TORSORS OVER AFFINE CURVES

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Abstract. The lectures are an introduction to torsors in Algebraic Geometry with special attention to the case of affine algebraic curves.

Contents

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1. Introduction

The theory of fibrations and principal fibrations is ubiquious in Topology and Differential Geometry. In 1955, Grothendieck investigated a general theory of fibrations focusing on functoriality issues [33]. In 1958, Grothendieck and Serre extended the setting of G–bundles in algebraic geometry by means of the étale topology [53].

For simplicity we shall present this theory over rings or equivalently over affine schemes. The general framework is close to that and can be found in other references [18, 38, 12, 5].

We shall focus on the case of an affine smooth curve over a field, starting with vector bundles and quadratic vector bundles. Important cases are the affine line and the affine punctured line. Panin and Stavrova provided also independently a more detailed survey of Harder's results of §5.1 [PS].

For further topics, we recommend the survey *Problems about torsors over regular rings* of K. Česnavičius [13].

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2. The Swan-Serre correspondence

This is the correspondence between locally free modules of finite rank and vector bundles, it arises from the case of a paracompact topological space [60].

We explain it in the setting of affine schemes following the book of Görtz-Wedhorn [29, ch. 11] up to slightly different conventions.

2.1. **Vector group schemes.** Let R be a ring (commutative, unital). The additive R–group scheme is $\mathbb{G}_{a,R} = \mathrm{Spec}(R[t])$ and is a part of a wider family.

(a) Let M be an R–module. We denote by $\mathbf{V}(M)$ the affine R–scheme defined by $\mathbf{V}(M) = \mathrm{Spec}\big(\mathrm{Sym}^\bullet(M)\big)$; it is affine over R and represents the R–functor $S \mapsto \mathrm{Hom}_S(M \otimes_R S, S) = \mathrm{Hom}_R(M, S)$ [19, 9.4.9]. Indeed for each R-ring B, we have

$$\mathbf{V}(M)(B) = \mathrm{Hom}_{R-ring}(\mathrm{Sym}^\bullet(M), B) = \mathrm{Hom}_{R-mod}(M, B).$$

It is called the *vector group scheme* attached to M, this construction commutes with arbitrary base change of rings $R \to R'$. We have $\mathbf{V}(R) = \mathbb{A}^1_R = \mathrm{Spec}(R[t])$, that is, the affine line over R. We can consider $\mathbf{V}(M)$ as an R-scheme, as a commutative R–group scheme or as a O_R–module (where O_R stands for the functor in R–rings defined by $\mathrm{O}_R(R') = R'$). Our default convention is that of O_R-modules and is justified by the following fact.

Proposition 2.1. [52, I.4.6.2] *The functor $M \to \mathbf{V}(M)$ induces a full contravariant embedding of the category of R–modules in the category of vector group schemes over R.*

That functor has a nice behaviour. For example if the R–module M is finitely presented, then the R-scheme $\mathbf{V}(M)$ is finitely presented [19, Cororllaire 9.4.7] and the converse holds by using the limit criterion [62, Tag 0G8P].

(b) We assume now that M is locally free of finite rank and denote by $M^\vee$ its dual. In this case $\mathrm{Sym}^\bullet(M)$ is of finite presentation [19, 9.4.11]. Also the R–functor $S \mapsto M \otimes_R S$ is representable by the affine R–scheme $\mathbf{V}(M^\vee)$ which is also denoted by $\mathbf{W}(M)$ [52, I.4.6].

Remark 2.2. Assuming that M is finitely presented, Romagny has shown that the finite locally freeness condition on M is a necessary condition for the representability of $\mathbf{W}(M)$ by a group scheme [47, Theorem 5.4.5]. The proof is one of the exercise. This extends a result of Nitsure in the noetherian setting [46, Corollary 2].

Let $r \geq 0$ be an integer.

Definition 2.3. *A vector bundle of rank r over $\mathrm{Spec}(R)$ is an affine R-scheme X such that there exists a partition $1 = f_1 + \cdots + f_n$ and isomorphisms $\phi_i : X \times_R R_{f_i} \xrightarrow{\sim} \mathbf{W}((R_{f_i})^r)$ such that $\phi_i \circ \phi_j^{-1} : \mathbf{W}((R_{f_i f_j})^r) \xrightarrow{\sim} \mathbf{W}\big((R_{f_i f_j})^r\big)$ is a linear automorphism of $\mathbf{W}\big((R_{f_i f_j})^r\big)$ for $i, j = 1, \dots, n$.*

