ and is functorial in $G$.

*4.19.* We extend $\theta_G$ to all connected $k$-groups. We denote by $r : G \to G^{\mathrm{red}}$ the canonical epimorphism. Using Corollary 4.17 and Lemma 3.18, we can construct $\theta_G$ for any connected $k$-group $G$ so that the following diagram is commutative:

$$\begin{array}{ccc} G(k)/R & \xrightarrow{\theta_G} & H^1(k, F_G) \\ \downarrow{\scriptstyle r_*} & & \downarrow{\scriptstyle r_*} \\ G^{\mathrm{red}}(k)/R & \xrightarrow{\theta_{G^{\mathrm{red}}}} & H^1(k, F_{G^{\mathrm{red}}}) \end{array}$$

This isomorphism $\theta_G$ is functorial in $G$.

Theorem 4.8 is completely proved.

**Remark 4.20.** Theorem 4.8 also holds when $k$ is a non-archimedean local field of characteristic 0 or a totally imaginary number field. The assertion similar to Theorem 4.5 was proved for such fields by Gille (in [21, III.2.7] for local fields and in [21, III.4.1(1)] for totally imaginary number fields).

*4.21.* We now show how one can derive the formula of Theorem 4.5 from the formula of Theorem 4.8.

Let $k$ be a field of characteristic 0, $G$ a reductive $k$-group admitting a special covering

$$1 \to \mu \to G_0 \times N_0 \xrightarrow{\alpha} G \to 1$$

where $G_0$ is a simply connected group and $N_0$ is a quasi-trivial torus. We identify $G_0$ with $G^{\mathrm{sc}}$. Let

$$1 \to \mu \to F \to N \to 1 \tag{4.1}$$

be a flasque resolution of $\mu$, i.e., $F$ is a flasque torus and $N$ is a quasi-trivial torus. We wish to construct a coflasque resolution of $\pi_1(G)$ of the form

$$0 \to \mathbf{X}_*(F) \to P \to \pi_1(G) \to 0$$

where $P$ is a permutation module. Then we may take $F_G = F$, hence $H^1(k, F_G) = H^1(k, F)$.

Let $T \subset G$ be a maximal torus. We obtain an exact sequence

$$1 \to \mu \to T^{\mathrm{sc}} \times N_0 \to T \to 1$$

where $T^{\mathrm{sc}} \times N_0 = \alpha^{-1}(T)$ and $T^{\mathrm{sc}}$ is a maximal torus of $G^{\mathrm{sc}}$. Going over to cocharacters, we obtain an exact sequence of $\Gamma$-modules (cf. [21, Lemme A.3])

$$0 \to \mathbf{X}_*\big(T^{\mathrm{sc}}\big) \oplus \mathbf{X}_*(N_0) \to \mathbf{X}_*(T) \to \mu(-1) \to 0.$$

We now factor out $\mathbf{X}_*(T^{\mathrm{sc}})$ taking into account the definition of $\pi_1(G)$ (see 1.8). We obtain an exact sequence

$$0 \to \mathbf{X}_*(N_0) \to \pi_1(G) \to \mu(-1) \to 0.$$

Going over to cocharacters in (4.1), we obtain

$$0 \to \mathbf{X}_*(F) \to \mathbf{X}_*(N) \to \mu(-1) \to 0.$$

Let $P = \pi_1(G) \times_{\mu(-1)} \mathbf{X}_*(N)$ be the fibered product. We obtain exact sequences

$$0 \to \mathbf{X}_*(N_0) \to P \to \mathbf{X}_*(N) \to 0, \tag{4.2}$$

$$0 \to \mathbf{X}_*(F) \to P \to \pi_1(G) \to 0. \tag{4.3}$$

Sequence (4.2) splits because $\mathbf{X}_*(N_0)$ and $\mathbf{X}_*(N)$ are permutation modules (cf. [29, Proposition 1.2]), hence $P$ is a permutation module. Thus sequence (4.3) is a required coflasque resolution.

## 5. Weak approximation

In this section we compute $A_\Sigma(G)$ where $G$ is a field of type (gl) or (ll). But first we consider weak approximation in a more general setting.

*5.1.* Let $k$ be a field of characteristic 0. Let $\Sigma$ be a finite set of non-equivalent absolute values on $k$, cf. [28, Chapter XII, §1]. For $v \in \Sigma$ let $k_v$ denote the completion of $k$ with respect to $v$. Let $G$ be a connected linear $k$-group. We set $k_\Sigma = \prod_{v \in \Sigma} k_v$, then $G(k_\Sigma) = \prod_{v \in \Sigma} G(k_v)$. Let $\overline{G(k)}$ denote the closure of the image of $G(k)$ under the diagonal embedding $G(k) \to G(k_\Sigma)$. We say that $G$ has the weak approximation property with respect to $\Sigma$ if $G(k)$ is dense in $G(k_\Sigma)$, i.e., $\overline{G(k)} = G(k_\Sigma)$.

**Proposition 5.2** (stated in [35, Proposition 1.3]). *Let* $k, \ \Sigma, \ G$ *be as in* 5.1. *Then* $\overline{G(k)}$ *is an open subgroup of* $G(k_\Sigma)$.

**Proof.** Since $\mathrm{char}(k) = 0$, $G$ is a $k$-unirational variety, cf. [2, Theorem 18.2(ii)]. It follows that there exists a smooth morphism of $k$-varieties $\lambda : U \to G$, where $U$ is an open subvariety in $\mathbf{P}_k^n$ for some $n$. We have $\overline{U(k)} = U(k_\Sigma)$. Since $\lambda$ is a smooth morphism, the map $\lambda_v : U(k_v) \to G(k_v)$ is open for each $v$, cf. [23, Satz 1.1.1], hence the map $\lambda : U(k_\Sigma) \to G(k_\Sigma)$ is open. We see that the set $\lambda(U(k_\Sigma))$ is open in $G(k_\Sigma)$.

We have $\overline{G(k)} \supset \overline{\lambda(U(k))} \supset \lambda(\overline{U(k)}) = \lambda(U(k_\Sigma))$. We see that the subgroup $\overline{G(k)}$ of $G(k_\Sigma)$ contains the open set $\lambda(U(k_\Sigma))$. It follows that the subgroup $\overline{G(k)}$ is open in $G(k_\Sigma)$. $\square$

**Proposition 5.3** [35, Proposition 1.4]. *Let* $k$, $\Sigma$, *and* $G$ *be as in* 5.1. *Assume that* $\overline{G^{\mathrm{sc}}(k)} = G^{\mathrm{sc}}(k_\Sigma)$. *Then the closure* $\overline{G(k)}$ *of* $G(k)$ *in* $G(k_\Sigma)$ *is a normal subgroup, and the quotient* $A_\Sigma(G) := G(k_\Sigma)/\overline{G(k)}$ *is an abelian group.*

**Proof.** We use an idea of [26, proof of Satz 6.1]. It suffices to prove that $\overline{G(k)}$ contains the commutator subgroup of $G(k_\Sigma)$.

First we assume that $G$ is reductive. Consider the homomorphism $\rho : G^{\mathrm{sc}} \to G$. By [26, Hilfssatz 6.2] (see also [18, 2.0.3]) the commutator subgroup $[G(k_v), G(k_v)]$ is contained in $\rho(G^{\mathrm{sc}}(k_v))$ for each $v$. It follows that $[G(k_\Sigma), G(k_\Sigma)]$ is contained in $\rho(G^{\mathrm{sc}}(k_\Sigma))$. But $G^{\mathrm{sc}}(k_\Sigma) = \overline{G^{\mathrm{sc}}(k)}$ by assumption. Since $\rho(\overline{G^{\mathrm{sc}}(k)}) \subset \overline{G(k)}$, we conclude that $[G(k_\Sigma), G(k_\Sigma)] \subset \overline{G(k)}$, which was to be proved.

We now consider the case of general $G$ (not necessarily reductive). Consider the canonical map $r : G \to G^{\mathrm{red}}$. Let $s : G^{\mathrm{red}} \to G$ be the splitting corresponding to a Levi decomposition $G \simeq G^{\mathrm{u}} \rtimes G^{\mathrm{red}}$. Then $G(k) = G^{\mathrm{u}}(k) \cdot s(G^{\mathrm{red}}(k))$. Clearly we have $\overline{G(k)} = \overline{G^{\mathrm{u}}(k)} \cdot s(\overline{G^{\mathrm{red}}(k)})$. Since $G^{\mathrm{u}}$ is $k$-rational, $\overline{G^{\mathrm{u}}(k)} = G^{\mathrm{u}}(k_\Sigma)$, hence $\overline{G(k)} = G^{\mathrm{u}}(k_\Sigma) \cdot s(\overline{G^{\mathrm{red}}(k)})$. Clearly $r^{-1}(\overline{G^{\mathrm{red}}(k)}) = G^{\mathrm{u}}(k_\Sigma) \cdot s(\overline{G^{\mathrm{red}}(k)})$, and therefore $r^{-1}(\overline{G^{\mathrm{red}}(k)}) = \overline{G(k)}$. Since $[G^{\mathrm{red}}(k_\Sigma), G^{\mathrm{red}}(k_\Sigma)] \subset \overline{G^{\mathrm{red}}(k)}$, we conclude that $[G(k_\Sigma), G(k_\Sigma)] \subset \overline{G(k)}$, which was to be proved. $\square$

*5.4.* Let now $k$ be a field of one of types (ll) or (gl). Let $\Omega$ denote the associated set of discrete valuations of $k$, see [12, §1]. Let $\Sigma \subset \Omega$ be a finite subset. Let $G$ be a connected linear $k$-group. In the case (gl) assume that $G$ has no $E_8$-factor. By [12, Theorem 4.7], $G^{\mathrm{sc}}(k)$ is dense in $G^{\mathrm{sc}}(k_\Sigma)$. By Proposition 5.3, $\overline{G(k)}$ is a normal subgroup of $G(k_\Sigma)$, and the quotient $A_\Sigma(G) := G(k_\Sigma)/\overline{G(k)}$ is an abelian group (this was earlier proved in [12, Theorem 4.13(i)]).