We say that the Zariski cover $\big(\mathrm{Spec}(R_{f_i})\big)_{i=1,..,n}$ trivializes the vector bundle X and any finer cover trivializes as well X. A homomorphism of vector bundles $X \to X'$ of respective rank r, r', is a morphism of R-schemes $f : X \to X'$ built locally from linear maps. More precisely, we require that there exists a trivializing cover $(\mathrm{Spec}(R_{f_i})_{i=1,..,n}$ for X and X' with maps $\phi_i : X \times_R R_{f_i} \xrightarrow{\sim} \mathbf{W}((R_{f_i})^r)$ and $\phi'_i : X' \times_R R_{f_i} \xrightarrow{\sim} \mathbf{W}((R_{f_i})^{r'})$ as is the definition such that each

${\phi'}_j \circ f_i \circ \phi_i^{-1} : \mathbf{W}((R_{f_i})^r) \to \mathbf{W}((R_{f_i})^{r'})$ arises from a R_{f_i}-linear map $(R_{f_i})^r \to (R_{f_i})^{r'}$. The notion of isomorphisms is clear and leads to the groupoid of vector bundles of rank r over R whose objects are vector bundles of rank r and whose morphisms are the isomorphisms. We denote by $\mathrm{Vect}_r(R)$ this category.

Theorem 2.4. *(Swan-Serre's correspondence) The above functor $M \mapsto \mathbf{W}(M)$ induces an equivalence of categories between the groupoid of locally free R-modules of rank r and* $\mathrm{Vect}_r(R)$.

Proof. See [29, Proposition 11.7] for the general case (i.e. over a base scheme). We check first that the functor is well-defined. If M is locally free of rank r, there exists a partition $1 = f_1 + \cdots + f_n$ and trivializations $\psi_i : M_{f_i} \xrightarrow{\sim} (R_{f_i})^r$. It follows that each map $\psi_j \circ \psi_i^{-1} : (R_{f_i f_j})^r \xrightarrow{\sim} (R_{f_i f_j})^r$ is a linear isomorphism for $i, j = 1, \dots, n$. By applying the functor $\mathbf{W}$, we get that $\mathbf{W}(M)$ is a vector bundle of rank r and the trivializations are the

$$\phi_i = \psi_{i,*} : \mathbf{W}(M) \times_R R_{f_i} \xrightarrow{\sim} \mathbf{W}((R_{f_i})^r).$$

It follows that the R-functor $\mathbf{W}$ is well-defined and is fully faithful. We need to check that the functor is essentially surjective. We are given a vector bundle X over $\mathrm{Spec}(R)$ of rank r and we need to construct an R–module M which is locally free of rank r together with an isomorphism $X \xrightarrow{\sim} \mathbf{W}(M)$ of vector bundles.

As in the definition consider a partition $1 = f_1 + \cdots + f_n$ and isomorphisms $\phi_i : \mathbf{W}((R_{f_i})^r) \xrightarrow{\sim} X \times_R R_{f_i}$. We put $M_i = (R_{f_i})^r$ and we have linear isomorphisms $\psi_{i,j} = \phi_j \circ \phi_i^{-1} : (M_i)_{f_j} \xrightarrow{\sim} (M_i)_{f_j}$ satisfying the compatibilities $\psi_{i,j} \circ \psi_{j,k} = \psi_{i,k}$. We define the R-module

$$M = \mathrm{Ker}\Big(\bigoplus_{1 \le i \le n} M_i \quad \to \bigoplus_{1 \le i,j \le n} (M_i)_{f_j} \Big)$$

where $(m_1, \dots, m_n)$ maps to the element whose (i,j)-th entry is $\frac{m_i}{1} - \psi_{j,i}(\frac{m_j}{1})$. According to [62, Tag 00EQ], the map $M \to M_i$ induces an isomorphism $\gamma_i : M_{f_i} \xrightarrow{\sim} M_i$ for each i and we have $\psi_{i,j} = \gamma_j^{-1} \circ \gamma_i$ for all i, j.

In particular M is locally free of rank r. Since $\psi_{i,j,*} = \gamma_{j,*}^{-1} \circ \gamma_{i,*} = \phi_j^{-1} \phi_i$ we can glue the isomorphisms

$$\mathbf{W}(M) \times_R R_{f_i} \underset{\sim}{\xrightarrow{\gamma_{i,*}}} \mathbf{W}(M_i) \underset{\sim}{\xleftarrow{\phi_i}} X \times_R R_{f_i}$$

in an isomorphism of vector bundles $\mathbf{W}(M) \xrightarrow{\sim} X$. □

A vector bundle of rank 1 is called a line bundle and a locally free R-module of rank 1 is called an invertible R-module.

Examples 2.1.1. (a) Given a smooth map of affine schemes $X = \mathrm{Spec}(S) \to Y = \mathrm{Spec}(R)$ of relative dimension $r \geq 1$, the tangent bundle $T_{X/Y} = \mathbf{V}(\Omega^1_{S/R})$ is a vector bundle over $\mathrm{Spec}(S)$ of dimension r [20, §16.5.12]. Indeed the R-module $\Omega^1_{S/R}$ is locally free of rank r [5, §2, Proposition 5].

(b) The tangent bundle of the real sphere $Z = \mathrm{Spec}\Big(\mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)\Big)$ is an example of a vector bundle of dimension 2 which is not trivial. It can be proven by differential topology (hairy ball theorem) but there are also algebraic proofs, see for instance [61]. A consequence is that Z cannot be equipped with a structure of a real algebraic group.

(c) Note that this tangent bundle extended to $\mathbb{C}$ becomes free. This is a consequence of Murty-Swan's theorem [43] since it is stably free.

2.2. **Linear groups.** Let M be a locally free R–module of finite rank. We consider the R–algebra $\mathrm{End}_R(M^\vee) = M^\vee \otimes_R M$. It is a locally free R-module of finite rank so that we can consider the vector R–group scheme $\mathbf{V}\big(\mathrm{End}_R(M^\vee)\big)$ which is an R–functor with values in associative and unital algebras [19, 9.6.2]. It is isomorphic to $\mathbf{W}\big(\mathrm{End}_R(M)\big)$. Now we consider the R–functor $S \mapsto \mathrm{Aut}_S(M \otimes_R S)$. It is representable by an open R–subscheme of $\mathbf{W}\big(\mathrm{End}_R(M)\big)$ which is denoted by $\mathrm{GL}(M)$ (*loc. cit.*, 9.6.4). We bear in mind that the action of the group scheme $\mathrm{GL}(M)$ on $\mathbf{W}(M)$ (resp. $\mathbf{V}(M)$) is a left (resp. right) action.

In particular, we denote by $\mathrm{GL}_r = \mathrm{GL}(R^r)$.

Remark 2.5. For R noetherian, Nitsure has shown that the finite locally freeness condition on M is a necessary condition for the representability of $\mathrm{GL}(M)$ by a group scheme [45].

If $\mathcal{B}$ is a locally free R–algebra of finite rank, we recall that the functor of invertible elements of $\mathcal{B}$ is representable by an affine R-group scheme which is a principal open subset of $\mathbf{W}(\mathcal{B})$. It is denoted by $\mathrm{GL}_1(\mathcal{B})$ [12, 2.4.2.1].

2.3. **Cocycles.** Let M be a locally free R–module of rank r. There exists a partition $1 = f_1 + \cdots + f_n$ of R and isomorphisms $\phi_i : (R_{f_i})^r \xrightarrow{\sim} M \times_R R_{f_i}$. Then the $R_{f_i f_j}$–isomorphism $\phi_i^{-1}\phi_j : (R_{f_i f_j})^r \xrightarrow{\sim} (R_{f_i f_j})^r$ is linear so defines an element $g_{i,j} \in \mathrm{GL}_r(R_{f_i f_j})$. More precisely we have $(\phi_i^{-1}\phi_j)(v) = g_{i,j} \,.\, v$ for each $v \in (R_{f_i f_j})^r$ (in other words, $(R_{f_i f_j})^r$ is seen as column vectors).

Lemma 2.6. *The element $g = (g_{i,j})$ is a 1–cocycle, that is, satisfies the relation*

$$g_{i,j}\, g_{j,k} = g_{i,k} \in \mathrm{GL}_r(R_{f_i f_j f_k})$$

for all $i, j, k = 1, \dots, n$.

Proof. Over $R_{f_i f_j f_k}$ we have $\phi_i^{-1}\phi_k = (\phi_i^{-1}\phi_j) \circ (\phi_j^{-1}\phi_k) = L_{g_{i,j}} \circ L_{g_{j,k}} = L_{g_{i,j} g_{j,k}}$ where L stands for the left translation on GL_r. □

If we replace the ϕ_i's by the $\phi_i' = \phi_i \circ g_i$'s with elements g_i's in $\prod \mathrm{GL}_r(R_{f_i})$, we get $g'_{i,j} = g_i^{-1}\, g_{i,j}\, g_j$ and we say that $(g'_{i,j})$ is cohomologous to $(g_{i,j})$.