**Lemma 5.5.** *Let $k$ and $\Sigma$ be as in* 5.4*, and let*

$$1 \to G_1 \to G_2 \xrightarrow{\beta} G_3 \to 1$$

*be an exact sequence of connected linear $k$-groups. Assume that $H^1(k, G_1) = 1$, $H^1(k_v, G_1) = 1$ for each $v \in \Sigma$. Assume that $G_1$ has the weak approximation property with respect to $\Sigma$. Then the induced homomorphism $\beta_* : A_\Sigma(G_2) \to A_\Sigma(G_3)$ is an isomorphism.*

**Proof.** We use an idea of [36, proof of Lemma 3.8]. First we prove that $\beta(\overline{G_2(k)}) = \overline{G_3(k)}$. By Lemma 5.2, $\overline{G_2(k)}$ is open in $G_2(k_\Sigma)$. Since the homomorphism $\beta : G_2 \to G_3$ is surjective, it is a smooth morphism, hence the map $\beta : G_2(k_\Sigma) \to G_3(k_\Sigma)$ is open (cf. [23, Satz 1.1.1]), and therefore the group $\beta(\overline{G_2(k)})$ is open in $G_3(k_\Sigma)$. But any open subgroup of a topological group is closed, hence the group $\beta(\overline{G_2(k)})$ is closed in $G_3(k_\Sigma)$. Since $H^1(k, G_1) = 1$, we have $\beta(G_2(k)) = G_3(k)$, hence $\beta(\overline{G_2(k)}) \supset G_3(k)$. Thus $\beta(\overline{G_2(k)})$ is a closed subgroup of $G_3(k_\Sigma)$ containing $G_3(k)$, and we see that $\beta(\overline{G_2(k)}) = \overline{G_3(k)}$.

Next we prove that $\beta^{-1}(\overline{G_3(k)}) = \overline{G_2(k)}$. Let $g_2 \in G_2(k_\Sigma)$ be such that $\beta(g_2) \in \overline{G_3(k)}$. Then there exists $g_2' \in \overline{G_2(k)}$ such that $\beta(g_2') = \beta(g_2)$. We have $g_2(g_2')^{-1} \in G_1(k_\Sigma)$ (we assume that $G_1 \subset G_2$). By assumption $G_1(k_\Sigma) = \overline{G_1(k)}$, hence $g_2 \in \overline{G_1(k)} \cdot \overline{G_2(k)} = \overline{G_2(k)}$. Thus $\beta^{-1}(\overline{G_3(k)}) = \overline{G_2(k)}$.

Consider the homomorphism of abelian groups $\beta_* : A_\Sigma(G_2) \to A_\Sigma(G_3)$. We prove that $\beta_*$ is bijective. Since $H^1(k_v, G_1) = 1$ for each $v \in \Sigma$, we have $\beta(G_2(k_\Sigma)) = G_3(k_\Sigma)$, hence $\beta_*$ is surjective. Since $\beta^{-1}(\overline{G_3(k)}) = \overline{G_2(k)}$, the map $\beta_*$ is injective. Thus $\beta_*$ is bijective. □

**Corollary 5.6.** *Let $k$ and $\Sigma$ be as in* 5.4*, and let*

$$1 \to G_1 \to G_2 \xrightarrow{\beta} G_3 \to 1$$

*be an exact sequence of connected linear $k$-groups. If $G_1$ is a quasi-trivial torus or a unipotent group, then $\beta_* : A_\Sigma(G_2) \to A_\Sigma(G_3)$ is an isomorphism.* □

**Proof.** The corollary follows from Lemma 5.5. We give another proof. We use the fact that $\beta$ admits a rational section.

Since $H^1(K, G_1) = 1$ for any field extension $K/k$, the map $\beta$ admits a rational section. This means that there exist a Zariski open subset $U_3 \subset G_3$ and a regular map $s : U_3 \to U_2$, where $U_2 = \beta^{-1}(U_3)$, such that $\beta|_{U_2} \circ s = \mathrm{id}_{U_3}$ (all defined over $k$).

Let $g_3 \in G_3(k)$. Since $H^1(k, G_1) = 1$, there exists $g_2 \in G_2(k)$ such that $g_3 = \beta(g_2)$. Consider the open set $g_3 U_3$ and define a map $g_{2*}s : g_3 U_3 \to g_2 U_2$ by

$$(g_{2*}s)(x) = g_2 s\bigl(g_3^{-1} x\bigr) \quad (x \in g_3 U_3).$$

Clearly $\beta|_{g_2 U_2} \circ g_{2*}s = \mathrm{id}_{g_3 U_3}$, i.e., $g_{2*}s$ is a rational section of $\beta$.

We prove that $\overline{G_3(k)} \subset \beta(\overline{G_2(k)})$. Let $g_{3\Sigma} \in \overline{G_3(k)} \subset G_3(k_\Sigma)$. Using the fact that $G_3(k)$ is Zariski dense in $G_3$, one can show that there exists $g_3 \in G_3(k)$ such that

$g_{3\Sigma} \in (g_3U_3)(k_\Sigma)$. We shall write $U_3$ instead of $g_3U_3$ and $s$ instead of $g_{2*}s$. Set $g_{2\Sigma} = s(g_{3\Sigma})$. Since $g_{3\Sigma} \in \overline{G_3(k)}$, we have $g_{3\Sigma} \in \overline{U_3(k)}$, and $g_{2\Sigma} \in s(\overline{U_3(k)}) \subset \overline{G_2(k)}$. Thus $\overline{G_3(k)} \subset \beta(\overline{G_2(k)})$.

Next, using the fact that $G_1(k_\Sigma) = \overline{G_1(k)}$ (because $G_1$ is $k$-rational), we can prove that $\beta^{-1}(\overline{G_3(k)}) = \overline{G_2(k)}$. It follows that the map $\beta_* : A_\Sigma(G_2) \to A_\Sigma(G_3)$ is injective.

Since $H^1(k_v, G_1) = 1$ for any $v \in \Sigma$, the map $\beta : G_2(k_\Sigma) \to G_3(k_\Sigma)$ is surjective, hence $\beta_* : A_\Sigma(G_2) \to A_\Sigma(G_3)$ is surjective. Thus $\beta_*$ is bijective. □

**Corollary 5.7.** *Let $k$, $\Sigma$ and $G$ be as in* 5.4. *Then the canonical epimorphism $r : G \to G^{\mathrm{red}}$ induces an isomorphism $r_* : A_\Sigma(G) \to A_\Sigma(G^{\mathrm{red}})$.*

**Proof.** The corollary follows from Corollary 5.6. The second proof can be simplified in this case, using the fact that a Levi decomposition gives a splitting $s : G^{\mathrm{red}} \to G$ of the epimorphism $r$. □

*5.8.* Let $k$ and $\Sigma$ be as in 5.4. For the notation $Ч^1_\Sigma(k, T)$, where $T$ is a $k$-torus, see 0.4. Clearly the functor $T \mapsto Ч^1_\Sigma(k, T)$ satisfies conditions (1)–(3) of Introduction, so we have functors $G \mapsto Ч^1_\Sigma(k, F_G)$ and $X \mapsto Ч^1_\Sigma(k, S_X)$, as in Sections 1 and 2. We wish to construct an isomorphism of functors $\eta_G : A_\Sigma(G) \xrightarrow{\sim} Ч^1_\Sigma(k, F_G)$.

We start from tori.

**Proposition 5.9.** *Let $k$ and $\Sigma$ be as in* 5.4, *and let $T$ be a $k$-torus. Let $0 \to Q \to P \to \mathbf{X}_*(T) \to 0$ be a coflasque resolution of $\mathbf{X}_*(T)$, and let $1 \to F_T \to N \to T \to 1$ be the corresponding flasque resolution of $T$ (i.e., $\mathbf{X}_*(N) = P$, $\mathbf{X}_*(F_T) = Q$).*

*Then the epimorphisms $T(k) \to H^1(k, F_T)$ and $T(k_v) \to H^1(k_v, F_T)$ define a canonical isomorphism $\eta_T : A_\Sigma(T) \to Ч^1_\Sigma(k, F_T)$. This isomorphism is functorial in $T$.*

**Proof.** See [15, Proposition 18], [12, §3.3]. □

We now pass to the case of any connected linear algebraic $k$-group $G$.

**Theorem 5.10.** *Let $k$ be a field of type* (ll) *or* (gl). *In the case* (gl) *we assume that $G$ has no factor of type $E_8$. Let $\Sigma \subset \Omega$ be a finite set of places of $k$. Then the isomorphism of functors of Proposition* 5.9 *can be uniquely extended to an isomorphism of functors $\eta_G : A_\Sigma(G) \to Ч^1_\Sigma(k, F_G)$ from the category of connected linear $k$-groups to the category of abelian groups.*

**Corollary 5.11.** *For $k$, $G$, and $\Sigma$ as in Theorem* 5.10, *if the image of* $\mathrm{Gal}(\bar{k}/k)$ *in* $\mathrm{Aut}\,\pi_1(G)$ *is a metacyclic group, then $A_\Sigma(G) = 1$.*

**Proof.** The corollary follows from Theorem 5.10 and Proposition 1.10. □

**Corollary 5.12.** *Let $k$, $G$, and $\Sigma$ be as in Theorem* 5.10, *then*:

(i) [12, Theorem 4.13(i)] *$A_\Sigma(G)$ is finite*;
(ii) $A_\Sigma(G) \simeq \text{Ч}^1_\Sigma(k, S_G)$;
(iii) *the abelian group $A_\Sigma(G)$ is a stably $k$-birational invariant of $G$.*

**Proof.** By [12, Theorem 3.2], $H^1(k_v, F_G)$ is finite for any $v \in \Omega$, hence $\text{Ч}^1_\Sigma(k, F_G)$ is finite. Now the assertion (i) follows from Theorem 5.10. The assertion (ii) follows from Theorems 5.10 and 3.5. The assertion (iii) follows from (ii) and Proposition 2.14. $\square$

**Corollary 5.13.** *Let $k$, $G$, and $\Sigma$ be as in Theorem* 5.10. *Let $0 \to L_{-1} \to L_0 \to \pi_1(G) \to 0$ be a torsion-free resolution of $\pi_1(G)$. Let $T_{-1}$ and $T_0$ be the $k$-tori such that $\mathbf{X}_*(T_i) = L_i$, $i = -1, 0$. Then $A_\Sigma(G) \simeq \text{Ч}^0_\Sigma(k, T_{-1} \to T_0)$, where*

$$\text{Ч}^0_\Sigma(k, T_{-1} \to T_0) = \operatorname{coker}\left[\mathbb{H}^0(k, T_{-1} \to T_0) \to \prod_{v \in \Sigma} \mathbb{H}^0(k_v, T_{-1} \to T_0)\right],$$

*$\mathbb{H}^0$ denoting the* 0*-dimensional Galois hypercohomology.*

**Proof** (*idea*). Note that $\text{Ч}^0_\Sigma(k, T_{-1} \to T_0)$ does not depend on the resolution. We take a coflasque resolution $0 \to Q \to P \to \pi_1(G) \to 0$ and prove that $\text{Ч}^0_\Sigma(k, F_G \to N) \simeq \text{Ч}^1_\Sigma(k, F_G)$, where $F_G$ and $N$ are the $k$-tori such that $\mathbf{X}_*(F_G) = Q$, $\mathbf{X}_*(N) = P$. $\square$

We prove Theorem 5.10 in the rest of this section. We use the method of Kottwitz. We need two lemmas.