We denote by $\mathcal{U} = \big(\mathrm{Spec}(R_{f_i})\big)_{i=1,..,n}$ the affine cover of $\mathrm{Spec}(R)$, by $Z^1(\mathcal{U}/R, \mathrm{GL}_r)$ the set of 1-cocycles. We consider the following equivalence relation on $Z^1(\mathcal{U}/R, \mathrm{GL}_r)$: the cocycle $(g'_{i,j})$ is equivalent to the cocycle $(g_{i,j})$ if $g'_{i,j} = g_i^{-1}\, g_{i,j}\, g_j$ for some $(g_i)_{i=1,..,n} \in \prod\limits_{i=1,..,n} G(R_{f_i})$. We denote by $H^1(\mathcal{U}/R, \mathrm{GL}_r) = Z^1(\mathcal{U}/R, \mathrm{GL}_r)/\sim$ the set of 1-cocycles modulo equivalence relation The set $H^1(\mathcal{U}/R, \mathrm{GL}_r)$ is called the pointed set of Čech cohomology with respect to $\mathcal{U}$.

Summarizing we attached to the vector bundle $\mathbf{W}(M)$ of rank r a class $\gamma(M) \in H^1(\mathcal{U}/R, \mathrm{GL}_r)$.

Conversely by Zariski glueing, we can attach to a cocycle $(g_{i,j})$ a vector bundle $\mathbf{W}_g$ over R of rank r equipped with trivializations $\phi_i : \mathbf{W}(R^r_{f_i}) \xrightarrow{\sim} \mathbf{W}_g \times_R R_{f_i}$ such that $\phi_i^{-1}\phi_j = g_{i,j}$.

Lemma 2.7. *The pointed set $H^1(\mathcal{U}/R, \mathrm{GL}_r)$ classifies the isomorphism classes of vector bundles of rank r over $\mathrm{Spec}(R)$ which are trivialized by $\mathcal{U}$.*

For the proof, see [29, 11.15]. We can pass this construction to the limit over all affine open covers of X. We define the pointed set $\check{H}^1_{\mathrm{Zar}}(R, \mathrm{GL}_r) = \varinjlim_{\mathcal{U}} H^1(\mathcal{U}/R, \mathrm{GL}_r)$ of non-abelian Čech cohomology of GL_n with respect to the Zariski topology of $\mathrm{Spec}(R)$. By passage to the limit, Lemma 2.7 implies that $\check{H}^1_{\mathrm{Zar}}(R, \mathrm{GL}_r)$ classifies the isomorphism classes of vector bundles of rank r over $\mathrm{Spec}(R)$.

We will refer to the map $\mathrm{Vect}_r(R) \to \check{H}^1_{\mathrm{Zar}}(R, \mathrm{GL}_r), X \mapsto [X]$ as the class map.

2.4. **Functoriality.** The principle is that nice constructions for vector bundles arise from homomorphisms of group schemes. Given a map $f : \mathrm{GL}_r \to \mathrm{GL}_s$, we can attach to a vector bundle $\mathbf{W}_g$ of rank r (where $g = (g_{i,j})$ is a cocycle) the vector bundle $\mathbf{W}_{f(g)}$ of rank s where $f(g) = (f(g_i))$. It can be shown (as an extra exercise) that it extends to a functor $X \mapsto f_*(X)$ from vector bundles of rank r to vector bundles of rank s and is compatible to class maps. We mean that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Vect}_r(R) & \xrightarrow{\quad f_* \quad} & \mathrm{Vect}_s(R) \\ \downarrow & & \downarrow \\ \check{H}^1_{\mathrm{Zar}}(R, \mathrm{GL}_r) & \xrightarrow{\quad f_* \quad} & \check{H}^1_{\mathrm{Zar}}(R, \mathrm{GL}_r). \end{array} \tag{2.1}$$

where the vertical arrows are the class maps. We limit ourselves here to the following three cases for which we have an explicit description of f_*.

(a) *Direct sum.* If $r = r_1 + r_2$, we consider the map $f : \mathrm{GL}_{r_1} \times \mathrm{GL}_{r_2} \to \mathrm{GL}_r$, $(A_1, A_2) \mapsto A_1 \oplus A_2$. We then have $f_*(\mathbf{W}_1, \mathbf{W}_2) = \mathbf{W}_1 \oplus \mathbf{W}_2$.