**Lemma 5.14.** *Let $G$ be as in Theorem* 5.10, *and assume that $G$ is reductive and $G^{ss}$ is simply connected. Then the canonical homomorphism $t : G \to G^{tor}$ induces an isomorphism $t_* : A_\Sigma(G) \to A_\Sigma(G^{tor})$.*

**Proof.** We give two proofs.

(1) By [12, Theorem 4.7], $\overline{G^{ss}(k)} = G^{ss}(k_\Sigma)$. Since $k$ is of type (gl) or (ll), $k_v$ is of type (sl) for any $v \in \Omega$ (see [12, end of §1]), and by the results of [12, §1] we have $H^1(k, G^{ss}) = 1$ and $H^1(k_v, G^{ss}) = 1$ for any $v$. The lemma now follows from Lemma 5.5.

(2) Similar to the second proof of Lemma 4.12, but using [12, Theorem 4.13] instead of Corollary 4.6 [12, Theorem 4.12]. $\square$

**Lemma 5.15.** *Let $G$ be as in Lemma* 5.14, *then the canonical homomorphism $t : G \to G^{tor}$ induces an isomorphism $t_* : \text{Ч}^1_\Sigma(k, F_G) \to \text{Ч}^1_\Sigma(k, F_{G^{tor}})$.*

**Proof.** This is an immediate consequence of Corollary 3.8. $\square$

*5.16.* We can now extend the isomorphism of functors $\eta_G : A_\Sigma(G) \to \text{Ч}^1_\Sigma(k, F_G)$ from the category of $k$-tori to the category of reductive $k$-groups $G$ such that $G^{ss}$ is simply connected (and in the case (gl) $G$ has no $E_8$-factor). Namely, we must define an isomorphism $\eta_G : A_\Sigma(G) \to \text{Ш}^1(k, F_G)$ so that the following diagram is commutative:

$$\begin{array}{ccc} A_\Sigma(G) & \xrightarrow{\eta_G} & \text{Ш}^1_\Sigma(k, F_G) \\ \downarrow{\scriptstyle t_*} & & \downarrow{\scriptstyle t_*} \\ A_\Sigma(G^{\mathrm{tor}}) & \xrightarrow{\eta_{G^{\mathrm{tor}}}} & \text{Ш}^1_\Sigma(k, F_{G^{\mathrm{tor}}}) \end{array}$$

By Proposition 5.9, Lemmas 5.14 and 5.15, all the other three arrows in the diagram are isomorphisms. The isomorphism $\eta_G$ is functorial in $G$.

*5.17.* We can now extend $\eta_G$ to the category of all connected reductive $k$-groups $G$ such that in the case (gl) $G$ has no factor of type $E_8$. Choose a $z$-extension $H \xrightarrow{\beta} G$. We must define $\eta_G$ so that the following diagram is commutative:

$$\begin{array}{ccc} A_\Sigma(H) & \xrightarrow{\eta_H} & \text{Ш}^1_\Sigma(k, F_H) \\ \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_*} \\ A_\Sigma(G) & \xrightarrow{\eta_G} & \text{Ш}^1_\Sigma(k, F_G) \end{array}$$

The left vertical arrow is an isomorphism by Corollary 5.6, the right vertical arrow is an isomorphism by Lemma 3.15, and the top horizontal arrow is an isomorphism by 5.16, and $\eta_G$ is thus defined. Using Lemma 3.16, one can easily check that $\eta_G$ does not depend on the choice of a $z$-extension $H \xrightarrow{\beta} G$ and is functorial in $G$.

*5.18.* We can now extend $\eta_G$ to the category of all connected linear algebraic $k$-groups $G$ such that in the case (gl) $G^{\mathrm{red}}$ has no factor of type $E_8$. We must define $\eta_G$ so that the following diagram is commutative:

$$\begin{array}{ccc} A_\Sigma(G) & \xrightarrow{\eta_G} & \text{Ш}^1_\Sigma(k, F_G) \\ \downarrow{\scriptstyle r_*} & & \downarrow{\scriptstyle r_*} \\ A_\Sigma\big(G^{\mathrm{red}}\big) & \xrightarrow{\eta_{G^{\mathrm{red}}}} & \text{Ш}^1_\Sigma(k, F_{G^{\mathrm{red}}}) \end{array}$$

The left vertical arrow is an isomorphism by Corollary 5.7, the right vertical arrow is an isomorphism by Lemma 3.18, and the bottom horizontal arrow is an isomorphism by 5.17, and $\eta_G$ is thus defined. The isomorphism $\eta_G$ is functorial in $G$.

Theorem 5.10 is completely proved.

## 6. Galois cohomology

In this section for a connected linear algebraic group $G$ over a field $k$ of one of types (gl), (sl), (ll) we compute $H^1(k,G)$ in terms of $\pi_1(G)$.

*6.1.* Let $k$ be a field of characteristic 0. Let $G$ be a connected linear $k$-group. Let

$$0 \to L_{-1} \to L_0 \to \pi_1(G) \to 0$$

be any (not necessarily coflasque) resolution of $\pi_1(G)$, where $L_{-1}$ and $L_0$ are finitely generated torsion-free $\Gamma$-modules. Let $T_{-1}$, $T_0$ be the $k$-tori such that $\mathbf{X}_*(T_{-1}) = L_{-1}$, $\mathbf{X}_*(T_0) = L_0$. Set $H^n_{\mathrm{ab}}(k,G) = \mathbb{H}^n(k, T_{-1} \to T_0)$ (hypercohomology) for $n \geqslant -1$. Then $H^n_{\mathrm{ab}}(k,G)$ does not depend on the choice of a resolution (see [6, 2.6.2]), and it is a functor of $G$.

In the case when $G$ is a reductive group, there exist abelianization maps

$$\mathrm{ab}^n : H^n(k,G) \to H^n_{\mathrm{ab}}(k,G) \quad (n = 0,1)$$

constructed in [6, Section 3]. When $G$ is any connected linear $k$-group, we consider the canonical map $r : G \to G^{\mathrm{red}}$ and define $\mathrm{ab}^n$ as the composed maps

$$\mathrm{ab}^n : H^n(k,G) \xrightarrow{r_*} H^n\big(k, G^{\mathrm{red}}\big) \to H^n_{\mathrm{ab}}(k,G) \quad (n = 0,1).$$

We wish to prove that the map $\mathrm{ab}^1$ is bijective over certain fields. Since the map $r_* : H^1(k,G) \to H^1(k, G^{\mathrm{red}})$ is bijective (cf. [39, Lemme 1.13]), we may and shall assume in the rest of this section that $G$ is reductive.

There is a canonical exact sequence

$$H^1\big(k, G^{\mathrm{sc}}\big) \to H^1(k,G) \xrightarrow{\mathrm{ab}^1} H^1_{\mathrm{ab}}(k,G).$$

Moreover we can describe the fibers of the map $\mathrm{ab}^1$. Let $\psi \in Z^1(k,G)$, and let $\xi$ denote the cohomology class of the cocycle $\psi$. Then by [6, 3.17(ii)]

$$\big(\mathrm{ab}^1\big)^{-1}\big(\mathrm{ab}^1(\xi)\big) \simeq H^0_{\mathrm{ab}}(k,G)\backslash H^1\big(k, {}_\psi G^{\mathrm{sc}}\big) \tag{6.1}$$

where the abelian group $H^0_{\mathrm{ab}}(k,G)$ acts on the set $H^1(k, {}_\psi G^{\mathrm{sc}})$ and ${}_\psi G^{\mathrm{sc}}$ denotes the twisted form of $G^{\mathrm{sc}}$ defined by $\psi$.

**Proposition 6.2.** *Let $G$ be a connected linear algebraic group over a field $k$ of characteristic 0 such that $H^1(k, {}_\psi G^{\mathrm{sc}}) = 1$ for any twisted form ${}_\psi G^{\mathrm{sc}}$ of $G^{\mathrm{sc}}$. Then the map $\mathrm{ab}^1 : H^1(k,G) \to H^1_{\mathrm{ab}}(k,G)$ is injective.*

**Proof.** The proposition follows from (6.1). □

We wish to describe the image of the map $\mathrm{ab}^1$ in terms of the second non-abelian Galois cohomology.

*6.3.* A crossed module of $k$-groups is a homomorphism of $k$-groups $\alpha : H \to G$ together with an action of $G$ on $H$ satisfying certain conditions (see, e.g. [6, Definition 3.2.1]). In [6, Section 3] Galois hypercohomology $\mathbb{H}^n(k, H \to G)$ $(n = 0, 1)$ with coefficients in a crossed module was defined (Breen [8] defined hypercohomology with coefficients in a crossed module in a very general setting). A 1-cocycle $(u, \psi) \in Z^1(k, H \to G)$ is a pair of continuous mappings

$$u : \Gamma \times \Gamma \to H(\bar{k}), \qquad \psi : \Gamma \to G(\bar{k}),$$

such that

$$\psi_{\sigma\tau} = \alpha(u_{\sigma,\tau}) \cdot \psi_\sigma \cdot {}^\sigma\psi_\tau,$$
$$u_{\sigma,\tau\upsilon} \cdot {}^{\psi_\sigma \sigma}u_{\tau,\upsilon} = u_{\sigma\tau,\upsilon} \cdot u_{\sigma,\tau}$$

where $\sigma, \tau, \upsilon \in \Gamma$. Two cocycles $(u, \psi)$ and $(u', \psi')$ are called cohomologous, if there exist a continuous map $a : \Gamma \to H(\bar{k})$ and an element $g \in G(\bar{k})$ such that

$$\psi'_\sigma = g^{-1} \cdot \alpha(a_\sigma) \cdot \psi_\sigma \cdot {}^\sigma g,$$
$$u'_{\sigma,\tau} = {}^{g^{-1}}\big(a_{\sigma\tau} \cdot u_{\sigma,\tau} \cdot {}^{\psi_\sigma \sigma}a_\tau^{-1} \cdot a_\sigma^{-1}\big).$$

The first Galois hypercohomology set is denoted by $\mathbb{H}^1(k, H \to G)$. The short exact sequence

$$1 \to (1 \to G) \to (H \to G) \to (H \to 1) \to 1$$

gives rise to an exact sequence

$$H^1(k, H) \to H^1(k, G) \xrightarrow{\gamma} \mathbb{H}^1(k, H \to G), \tag{6.2}$$

cf. [6, 3.4.3(i)].

*6.4. Construction*

Let $(u, \psi) \in Z^1(k, H \to G)$. We wish to construct a 2-cohomology class $\Delta(u, \psi)$ with coefficients in $\overline{H}$. For every $\sigma \in \Gamma$ we define a $\sigma$-semialgebraic automorphism of $\overline{H}$

$$f_\sigma \in \operatorname{SAut} \overline{H}, \quad f_\sigma(h) = {}^{\psi_\sigma \sigma}h \quad \text{for } h \in H$$

(see [5, 1.1] for the definition of semialgebraic automorphisms). Then

$$f_{\sigma\tau} = \operatorname{int}(u_{\sigma,\tau}) \circ f_\sigma \circ f_\tau,$$
$$u_{\sigma,\tau\upsilon} \cdot f_\sigma(u_{\tau,\upsilon}) = u_{\sigma\tau,\upsilon} \cdot u_{\sigma,\tau}.$$

Thus $(u, f)$ is a non-abelian 2-cocycle in the sense of [5, 1.5]. Let

$$\kappa_\sigma = f_\sigma \ \big(\operatorname{mod} \operatorname{Aut} \overline{H}\big) \in \operatorname{SOut} G$$

(see [5, 1.2] for the notation). We obtain a homomorphism $\kappa : \Gamma \to \operatorname{SOut} \overline{H}$. Then $(u, f) \in Z^2(k, \overline{H}, \kappa)$. Set $\Delta(u, \psi) = \operatorname{Cl}(u, f) \in H^2(k, \overline{H}, \kappa)$, where Cl denotes the cohomology class.

**Proposition 6.5.** *A hypercohomology class* $\operatorname{Cl}(u, \psi) \in \mathbb{H}^1(k, H \to G)$ *comes from* $H^1(k, G)$ *if and only if* $\Delta(u, \psi)$ *is a neutral element of* $H^2(k, \overline{H}, \kappa)$.

**Proof.** If $\operatorname{Cl}(u, \psi)$ comes from $H^1(k, G)$, then there exist $a : \Gamma \to H(\bar{k})$ and $g \in G(\bar{k})$ such that

$$^{g^{-1}}\big(a_{\sigma\tau} \cdot u_{\sigma,\tau} \cdot {}^{\psi_\sigma \sigma} a_\tau^{-1} \cdot a_\sigma^{-1}\big) = 1.$$

Then

$$a_{\sigma\tau} \cdot u_{\sigma,\tau} \cdot f_\sigma(a_\tau)^{-1} \cdot a_\sigma^{-1} = 1,$$

hence $\Delta(u, \psi) = \operatorname{Cl}(u, f)$ is neutral, cf. [5, 1.6, 1.5].

Conversely, if $\Delta(u, \psi) = \operatorname{Cl}(u, f)$ is neutral, then there exists $a : \Gamma \to H(\bar{k})$ such that

$$a_{\sigma\tau} \cdot u_{\sigma,\tau} \cdot f_\sigma(a_\tau)^{-1} \cdot a_\sigma^{-1} = 1.$$

Then

$$a_{\sigma\tau} \cdot u_{\sigma,\tau} \cdot {}^{\psi_\sigma \sigma} a_\tau^{-1} \cdot a_\sigma^{-1} = 1,$$

hence $\operatorname{Cl}(u, \psi)$ comes from $H^1(k, G)$. □

**Proposition 6.6.** *Let $G$ be a connected linear algebraic group over a field $k$ of characteristic* 0. *Assume that for any $k$-kernel of the form $L = (\overline{G}^{\mathrm{sc}}, \kappa)$ all the elements of $H^2(k, L)$ are neutral. Then the map* $\mathrm{ab}^1 : H^1(k, G) \to H^1_{\mathrm{ab}}(k, G)$ *is surjective.*

**Proof.** First we describe the construction of the map $\mathrm{ab}^1$ in [6, Section 3]. We assume that $G$ is reductive. Consider the map $\rho : G^{\mathrm{sc}} \to G$. Let $T \subset G$ be a maximal torus. Set $T^{\mathrm{sc}} = \rho^{-1}(T) \subset G^{\mathrm{sc}}$. Then $H^1_{\mathrm{ab}}(k, G) = \mathbb{H}^1(k, T^{\mathrm{sc}} \to T)$. The embedding

$$\lambda : \big(T^{\mathrm{sc}} \to T\big) \to \big(G^{\mathrm{sc}} \to G\big)$$

is a quasi-isomorphism of crossed modules, hence

$$\lambda_* : \mathbb{H}^1\big(k, T^{\mathrm{sc}} \to T\big) \to \mathbb{H}^1\big(k, G^{\mathrm{sc}} \to G\big)$$

is a bijection, cf. [6, Theorem 3.5.3]. We have a canonical map of (6.2) $\gamma : H^1(k, G) \to \mathbb{H}^1(k, G^{sc} \to G)$. By definition

$$\mathrm{ab}^1 = \lambda_*^{-1} \circ \gamma : H^1(k, G) \to H^1_{\mathrm{ab}}(k, G).$$

It suffices to prove that $\gamma$ is surjective. But this follows from Proposition 6.5, because by assumption all the elements of $H^2(k, \overline{G}^{sc}, \kappa)$ are neutral for any $\kappa$. □

**Theorem 6.7.** *Let $k$ be a field of one of types* (gl), (sl), (ll) *and let $G$ be a connected linear $k$-group. In the case* (gl) *assume that $G$ has no factors of type $E_8$. Then the abelianization map* $\mathrm{ab}^1 : H^1(k, G) \to H^1_{\mathrm{ab}}(k, G)$ *is bijective.*

**Proof.** By [12, §1] we have $H^1(k, {}_\psi G^{sc}) = 1$ for any twisted form ${}_\psi G^{sc}$ of $G^{sc}$, and by [12, Remark after Proposition 5.3] all the elements of $H^2(k, \overline{G}^{sc}, \kappa)$ are neutral for any $\kappa$. The theorem now follows from Propositions 6.2 and 6.6. □

**Remark 6.8.** (i) In the case when $k$ is a non-archimedean local field, Theorem 6.7 was proved in [6, Corollary 5.4.1]. The surjectivity of $\mathrm{ab}^1$ was proved by a different method. This result is essentially due to Kottwitz [27, Proposition 6.4]. Note that the method of the present paper also works. The assertion that all the elements of $H^2(k, L)$ are neutral when $L = (\overline{G}, \kappa)$ with $\overline{G}$ semisimple simply connected, was proved in [19, Theorem 1.1], see also [5, Corollary 5.6].

(ii) Theorem 6.7 also holds when $k$ is a totally imaginary number field. The injectivity of $\mathrm{ab}^1$ follows from the Hasse principle for simply connected $k$-groups (Kneser–Harder–Chernousov). The surjectivity holds for any number field $k$, see Theorem 8.14 below.

## 7. Hasse principle

In this section we consider the case where $k$ is a field of type (ll). For a connected linear algebraic $k$-group $G$ one can define the Tate–Shafarevich kernel

$$Ш^1(k, G) = \ker\left[H^1(k, G) \to \prod_{v \in \Omega} H^1(k_v, G)\right]$$

which is a finite set [12, Theorem 5.1]. Here $\Omega$ is the associated set of places, see [12, §1]. We compute $Ш^1(k, G)$ in terms of $\pi_1(G)$ and in terms of $S_G$, and prove that the cardinality of $Ш^1(k, G)$ is a stably $k$-birational invariant of $G$.

We define $Ш^1_{\mathrm{ab}}(k, G) = \ker[H^1_{\mathrm{ab}}(k, G) \to \prod_{v \in \Omega} H^1_{\mathrm{ab}}(k_v, G)]$.

**Theorem 7.1.** *Let $k$ be a field of type* (ll)*, and $G$ a connected linear $k$-group. Then the abelianization map* $\mathrm{ab}^1 : H^1(k, G) \to H^1_{\mathrm{ab}}(k, G)$ *induces a bijection* $Ш^1(k, G) \xrightarrow{\sim} Ш^1_{\mathrm{ab}}(k, G)$.

**Proof.** Since $k$ is of type (ll), $k_v$ is of type (sl) for any $v \in \Omega$. By Theorem 6.7, in the commutative diagram

$$\begin{array}{ccc} H^1(k,G) & \longrightarrow & H^1_{\mathrm{ab}}(k,G) \\ \downarrow & & \downarrow \\ \prod_{v\in\Omega} H^1(k_v,G) & \longrightarrow & \prod_{v\in\Omega} H^1_{\mathrm{ab}}(k_v,G) \end{array}$$

both horizontal arrows are bijections, and our theorem follows. □

*7.2.* We now take a *coflasque* resolution

$$0 \to Q \to P \to \pi_1(G) \to 0$$

where $P$ is a permutation module and $Q$ is a coflasque module. Let $F$ and $N$ be the $k$-tori such that $\mathbf{X}_*(F) = Q$, $\mathbf{X}_*(N) = P$. The exact sequence of complexes of tori

$$1 \to (1 \to N) \to (F \to N) \to (F \to 1) \to 1$$

induces an exact sequence

$$0 = H^1(k,N) \to \mathbb{H}^1(k, F \to N) \xrightarrow{\Delta} H^2(k,F) \to H^2(k,N). \tag{7.1}$$

Clearly we have $\mathbb{H}^1(k, F \to N) = H^1_{\mathrm{ab}}(k,G)$.