Of course, it can be done with $r = r_1 + \cdots + r_l$, in particular we have in the case $r = 1 + \cdots + 1$ the diagonal map $(\mathbb{G}_m)^r \to \mathrm{GL}_r$ which leads to decomposable vector bundles, that is, direct sum of line bundles.

(b) *Tensor product.* If $r = r_1 r_2$, we consider the map $f : \mathrm{GL}_{r_1} \times \mathrm{GL}_{r_2} \to \mathrm{GL}_r$, $(A_1, A_2) \mapsto A_1 \otimes A_2$ (called the Kronecker product). We then have $f_*(\mathbf{W}_1, \mathbf{W}_2) = \mathbf{W}_1 \otimes \mathbf{W}_2$.

(c) *Determinant.* We put $\det(\mathbf{W}) = \det_*(\mathbf{W})$, this is the determinant line bundle.

2.5. **The case of a Dedekind ring.** Let R be a Dedekind ring, that is, a noetherian domain such that the localization at each maximal ideal is a discrete valuation ring. The next result is a classical fact of commutative algebra, see [31, II.4, Theorem 13].

Theorem 2.8. *A locally free R–module of rank $r \geq 1$ is isomorphic to $R^{r-1} \oplus I$ for I an invertible R–module which is unique up to isomorphism.*

Since $I \cong \Lambda^r(R^{r-1} \oplus I)$ is the determinant of $R^{r-1} \oplus I$, the last assertion is clear. Our goal is to discuss this statement with cohomological methods in view of possible generalizations. The key input is the strong approximation theorem for the Dedekind ring R.

Let R_f be localization of R and denote by $\{\mathfrak{p}_1, \dots, \mathfrak{p}_c\} = \mathrm{Spec}(R) \setminus \mathrm{Spec}(R_f)$ and by v_i the discrete valuation of K attached to $\mathfrak{p}_i$. We denote by $\widehat{K}_i$ the completion of K with respect to v_i and by $\widehat{R}_i$ its valuation ring.

Theorem 2.9. *(1) (Weak Approximation) The image of the diagonal embedding $K \hookrightarrow \prod_{i=1,\dots,c} \widehat{K}_i$ is dense.*

(2) (Chinese remainder) For each c-uple $(e_1, \dots, e_c)$ of positive integers, the map $R \to \prod\limits_{i=1}^{c} R/\mathfrak{p}_i^{e_i}$ is onto and its kernel is $\prod\limits_{i=1}^{c} \mathfrak{p}_i^{e_i}$.

(3) (Strong approximation) Let $x_1, \dots, x_c \in K$ and let $e = (e_1, \dots, e_c)$ be a c–uple of integers. Then there exists $x \in K$ such that $v_i(x - x_i) \geq e_i$ for $i = 1, .., c$ and and $v_{\mathfrak{p}}(x) \geq 0$ for each maximal ideal $\mathfrak{p}$ of R satisfying $\mathfrak{p} \neq \mathfrak{p}_i$ for $i = 1, \dots, n$.

Proof. Part (3) implies clearly (1) and (2). For a proof of (3), see [55, §I.3] or [7, §VII.2.4]. For a direct proof of (2), see [62, Tag 00DT]. For a direct proof of (1), see [7, §VI.7.2]. □

The Chinese remainder theorem is then a special case of strong approximation. Coming back to Theorem 2.8, it states firstly that vector bundles over R are decomposable and secondly that vector bundles over R are classified by their determinant. We limit ourselves to prove the following corollary by using strong approximation.

Corollary 2.10. *A locally free R-module of rank $r \geq 1$ is trivial if and only if its determinant is trivial.*

Proof. We are given a vector bundle $\mathbf{W}(M)$. It trivializes over an open affine subset $\mathrm{Spec}(R_f)$. We consider its class

$$[M] \in \ker\Big(H^1_{\mathrm{Zar}}(R, \mathrm{GL}_r) \to H^1_{\mathrm{Zar}}(R_f, \mathrm{GL}_r) \Big)$$

and the right handside classifies isomorphism classes of vector bundles of rank r which trivializes over R_f. We put $\Sigma = \mathrm{Spec}(R) \setminus \mathrm{Spec}(R_f) = \{\mathbf{p}_1, \dots, \mathbf{p}_c\}$. We use then the above notation and consider for each i the diagram

$$\begin{array}{ccc} \widehat{R}_{\mathfrak{p}_i} & \hookrightarrow & \widehat{K}_{\mathfrak{p}_i} \\ \uparrow & & \uparrow \\ R & \hookrightarrow & K \end{array}$$

According to Nakayama's lemma[1], the $\widehat{R}_{\mathbf{p}_i}$–module $M \otimes_R \widehat{R}_{\mathbf{p}_i}$ is free so we can pick a trivialization $\widehat{\phi}_i : (\widehat{R}_{\mathbf{p}_i})^r \xrightarrow{\sim} M \times_R \widehat{R}_{\mathbf{p}_i}$; we bear in mind that the choice $\widehat{\phi}_i$ is up to precomposing with an element of $\mathrm{GL}_r(\widehat{R}_{\mathbf{p}_i})$.