**Lemma 7.3.** *For any quasi-trivial torus $N'$ over a field $k$ of type* (ll) *we have* $Ш^2(k,N') = 0$.

**Proof.** The lemma follows from [12, Theorem 1.6] (proved in [14, Corollary 1.10]) and Shapiro's lemma. □

**Proposition 7.4.** *The map $\Delta : H^1_{\mathrm{ab}}(k,G) \to H^2(k,F)$ of* (7.1) *induces an isomorphism* $Ш^1_{\mathrm{ab}}(k,G) \to Ш^2(k,F)$.

**Proof.** We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_{\mathrm{ab}}(k,G) & \xrightarrow{\Delta} & H^2(k,F) & \longrightarrow & H^2(k,N) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \prod_v H^1_{\mathrm{ab}}(k,G) & \longrightarrow & \prod_v H^2(k,F) & \longrightarrow & \prod_v H^2(k,N) \end{array}$$

By Lemma 7.3, $Ш^2(k,N) = 0$. Now the proposition can be proved by easy diagram chasing. □

*7.5.* Over a field $k$ of type (ll) the functor $T \mapsto Ш^2(k, T)$ on the category of $k$-tori satisfies the properties (1)–(3) of Introduction (by Lemma 7.3), so we have functors $G \mapsto Ш^2(k, F_G)$ and $X \mapsto Ш^2(k, S_X)$ as in Sections 1 and 2.

**Theorem 7.6.** *Let $G$ be a connected linear algebraic group over a field $k$ of type* (ll). *Then*

(i) *there exist canonical isomorphisms $Ш^1_{ab}(k, G) \simeq Ш^2(k, F_G) \simeq Ш^2(k, S_G)$ and a canonical bijection $Ш^1(k, G) \simeq Ш^2(k, S_G)$;*
(ii) *the group $Ш^1_{ab}(k, G)$ and the set $Ш^1(k, G)$ are stably $k$-birational invariants of a connected linear $k$-group $G$.*

**Proof.** (i) In 7.2 we may write $F_G$ instead of $F$, then by Proposition 7.4 we have a canonical isomorphism $Ш^1_{ab}(k, G) \simeq Ш^2(k, F_G)$. By Theorem 3.5 there exists a canonical isomorphism $Ш^2(k, F_G) \simeq Ш^2(k, S_G)$. From these isomorphisms and the bijection of Theorem 7.1 we obtain a canonical bijection $Ш^1(k, G) \simeq Ш^2(k, S_G)$.

(ii) The assertion follows from (i) and Proposition 2.14. □

**Corollary 7.7.** *If $G$ is a stably $k$-rational group, then $Ш^1(k, G) = 1$.*

**Corollary 7.8.** *If the image of* $\mathrm{Gal}(\bar{k}/k)$ *in* $\mathrm{Aut}\,\pi_1(G)$ *is metacyclic, then* $Ш^1(k, G) = 1$.

**Theorem 7.9.** *Let $k$ be a field of type* (ll), *$G$ a connected linear $k$-group. If there exists a $k$-variety $X$ such that $G \times X$ is $k$-rational, then $Ш^1(k, G) = 1$.*

**Proof.** Assume that $G \times X$ is $k$-rational. We argue as in [12, proof of Theorem 5.2]. By [17, Proposition 2.A.1, p. 461] there exists a $\Gamma$-module $M$ such that $\mathrm{Pic}\,\overline{V}_G \oplus M$ is a permutation module. Let $T$ be the torus such that $\mathbf{X}^*(T) = M$, then we obtain that $S_G \times T$ is a quasi-trivial torus. By Lemma 7.3 we have $Ш^2(k, S_G) \times Ш^2(k, T) = 0$, whence $Ш^2(k, S_G) = 0$. By Theorem 7.6(i), $Ш^1(k, G) = 1$. □

**Remark 7.10.** (i) Theorem 7.9 generalizes [12, Theorem 5.2(b)(ii)].

(ii) Corollary 7.7 proves a conjecture of [12, Remark (i) after Theorem 5.2].

(iii) The canonical bijection $Ш^1(k, G) \to Ш^1_{ab}(k, G)$ of Theorem 7.1 defines a canonical and functorial structure of an abelian group on $Ш^1(k, G)$, and this abelian group is a stably $k$-birational invariant of $G$.

(iv) The results of this section also hold when $k$ is a totally imaginary number field.

## 8. Case of a number field

All the results of Sections 4–7 hold when $k$ is a totally imaginary number field. In this section we suppose that $k$ is *any* number field, not necessarily totally imaginary. We prove analogues of the results of Sections 4, 5, 7.

We start from $R$-equivalence.

**Proposition 8.1.** *Let $G$ be a semisimple simply connected group over a number field $k$. Assume that $G$ has no anisotropic factors of type $E_6$. Then $G(k)/R = 1$.*

**Proof.** First assume that $G$ is isotropic. Let $S$ be a maximal split torus of $G$, and $Z_G(S)$ the centralizer of $S$ in $G$. Then $Z_G(S)$ is connected reductive, and $Z_G(S)^{\mathrm{ss}}$ is a $k$-anisotropic simply connected semisimple group. By Appendix by P. Gille (Corollary) the map $Z_G(S)^{\mathrm{ss}}(k)/R \to G(k)/R$ is surjective. This reduces the proposition to the case of an anisotropic group.

The anisotropic groups were treated, case by case, by many people, see [10] and [38, Chapter 9]. A difficult case of ${}^{3,6}D_4$ was treated in [10]. □

**Proposition 8.2.** *Let $G$ be a connected reductive group over a number field $k$ without anisotropic factors of type $E_6$. Assume that $G$ admits a special covering*

$$1 \to \mu \to G_0 \times N_0 \to G \to 1$$

*where $G_0$ is simply connected and $N_0$ is a quasi-trivial torus. Let*

$$1 \to \mu \to F \to N \to 1$$

*be a flasque resolution of $\mu$. Then Galois cohomology exact sequences induce an isomorphism of groups $G(k)/R \simeq H^1(k, F)$.*

**Proof.** By [21, Theorem III.3.1] Galois cohomology exact sequences induce an exact sequence

$$G_0(k)/R \times N_0(k)/R \to G(k)/R \to H^1(k, F) \to 1.$$

We have $N_0(k)/R = 1$ because $N_0$ is $k$-rational, and $G_0(k)/R = 1$ by Proposition 8.1. Thus we obtain an isomorphism $G(k)/R \simeq H^1(k, F)$. □

**Lemma 8.3.** *Let $G$ be a connected reductive group over a number field $k$ without anisotropic factors of type $E_6$. Assume that $G^{\mathrm{ss}}$ is simply connected. Then the canonical map $t: G \to G^{\mathrm{tor}}$ induces an isomorphism $t_*: G(k)/R \to G^{\mathrm{tor}}(k)/R$.*

**Proof.** We outline two proofs.

(1) The lemma follows from Theorem 1(a) of the Appendix by P. Gille, and Proposition 8.1.

(2) Similar to the second proof of Lemma 4.12. Instead of Corollary 4.6 of Theorem 4.5 we use Proposition 8.2. □

**Theorem 8.4.** *Let $k$ be a number field, and consider the category of connected linear $k$-groups $G$ such that $G$ has no anisotropic factors of type $E_6$. Then the isomorphism of functors $\delta_{T*}$ of Theorem 4.3 extends uniquely to an isomorphism of functors $\theta_G: G(k)/R \to H^1(k, F_G)$.*

**Proof.** Similar to that of Theorem 4.8. □

**Corollary 8.5.** *Let $k$ and $G$ be as in Theorem* 8.4. *Then*

(i) *there is a canonical isomorphism $G(k)/R \simeq H^1(k, S_G)$;*
(ii) *the group $G(k)/R$ is a stably $k$-birational invariant of $G$.*

*8.6.* We pass to weak approximation over a number field $k$. Let $\Omega$ be the set of all places of $k$. We write $\Omega_\infty$ (respectively $\Omega_f$) for the set of all infinite (respectively finite) places of $k$.

**Lemma 8.7** (stated in [39, Proposition 3.3]). *Let $G$ be a connected linear algebraic group over a number field $k$. Let $\Sigma$ be a finite set of places of $k$. Then the canonical epimorphism $r : G \to G^{\mathrm{red}}$ induces an isomorphism $r_* : A_\Sigma(G) \to A_\Sigma(G^{\mathrm{red}})$.*

**Proof.** Similar to that of Corollary 5.7, second proof. □

**Lemma 8.8.** *Let $G$ and $k$ be as in Lemma* 8.7, *and let $\Sigma' \supset \Sigma$, where $\Sigma' - \Sigma \subset \Omega_\infty$. Then the projection $G(k_{\Sigma'}) \to G(k_\Sigma)$ induces an isomorphism $A_{\Sigma'}(G) \to A_\Sigma(G)$.*

**Proof.** The lemma follows from results of Sansuc [39]. Indeed, by Lemma 8.7 we may assume that $G$ is reductive. By [39, Lemme 1.10] we may assume that $G$ admits a special covering

$$1 \to \mu \to G_0 \times N_0 \to G \to 1$$

where $G_0$ is a simply connected group and $N_0$ is a quasi-trivial torus. By [39, (3.3.1)] there is a canonical and functorial in $\Sigma$ isomorphism $A_\Sigma(G) \to Ч^1_\Sigma(k, \mu)$. By [39, formula after Lemma 1.5], the canonical map $Ч^1_{\Sigma'}(k, \mu) \to Ч^1_\Sigma(k, \mu)$ is an isomorphism. Hence the map $A_{\Sigma'}(G) \to A_\Sigma(G)$ is an isomorphism. □

**Lemma 8.9.** *Let*

$$1 \to G_1 \to G_2 \xrightarrow{\beta} G_3 \to 1$$

*be an exact sequence of connected linear algebraic groups over a number field $k$. Assume that $H^1(k_v, G_1) = 1$ for all $v \in \Omega_f$ and that $Ш^1(k, G_1) = 1$. Let $\Sigma \subset \Omega$ be a finite set. Assume that $A_\Sigma(G_1) = 1$. Then the map $\beta_* : A_\Sigma(G_2) \to A_\Sigma(G_3)$ is an isomorphism.*

**Proof.** We prove that $\beta_*$ is surjective. By Lemma 8.8 we may assume that $\Sigma \subset \Omega_f$. Then $H^1(k_v, G_1) = 1$ for all $v \in \Sigma$, hence the map $\beta : G_2(k_\Sigma) \to G_3(k_\Sigma)$ is surjective, thus $\beta_*$ is surjective.