On the other hand, let $\phi_f : (R_f)^r \xrightarrow{\sim} M \times_R R_f$ be a trivialization, similarly its choice is up to precomposing with an element of $\mathrm{GL}_r(R_f)$. By extending the scalars to $\widehat{K}_{\mathbf{p}_i}$, we obtain then two trivializations

$$\phi_{f,\widehat{K}_{\mathbf{p}_i}} : (\widehat{K}_{\mathbf{p}_i})^r \xrightarrow{\sim} M \times_R \widehat{K}_{\mathbf{p}_i},$$
$$\widehat{\phi}_{i,\widehat{K}_{\mathbf{p}_i}} : (\widehat{K}_{\mathbf{p}_i})^r \xrightarrow{\sim} M \times_R \widehat{K}_{\mathbf{p}_i}.$$

The linear map

$$\phi^{-1}_{f,\widehat{K}_{\mathbf{p}_i}} \circ \widehat{\phi}_{i,\widehat{K}_{\mathbf{p}_i}} : (\widehat{K}_{\mathbf{p}_i})^r \xrightarrow{\sim} (\widehat{K}_{\mathbf{p}_i})^r$$

gives rise to an element $g_i \in \mathrm{GL}_r(\widehat{K}_{\mathbf{p}_i})$. Taking into account the choices, we attached to M an element of the double coset

$$c_\Sigma(R, \mathrm{GL}_r) := \mathrm{GL}_r(R_f) \ \backslash \ \prod_{i=1,\dots,c} \mathrm{GL}_r(\widehat{K}_{\mathbf{p}_i}) \, / \, \mathrm{GL}_r(\widehat{R}_{\mathbf{p}_i}).$$

If we consider an isomorphic R-module $\psi : M \xrightarrow{\sim} M'$, we can deal with the trivializations $\phi'_f : (R_f)^r \underset{\sim}{\xrightarrow{\phi_f}} M_f \underset{\sim}{\xrightarrow{\psi_f}} M'_f$ and $\widehat{\phi}'_i : (\widehat{R}_{\mathbf{p}_i})^r \underset{\sim}{\xrightarrow{\widehat{\phi}_i}} M \times_R \widehat{R}_{\mathbf{p}_i} \underset{\sim}{\xrightarrow{\psi_{\widehat{R}_{\mathbf{p}_i}}}} M' \times_R \widehat{R}_{\mathbf{p}_i}$ and we observe that $(\phi')^{-1}_{f,\widehat{K}_{\mathbf{p}_i}} \circ (\widehat{\phi}')_{i,\widehat{K}_{\mathbf{p}_i}} = \phi^{-1}_{f,\widehat{K}_{\mathbf{p}_i}} \circ \widehat{\phi}_{i,\widehat{K}_{\mathbf{p}_i}}$. This shows that we defined actually a class map

$$\ker\Big(H^1_{\mathrm{Zar}}(R, \mathrm{GL}_r) \to H^1_{\mathrm{Zar}}(R_f, \mathrm{GL}_r)\Big) \to c_\Sigma(R, \mathrm{GL}_r). \tag{2.2}$$

Claim 2.11. *The class map* (2.2) *is injective.*

For the sequel we need only to know that it has trivial kernel. We consider only this special case and let the reader to deal with the general case. Indeed if $[M]$ belongs in the kernel, it means that we can adjust the trivializations in order to get $g_i = 1$ for

[1] We could do it over the local ring $R_{\mathbf{p}_i}$ but we want to emphasize the approach involving completions.