We prove that $\beta_*$ is injective. We use an idea of [36, proof of Lemma 3.8]. By Lemma 8.8 we may assume that $\Sigma \supset \Omega_\infty$.

First we prove that $\beta(G_2(k_\Sigma)) \cap \overline{G_3(k)} \subset \beta(\overline{G_2(k)})$. Let $g_{2\Sigma} \in G_2(k_\Sigma)$ and assume that $\beta(g_{2\Sigma}) \in \overline{G_3(k)}$. By Lemma 5.2 the subgroup $\overline{G_2(k)}$ is open in $G_2(k_\Sigma)$, and the map $\beta : G_2(k_\Sigma) \to G_3(k_\Sigma)$ is open, hence the group $U_3 := \beta(\overline{G_2(k)})$ is open in $G_3(k_\Sigma)$. The open set $\beta(g_{2\Sigma})U_3$ is an open neighborhood of $\beta(g_{2\Sigma})$. Since $\beta(g_{2\Sigma}) \in \overline{G_3(k)}$, there exists $g_{3k} \in \beta(g_{2\Sigma})U_3 \cap G_3(k)$. Then $g_{3k} = \beta(g_{2\Sigma})\beta(\bar{g}_2)$, where $\bar{g}_2 \in \overline{G_2(k)}$. Thus $g_{3k} = \beta(g_{2\Sigma}\bar{g}_2)$ where $g_{2\Sigma}\bar{g}_2 \in G_2(k_\Sigma)$. Since $\Sigma \supset \Omega_\infty$, we see that $g_{3k}$ lifts to $G_2(k_v)$ for every $v \in \Omega_\infty$, and by assumptions $g_{3k}$ lifts to $G_2(k)$. Thus $g_{3k} = \beta(g_{2k})$ for some $g_{2k} \in G_2(k)$. We obtain that $\beta(g_{2\Sigma}) = \beta(g_{2k}\bar{g}_2^{-1})$ where $g_{2k}\bar{g}_2^{-1} \in G_2(k)\overline{G_2(k)} = \overline{G_2(k)}$. Thus $\beta(g_{2\Sigma}) \in \beta(\overline{G_2(k)})$, which was to be proved.

Then we prove that $\beta^{-1}(\overline{G_3(k)}) \subset \overline{G_2(k)}$. The proof is similar to the argument in the proof of Lemma 5.5 (we use the assumption that $A_\Sigma(G_1) = 1$). From the inclusion $\beta^{-1}(\overline{G_3(k)}) \subset \overline{G_2(k)}$ it follows immediately that the map $\beta_* : A_\Sigma(G_2) \to A_\Sigma(G_3)$ is injective. Thus $\beta_*$ is bijective. □

**Corollary 8.10.** *Let*

$$1 \to G_1 \to G_2 \xrightarrow{\beta} G_3 \to 1$$

*be an exact sequence of connected linear algebraic groups over a number field $k$. Let $\Sigma \subset \Omega$ be a finite set. Assume that $G_1$ is a quasi-trivial $k$-torus or a unipotent group. Then the map $\beta_* : A_\Sigma(G_2) \to A_\Sigma(G_3)$ is an isomorphism.*

**Proof.** The corollary follows from Lemma 8.9. For another proof see Corollary 5.6, second proof. □

**Theorem 8.11.** *Let $k$ be a number field, $\Sigma \subset \Omega$ a finite set of places. Then the isomorphism of functors $\eta_T : A_\Sigma(T) \to Ч^1_\Sigma(k, F_T)$ of Proposition* 5.9 *can be uniquely extended to an isomorphism of functors $\eta_G : A_\Sigma(G) \to Ч^1_\Sigma(k, F_G)$ from the category of connected linear $k$-groups to the category of abelian groups.*

**Proof.** Similar to that of Theorem 5.10. We use Lemma 8.9, Corollary 8.10, and Lemma 8.7. □

**Corollary 8.12.** *Let $k$, $\Sigma$, and $G$ be as in Theorem* 8.11. *Then*:

(i) $A_\Sigma(G) \simeq Ч^1_\Sigma(k, S_G)$;
(ii) [37, Theorem 2.1(3)] *the abelian group $A_\Sigma(G)$ is a stably $k$-birational invariant of $G$*;
(iii) $A_\Sigma(G) \simeq Ч^0_\Sigma(k, T_{-1} \to T_0)$ *with the notation of Corollary* 5.13.

**Remark 8.13.** In the proof of Theorem 8.11 we actually proved that $A_\Sigma(G) = A_\Sigma(T)$, where $T = H^{\mathrm{tor}}$ and $H$ is a $z$-extension of $G^{\mathrm{red}}$. This result was earlier proved in [36, Lemma 3.8].

Now we pass to Galois cohomology and Hasse principle.

**Theorem 8.14.** *Let $G$ be a connected linear algebraic group over a number field $k$. Then the map* $\mathrm{ab}^1 : H^1(k, G) \to H^1_{\mathrm{ab}}(k, G)$ *is surjective.*

**Proof.** By [39, Lemme 1.13] the map $r_* : H^1(k, G) \to H^1(k, G^{\mathrm{red}})$ is bijective, hence we may assume that $G$ is reductive. In this case the assertion was proved in [6, Theorem 5.7]. We give here another proof (assuming that $G$ is reductive). By Douai's theorem [20, Theorem 5.1], see also [5, Corollary 5.6], for any $k$-kernel of the type $L = (\overline{G}^{\mathrm{sc}}, \kappa)$, all the elements of $H^2(k, L)$ are neutral. By Proposition 6.6 the map $\mathrm{ab}^1$ is surjective. □

Note that over a number field the map $\mathrm{ab}^1$ can be non-injective.

We define $Ш^1_{\mathrm{ab}}(k, G)$ as in Section 7.

**Theorem 8.15** [6, Theorem 5.13]. *Let $G$ be a connected linear algebraic group over a number field $k$. Then the map* $\mathrm{ab}^1 : H^1(k, G) \to H^1_{\mathrm{ab}}(k, G)$ *induces a bijection* $Ш^1(k, G) \to Ш^1_{\mathrm{ab}}(k, G)$.

When $k$ is a number field, we have $Ш^2(k, N) = 0$ for any quasi-trivial $k$-torus $N$. Hence the functor $\mathcal{H}(T) = Ш^2(k, T)$ satisfies conditions (1)–(3) of Introduction. It follows that we can define functors $Ш^2(k, F_G)$ and $Ш^2(k, S_X)$ as in Sections 1 and 2.

**Theorem 8.16.** *Let $G$ be a connected linear algebraic group over a number field $k$. Then*

(i) *there exist canonical isomorphisms* $Ш^1_{\mathrm{ab}}(k, G) \simeq Ш^2(k, F_G) \simeq Ш^2(k, S_G)$ *and a canonical bijection* $Ш^1(k, G) \simeq Ш^2(k, S_G)$;
(ii) *the group* $Ш^1_{\mathrm{ab}}(k, G)$ *and the set* $Ш^1(k, G)$ *are stably $k$-birational invariants of a connected linear $k$-group $G$.*

**Proof.** Similar to that of Theorem 7.6. □

**Remark 8.17.** (i) The set $Ш^1(k, G)$ over a number field $k$ was computed in terms of $\pi_1(G)$ in [27, 4.2].

(ii) Sansuc [39, (9.5.1)] proved by a different method that there exists a bijection $Ш^1(k, G) \simeq Ш^2(k, S_G)$, and he deduced that the set $Ш^1(k, G)$ is a stably $k$-birational invariant of $G$.

**Corollary 8.18.** *Let $G$ be a connected linear algebraic group over a number field $k$. If the image of* $\mathrm{Gal}(\bar{k}/k)$ *in* $\mathrm{Aut}\,\pi_1(G)$ *is metacyclic, then $G(k)/R = 1$, $A_\Sigma(G) = 1$ for any finite $\Sigma$, and* $Ш^1(k, G) = 1$.

**Proof.** We use Proposition 1.10. □

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## Appendix[4]

In the papers [21,22] we passed from semisimple groups to reductive groups using Sansuc's special isogenies. Some results can be formulated and proved in the setting of this paper, which roughly speaking replaces isogenies by morphisms $G \to T$ from reductive groups to tori and uses $z$-extensions. We give here other formulations of Theorem III.3.1 of [21], Theorem 6 of [22] and Theorem 4.12 of [12] on $R$-equivalence.

Let $k$ be a field. Let $F$ be a covariant functor from commutative $k$-algebras to sets. Let $O$ denote the semilocal ring of the polynomial algebra $k[t]$ at the points $t = 0$ and $t = 1$. Let us say that two elements $a, b \in F(k)$ are elementarily related if there exists $\xi \in F(O)$ such that $\xi(0) = a$ and $\xi(1) = b$. By definition, $R$-equivalence on $F(k)$ is the equivalence relation generated by the previous elementary relation. Thus two elements $a, b \in F(k)$ are $R$-equivalent if and only if there exists a finite set of elements $x_0, \ldots, x_{n+1} \in F(k)$, with $x_0 = a$ and $x_{n+1} = b$, such that $x_i$ is elementarily related to $x_{i+1}$ for $0 \leqslant i \leqslant n$. We let $F(k)/R$ denote the quotient set for this equivalence relation. For any field $K$ containing $k$, we define a similar equivalence relation on $F(K)$ by using the semilocal ring of $K[t]$ at the points $t = 0$ and $t = 1$. There is a natural, functorial map $F(k)/R \to F(K)/R$. If $F$ goes from commutative $k$-algebras to groups, the class of $R$-equivalence of $e$, denoted by $RF(k)$, is a normal subgroup of $F(k)$, and $F(k)/R = F(k)/RF(k)$; any element of $RF(k)$ is elementarily related to 1 (cf. [21, Lemme II.1.1]). If $F = F_X$ is the functor associated to a $k$-variety $X$, namely $F_X(A) = X(A)$, we get the $R$-equivalence on $X(k)$, as defined by Manin.