$i=1,\dots,c$. We claim that the isomorphism $\phi_f : M_f \xrightarrow{\sim} (R_f)^r$ extends (uniquely) to an isomorphism $M \xrightarrow{\sim} R^r$. The point is that $\phi_f \otimes_{R_f} \widehat{K}_{\mathbf{p}_i} : M_f \otimes_{R_f} \widehat{K}_{\mathbf{p}_i} \xrightarrow{\sim} (\widehat{K}_{\mathbf{p}_i})^r$ is extended from $\widehat{\phi}_i$ by base change from $\widehat{R}_{\mathbf{p}_i}$ to $\widehat{K}_{\mathbf{p}_i}$. It means that there are no denominators involved so that the map extends ϕ_f to an R-linear mapping $\psi : M^r \to R^r$. For the same reason $(\phi_f)^{-1}$ extends as well and we conclude that ϕ_f extends to an R-linear isomorphism $\psi : M^r \xrightarrow{\sim} R^r$.

We assume now that the determinant of $\mathbf{W}(M)$ is trivial. Using the diagram (2.1), it follows that (g_i) belongs by functoriality to the kernel of the map $\det_* : c_\Sigma(R, \mathrm{GL}_r) \to c_\Sigma(R, \mathbb{G}_m) = R_f^\times \backslash \prod_{j=1,\dots,c} (\widehat{K}_{\mathbf{p}_i}^\times / \widehat{R}_{\mathbf{p}_i}^\times)$. After changing the trivializations we can[2] then assume that $g_i \in \mathrm{SL}_r(\widehat{K}_{\mathbf{p}_i})$ for $i=1,\dots,c$. Since $\mathrm{SL}_r(\widehat{K}_{\mathbf{p}_i})$ is generated by elementary matrices [59, lemme 64] and since R_f is dense in $\prod_i \widehat{K}_{\mathbf{p}_i}$ by the strong approximation therorem 2.8, it follows that $\mathrm{SL}_r(R_f)$ is dense in $\prod_{i=1,\dots,c} \mathrm{SL}_r(\widehat{K}_{\mathbf{p}_i})$ (this goes by decomposing elements in a product of N elementary matrices for $N >> 0$). On the other hand, each group $\mathrm{SL}_r(\widehat{R}_{\mathbf{p}_i})$ is open (actually clopen) in $\mathrm{SL}_r(\widehat{K}_{\mathbf{p}_i})$ so that $c_\Sigma(R, \mathrm{SL}_r) = 1$. The Claim 2.11 enables us to conclude that $\mathbf{W}(M)$ is a trivial vector bundle. □

Remarks 2.12. (a) The general case is quite close; we need to apply the previous argument to $\mathrm{GL}(R^{r-1} \oplus I)$ for an invertible R–module I and strong approximation with respect to the R–group scheme $\mathrm{GL}(R^{r-1} \oplus I)$.

(b) In the case $r=1$, $c_\Sigma(R, \mathbb{G}_m) = \mathrm{Div}_\Sigma(R)/R_f^\times$ is isomorphic to $\ker\big(\mathrm{Pic}(R) \to \mathrm{Pic}(R_f)\big)$ [30, Theorem 6.2.4], in other words the class map is bijective. This is a general fact, i.e. the map of Claim 2.11 is surjective. This can be seen by using patching techniques (see [5, §6.2, D.4]).

(c) The density of $\mathrm{SL}_r(R_f)$ in $\prod_{i=1,\dots,c} \mathrm{SL}_r(\widehat{K}_{\mathbf{p}_i})$ is an example of strong approximation. This argument comes from Harder [35, Korollar 2.3.2] and is used further (see 5.1).

3. Zariski topology is not fine enough

The above definition of non–abelian Čech cohomology extends to an arbitrary group scheme. There are several complementary reasons for trying to extend this theory.

3.1. **The example of quadratic bundles.** A quadratic form on an R–module M is a map $q : M \to R$ which satisfies

(i) $q(\lambda x) = \lambda^2 q(x)$ for all $\lambda \in R$, $x \in M$.

(ii) The form $M \times M \to R$, $(x,y) \mapsto b_q(x,y) = q(x+y) - q(x) - q(y)$ is (symmetric) bilinear.

[2]we use the surjectivity of the determinant.

This concept is stable under arbitrary base change. The form q is *regular* if b_q induces an isomorphism $M \xrightarrow{\sim} M^\vee$. A fundamental example is the hyperbolic form $(V \oplus V^\vee, hyp)$ attached to a locally free R-module of finite rank defined by $hyp(v, \phi) \to \phi(v)$.

Suppose we are given a regular quadratic form (M, q) where M is locally free of rank r. It is tempting to make analogies with vector bundles and to use the orthogonal group scheme $O(q, M)$ which is a closed subgroup scheme of $\mathrm{GL}(M)$. More precisely, we have

$$O(q, M)(S) = \bigl\{ g \in \mathrm{GL}(M)(S) \mid q_S \circ g = q_S \bigr\}$$

for each R–ring S. For an open cover $\mathcal{U}$ of R we define $Z^1(\mathcal{U}/R, O(q, M))$ and $H^1(\mathcal{U}/R, O(q, M))$ in the same way as in section 2 (it makes sense actually for any R–group scheme). What we get is the following.