**Lemma 1** (see [21, Proposition II.1.3]). *Let* $1 \to \widetilde{G} \xrightarrow{i} G \xrightarrow{\lambda} T \to 1$ *be an exact sequence of reductive $k$-groups where $T/k$ is a $k$-torus. We denote by $C_\lambda$ the functor $A \mapsto \lambda(G(A)) \subset T(A)$ from commutative $k$-algebras to groups. Then $\lambda(RG(k)) = RC_\lambda(k)$ and we have a natural exact sequence of groups*

$$\widetilde{G}(k)/R \to G(k)/R \to C_\lambda(k)/R \to 1.$$

**Proof.** We have to prove that $RC_\lambda(k) \subset \lambda(RG(k))$. Let $c \in RC_\lambda(k)$. Then there exists $c \in C_\lambda(O)$ such that $c(0) = 1$ and $c(1) = c$. By definition, there exists $g \in G(O)$ such

[4] By Philippe Gille.

that $\lambda(g) = c$. The sequence of groups $\widetilde{G}(k) \xrightarrow{i} G(k) \to T(k)$ is exact; so there exists $\widetilde{g}_0 \in \widetilde{G}(k)$ such that $i(\widetilde{g}_0) = g(0)$. We set $g' := gi(\widetilde{g}_0)^{-1} \in G(O)$. Then $g'(0) = 1$, so $g'(1) \in RG(k)$ and $\lambda(g'(1)) = \lambda(g)(1) = c(1) = c$ and $c \in \lambda(RG(k))$. □

**Theorem 1.** *Let $k$ be a field of one of the following types*:

(i) *$k$ is a number field,*
(ii) $\mathrm{char}(k) = 0$ *and* $\mathrm{cd}(k) \leqslant 2$.

*Let $1 \to \widetilde{G} \to G \xrightarrow{\lambda} T \to 1$ be an exact sequence of reductive $k$-groups where $\widetilde{G}/k$ is semisimple and simply connected and $T/k$ is a $k$-torus. In case* (ii), *we assume that Serre's Conjecture* II *holds for $\widetilde{G}/k$, i.e., $H^1(k, \widetilde{G}) = 1$.*

(a) *There is a natural exact sequence of groups*

$$\widetilde{G}(k)/R \longrightarrow G(k)/R \longrightarrow T(k)/R \to 1.$$

(b) *If moreover $k$ is a field as in* 0.1 *and $\widetilde{G}$ has no $E_8$-factor in the case* (gl), *then there is a natural isomorphism $G(k)/R \xrightarrow{\sim} T(k)/R$.*

We recall (cf. [44, §10]) that a torus $S/\mathbf{R}$ over the field of real numbers is isomorphic to a product $S = \mathbf{G}_m^r \times R_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m)^s \times R^1_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_m)^t$, so $S(\mathbf{R}) = (\mathbf{R}^\times)^r \times (\mathbf{C}^\times)^s \times (S^1)^t$ and we denote by $S(\mathbf{R})_+ := (\mathbf{R}^\times_+)^r \times (\mathbf{C}^\times)^s \times (S^1)^t$ the connected component of $S(\mathbf{R})$. If $S/k$ is a torus defined over a number field $k$, we denote by $S(k)_+$ the preimage of $\prod_{v \text{ real}} S(k_v)_+$ by the diagonal map $S(k) \to \prod_{v \text{ real}} S(k_v)$, where the product is taken over the real places of $k$.

**Lemma 2.** *Assume that $k$ is a number field. Let $1 \to S \to E \xrightarrow{f} T \to 1$ be a flasque resolution of $T$* (*where $S$ is a flasque torus and $E$ is a quasi-trivial torus*).

(a) *The group $RT(k)$ is dense in $\prod_{v \text{ real}} T(k_v)$ and we have $T(k) = T(k)_+.RT(k)$ and $T(k)_+ \cap RT(k) = f(E(k)_+)$.*
(b) *The group $RG(k)$ is dense in $\prod_{v \text{ real}} G(k_v)$.*
(c) *$T(k_v)_+ \subset C_\lambda(k_v) \subset T(k_v)$ for any real place $v$ of $k$.*
(d) *$T(k)_+ \subset \mathrm{Im}(G(k) \xrightarrow{\lambda} T(k))$.*

**Proof.** (a) As $R$-equivalence is the trivial relation over $\mathbf{R}$ (cf. [21, §III.2.c, p. 218]) we have the following commutative diagram

$$\begin{array}{ccccccc}
S(k) & \longrightarrow & E(k) & \xrightarrow{f_k} & T(k) & \longrightarrow & H^1(k,S) \\
\downarrow & & \downarrow & & \downarrow & & \\
\prod_{v \text{ real}} S(k_v) & \longrightarrow & \prod_{v \text{ real}} E(k_v) & \longrightarrow & \prod_{v \text{ real}} T(k_v) & \longrightarrow & 1.
\end{array}$$

The group $E(k)$ is dense in $\prod_{v \text{ real}} E(k_v)$, so the group $RT(k)$ is dense in $\prod_{v \text{ real}} T(k_v)$. For any real place $v$ of $k$, the map $E(k_v)_+ \to T(k_v)_+$ is surjective, and a diagram chase gives $f(E(k)).T(k)_+ = T(k)$, so $RT(k).T(k)_+ = T(k)$. We have to prove that $RT(k) \cap T(k)_+ \subset f(E(k)_+)$. Let $t \in RT(k) \cap T(k)_+$. Then there exists $e \in E(k)$ such that $f(e) = t$. There exists $e_v \in E(k_v)_+$ such that $f(e_v) = t \in T(k_v)$ and $s_v := ee_v^{-1} \in S(k_v)$. The weak approximation holds for $S$ at the real places [39, Lemme 1.8, p. 19], so there exists $s \in S(k)$ such that $ss_v^{-1} \in S(k_v)_+$. Then $f(es^{-1}) = t$ and $es^{-1} \in E(k)_+$.

(b) We have to show that any element $(g_v) \in \prod_{v \text{ real}} G(k_v)$ can be approximated by elements of $RG(k)$. As $RG(k)$ is a Zariski-dense subgroup of $G$, we can assume that some $g_v$ is semisimple regular. Let $\mathcal{U}$ be an open neighborhood of $(g_v)$. The group $G/k$ satisfies weak approximation at real places [39, Corollaire 3.5.c, p. 26] so there exists $g \in G(k)$ such that $g \in \mathcal{U}$ and we may assume that $g$ is semisimple regular. We consider the maximal torus $Z_G(g)$ of $G$. By the statement (a), the group $RZ_G(g)(k)$ is dense in the closed subgroup $\prod_{v \text{ real}} Z_G(g)(k_v)$ of $\prod_{v \text{ real}} G(k_v)$, so $\mathcal{U} \cap RZ_G(g)(k) \neq \emptyset$ and *a fortiori* $\mathcal{U} \cap RG(k) \neq \emptyset$.

(c) We consider the exact sequence of pointed sets

$$G(k_v) \xrightarrow{\lambda_{k_v}} T(k_v) \xrightarrow{\delta_v} H^1(k_v, \widetilde{G}).$$

The boundary map $\delta_v : T(k_v) \to H^1(k_v, \widetilde{G})$ is continuous for the real topology, and $H^1(k_v, \widetilde{G})$ is finite, so $\delta_v$ is trivial on $T(k_v)_+$ and $T(k_v)_+ \subset \lambda(G(k_v)) = C_\lambda(k_v)$.

(d) We use now the Hasse principle (Kneser–Harder–Chernousov) for the simply connected group $\widetilde{G}$ and we consider the following exact commutative diagram of pointed sets

$$\begin{array}{ccccc} G(k) & \xrightarrow{\lambda_k} & T(k) & \xrightarrow{\delta} & H^1(k, \widetilde{G}) \\ \downarrow & & \downarrow & & \wr\downarrow \\ \prod_{v \text{ real}} G(k_v) & \longrightarrow & \prod_{v \text{ real}} T(k_v) & \xrightarrow{\delta_v} & \prod_{v \text{ real}} H^1(k_v, \widetilde{G}). \end{array}$$

By assertion (c), the maps $\delta_v$'s are trivial on $T(k_v)_+$ and a diagram chase shows that $\delta$ is trivial on $T(k)_+$, i.e., $T(k)_+ \subset \mathrm{Im}(G(k) \xrightarrow{\lambda} T(k))$. □

**Proof of Theorem 1.** (a) *First step*: $\widetilde{G}$ *is quasi-split*. Then $\widetilde{G}$ contains a quasi-trivial maximal $k$-torus $E$. We consider the maximal $k$-torus $Z_G(E)$ of $G$, and we have the following exact sequence of $k$-tori

$$1 \to E \to Z_G(E) \to T \to 1.$$

By Hilbert 90 Theorem we have $H^1(k, E) = 0$, so the map $Z_G(E)(k) \to T(k)$ is surjective and *a fortiori* the map $G(k) \to T(k)$ is surjective. Moreover, since $E$ is quasi-trivial, by Lemma 4.15 of this paper we have an isomorphism

$$Z_G(E)(k)/R \xrightarrow{\sim} T(k)/R.$$

But according to Proposition 14(i) of [15] on quasi-split groups, we have $Z_G(E)(k)/R \xrightarrow{\sim} G(k)/R$, and we conclude that the map $G(k)/R \to T(k)/R$ is an isomorphism.

*Second step*: *the general case*. We do first the case of number fields (i) and shall explain after, how the proof works also for fields of kind (ii). Let $RG(k)$ denote the $R$-equivalence class of 1 in $G(k)$. We denote by $C_\lambda(k) \subset T(k)$ (respectively $RC_\lambda(k) \subset T(k)$ ) the image by $\lambda$ of $G(k)$ (respectively $RG(k)$). Let $1 \to S \to E \to T \to 1$ be a flasque resolution of $T$ (where $S$ is a flasque torus and $E$ is a quasi-trivial torus) and let us consider the map $f : E(k) \to T(k)$. We begin with the following

**Lemma 3.** $f(E(k)_+) \subset \mathrm{Im}(RG(k) \xrightarrow{\lambda_*} RT(k))$.

**Proof.** The torus $E$ is quasi-trivial, hence we have $E = \prod_{i=1,\dots,r} R_{k_i/k}(\mathbf{G}_m) = \prod_i E_i$ where the $k_i/k$'s are finite field extensions. We denote by $h_i : E_i \to E$ the morphism defined by $h_i(e_i) = (1, \dots, 1, e_i, 1, \dots, 1)$. As $E(k)_+ = \prod_i E_i(k)_+$, it is enough to prove that $f(h_i(E_i(k)_+)) \subset \mathrm{Im}(RG(k) \xrightarrow{\lambda_*} RT(k))$ for $i = 1, \dots, r$. We firstly recall the norm principle, i.e., Theorem 3.9 of [32], applied to the extension $1 \to \widetilde{G} \to G \xrightarrow{\lambda} T \to 1$. It states that for any finite field extension $L/k$ the norm map $N_{L/k} : T(L) \to T(k)$ preserves the image by $\lambda$ of $RG(L)$, i.e., $N_{L/k}(\lambda(RG(L))) \subset \lambda(RG(k)) \subset T(k)$. In other words, we have

$$N_{L/k}\bigl(RC_\lambda(L)\bigr) \subset RC_\lambda(k) \subset T(k). \tag{$*$}$$

(1) $k_i = k$ *and* $E_i = \mathbf{G}_m$. Let $L/k$ be a finite field extension such that $G_L$ is quasi-split, i.e., $L$ satisfies $X(L) \neq \emptyset$, where $X$ denotes the variety of the Borel subgroups of $G$. By the first step, we have $\lambda(RG(L)) = RT(L) = f(E(L))$, so $RC_\lambda(L) = RT(L) = f(E(L))$. By the norm principle, we get

$$N_{L/k}\bigl(f\bigl(E(L)\bigr)\bigr) \subset RC_\lambda(k) \subset RT(k).$$

The restriction to the factor $E_i$ yields

$$f\bigl(h_i\bigl(N_{L/k}\bigl(L^\times\bigr)\bigr)\bigr) = N_{L/k}\bigl(f\bigl(h_i\bigl(E_i(L)\bigr)\bigr)\bigr) \subset RC_\lambda(k).$$

By taking all the finite extensions $L$ such that $X(L) \neq \emptyset$, we get

$$f\bigl(h_i\bigl(N_X(k)\bigr)\bigr) \subset RC_\lambda(k) \subset RT(k),$$

where $N_X(k)$ denotes the normgroup of $X$, i.e., the subgroup of $k^\times$ generated by the $N_{L/k}(L^\times)$ for $L/k$ finite satisfying $X(L) \neq \emptyset$. We use now the Hasse principle of Kato–Saito [25, Theorem 4] for the normgroup of $X$

$$k^\times/N_X(k) \xrightarrow{\sim} \bigoplus_{v \in \Omega} k_v^\times/N_X(k_v),$$

where $\Omega$ denotes the set of places of $k$. For a finite place $v$ of $k$, one has $N_X(k_v) = k_v^\times$ [21, Lemme III.2.8], so $k_+^\times \subset N_X(k)$. We conclude that

$$f\big(h_i\big(k_+^\times\big)\big) \subset RC_\lambda(k).$$

(2) $E_i = R_{k_i/k}\mathbf{G}_m$. There exists an étale algebra $A_i/k_i$ such that $E_i \otimes_k k_i = \mathbf{G}_{m,k_i} \times R_{A_i/k_i}\mathbf{G}_m$. We consider the following commutative diagram of corestrictions

$$\begin{array}{ccc}
E_i(k_i) = k_i^\times \times A_i^\times & \xrightarrow{f_{k_i} \circ h_i} & T(k_i) \\
\Big\downarrow{\scriptstyle N_{k_i/k}} & & \Big\downarrow{\scriptstyle N_{k_i/k}} \\
E_i(k) = k_i^\times & \xrightarrow{f_k \circ h_i} & T(k_i).
\end{array}$$

The first case applied to $k_i$ and $\mathbf{G}_{m,k_i}$ gives

$$f_{k_i}\big(h_i\big(k_{i,+}^\times \times 1\big)\big) \subset RC_\lambda(k_i).$$

The norm principle (i.e., identity $(*)$ above) applied to the extension $k_i/k$ yields

$$N_{k_i/k}\big(RC_\lambda(k_i)\big) \subset RC_\lambda(k),$$

so

$$N_{k_i/k}\big((f_{k_i} \circ h_i)\big(k_{i,+}^\times \times 1\big)\big) \subset RC_\lambda(k).$$

But the norm $N_{k_i/k} : E_i(k_i) \to E_i(k)$ induces the identity on the first factor $k_i^\times$, so

$$(f_{k_i} \circ h_i)\big(E_i(k)_+\big) \subset RC_\lambda(k),$$

which completes the proof of the lemma.

By Lemma 1, we have an exact sequence

$$\widetilde{G}(k)/R \longrightarrow G(k)/R \longrightarrow C_\lambda(k)/RC_\lambda(k) \to 1.$$

So it remains to prove that the map $C_\lambda(k)/RC_\lambda(k) \to T(k)/R$ is an isomorphism.

*Surjectivity*: We have to check that $T(k) = RT(k).C_\lambda(k)$. According to Lemma 2(d), one has $T(k)_+ \subset C_\lambda(k)$, so

$$RT(k).T(k)_+ \subset RT(k).C_\lambda(k).$$

By Lemma 2(a), we have $RT(k).T(k)_+ = T(k)$, so $T(k) = RT(k).C_\lambda(k)$.

*Injectivity*: We have to check that $RC_\lambda(k) = C_\lambda(k) \cap RT(k)$. The inclusion $RC_\lambda(k) \subset C_\lambda(k) \cap RT(k)$ is obvious, let us show the converse by taking $t \in C_\lambda(k) \cap RT(k)$. By Lemma 2(c), we have the inclusions

$$\prod_{v \text{ real}} T(k_v)_+ \subset \prod_{v \text{ real}} C_\lambda(k_v) \subset \prod_{v \text{ real}} T(k_v),$$

and the group $\prod_{v \text{ real}} T(k_v)_+$ is open in $\prod_{v \text{ real}} C_\lambda(k_v)$. By Lemma 2(b), the group $RC_\lambda(k)$ is a dense subgroup of $\prod_{v \text{ real}} C_\lambda(k_v)$, so there exists $t_0 \in RC_\lambda(k)$ such that $tt_0^{-1} \in T(k)_+ = T(k) \cap \bigcap_{v \text{ real}} T(k_v)_+$. By Lemmas 2(a) and 3, one has

$$RT(k) \cap T(k)_+ = f\bigl(E(k)_+\bigr) \subset RC_\lambda(k),$$

so $tt_0^{-1} \in RC_\lambda(k)$ and finally $t = (tt_0^{-1})t_0 \in RC_\lambda(k)$. We conclude that $C_\lambda(k) \cap RT(k) = RC_\lambda(k)$ as desired.

The case of a field of type (ii) is much simpler and one replaces the Hasse principle of Kato–Saito by the fact that $N_X(k) = k^\times$ [22, Theorem 6.a]. In that case, Lemma 3 yields $RT(k) = RC_\lambda(k)$. Moreover, the assumption on the vanishing of $H^1(k, \widetilde{G})$ implies that the map $G(k) \to T(k)$ is surjective. So $C_\lambda(k)/RC_\lambda(k) = T(k)/RT(k)$ and the exact sequence (Lemma 1)

$$\widetilde{G}(k)/R \longrightarrow G(k)/R \longrightarrow C_\lambda(k)/RC_\lambda(k) \to 1,$$

induces the exact sequence

$$\widetilde{G}(k)/R \longrightarrow G(k)/R \longrightarrow T(k)/R \to 1$$

as desired. □

(b) If $k$ is as in 0.1, Serre's Conjecture II holds by Theorems 1.3, 1.4 and 1.5 of [12] and the group $\widetilde{G}$ satisfies $\widetilde{G}(k)/R = 1$ (*ibid.*, Corollary 4.6). In this case we deduce that the map $G(k)/R \to T(k)/R$ is an isomorphism. □

**Corollary.** *Let $k$ be a field as in Theorem* 1. *Let $G$ be a semisimple simply connected group and $S \subset G$ be a $k$-split torus of $G$. Then the group $Z_G(S)^{\mathrm{ss}}$ is semisimple and simply connected and the natural map*

$$Z_G(S)^{\mathrm{ss}}(k)/R \to G(k)/R$$

*is surjective.*

**Proof.** According to Theorem 4.15.a of [3], the centralizer $Z_G(S)$ is the Levi subgroup of some $k$-parabolic subgroup $P$ of $G$. The fact that $Z_G(S)^{\mathrm{ss}}/k$ is simply connected is well-known, it can be deduced from the presentation of standard parabolic subgroups [3, §4] and Corollary 4.4 of [4]. We denote by $U$ the unipotent radical of $P$ and we consider the opposite parabolic group $P^-$ of $P$ with respect to $Z_G(S)$; it is the unique $k$-parabolic subgroup $Q$ of $G$ containing $Z_G(S)$ such that $Q \cap P = Z_G(S)$ [3, §4.8]. Let $U^-$ be the unipotent radical of $P^-$. As $P \cap U^- = 1$, each fiber of the multiplication map $U \times Z_G(S) \times U^- \to G$ consists of a single point, so this map is an open immersion by [2, Proposition AG 18.4]. So by Proposition 11 of [15], we have $(U \times Z_G(S) \times U^-)(k)/R = G(k)/R$. But $U$ and $U^-$ are affine spaces, so one has an isomorphism $Z_G(S)(k)/R \xrightarrow{\approx} G(k)/R$. We denote by $T = Z_G(S)^{\mathrm{tor}} = Z_G(S)/Z_G(S)^{\mathrm{ss}}$ the coradical torus of $Z_G(S)$; the preceding theorem produces then the exact sequence

$$Z_G(S)^{\mathrm{ss}}(k)/R \to Z_G(S)(k)/R \to T(k)/R \to 1.$$

As $Z_G(S)$ is a Levi subgroup of $P$, we have a natural isomorphism $P^{\mathrm{tor}} \xrightarrow{\sim} Z_G(S)^{\mathrm{tor}} = T$. According to Lemma 5.6 of [12], the fact that $G$ is simply connected implies that the torus $T = P^{\mathrm{tor}}$ is quasi-trivial. So we have $T(k)/R = 1$ and the map $Z_G(S)^{\mathrm{ss}}(k)/R \to G(k)/R$ is surjective. □

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