Lemma 3.1. *The set $H^1_{\mathrm{Zar}}(\mathcal{U}/R, O(q, M))$ classifies the isometry classes of regular quadratic forms (q', M') which are locally isomorphic over $\mathcal{U}$ to (q, M).*

Proof. Let $\mathcal{U} = (U_i)_{i \in I}$ be the open cover. We define a class map from the set $\mathcal{S}$ of isomorphism classes of regular quadratic forms (q', M') which are locally isomorphic over $\mathcal{U}$ to (q, M). Let (q', M') be a regular quadratic form such that $(q', M')_{U_i}$ is isometric to $(q, M)_{U_i}$ for each i. In other words we have trivialization maps $\phi_i : (q, M)_{U_i} \xrightarrow{\sim} (q', M')_{U_i}$ for each i. On $U_{i,j} = U_i \cap U_j$, we have $g_{i,j} = \phi_i^{-1} \phi_j \in O(q, M)(U_{i,j})$. This is a 1-cocycle, i.e. $g_{i,j} = g_{i,j} g_{j,k}$ on $U_{i,j,k} = U_i \cap U_j \cap U_k$. By taking into account the choices, we obtain a well-defined map $\mathcal{S} \to H^1_{\mathrm{Zar}}(\mathcal{U}/R, O(q, M))$. □

This is nice, but the point is that regular quadratic forms over R of dimension r have no reason to be locally isomorphic to (M, q) (e.g. this occurs already with $R = \mathbb{R}$, the field of real numbers). So the set $H^1_{\mathrm{Zar}}(R, O(q, M))$ is only a piece of what we would like to obtain.

Remark 3.2. The above dictionnary is an example of the so-called "yoga of forms" which is of general nature. See [32, §III.2.5] and [12, §2.2.4] and §4.6.(d).

3.2. **Functoriality.** If we have a map $f : G \to H$ of group schemes, we would like to have some control on the map $f_* : H^1_{\mathrm{Zar}}(R, G) \to H^1_{\mathrm{Zar}}(R, H)$.

A basic example is the Kummer map $f_d : \mathbb{G}_m \to \mathbb{G}_m$, $t \mapsto t^d$ for an integer d. It gives rise to the multiplication by d map on the Picard group $\mathrm{Pic}(R)$. In terms of invertible modules, it corresponds to the map $M \mapsto M^{\otimes d}$.

We would like to understand its kernel and its image. We can already say something about the kernel. Given $[M] \in \ker\bigl(\mathrm{Pic}(R) \xrightarrow{\times d} \mathrm{Pic}(R)\bigr)$, then there exists a trivialization $\theta : R \xrightarrow{\sim} M^{\otimes d}$. We then define the commutative group $A_d(R)$ of isomorphism classes of couples (M, θ) where M is an invertible R–module equipped with a trivialization $\theta : R \xrightarrow{\sim} M^{\otimes d}$. The multiplication rule is given by $(M,\theta)\,.\,(M',\theta') = (M \otimes_R M', \widetilde{\theta})$ where $\widetilde{\theta}$ is defined by the composite

$$R \xrightarrow{\sim} R^{\otimes 2} \xrightarrow{\theta \otimes \theta'} M^{\otimes d} \otimes_R M'^{\otimes d} = (M \otimes_R M')^{\otimes d}.$$

The trivial element is (R,θ_0) where $\theta_0 : R \xrightarrow{\sim} R^{\otimes d}$. We have a forgetful map $A_d(R) \to \mathrm{Pic}(R)$.

Lemma 3.2.1. *We have $d\,A_d(R) = 0$ and an exact sequence*

$$1 \to R^\times/(R^\times)^d \xrightarrow{\phi} A_d(R) \to \mathrm{Pic}(R) \xrightarrow{\times d} \mathrm{Pic}(R)$$

with $\phi(a) = [(R,\theta_a)]$ where $\theta_a : R \xrightarrow{\sim} R^{\otimes d} = R$, $x \mapsto ax$.

Proof. Given $[(M,\theta)] \in A_d(R)$, its d–power is $[(M^{\otimes d},\theta_d)]$ where

$$\theta_d : R \xrightarrow{\sim} R^{\otimes d} \xrightarrow{\theta^{\otimes d}} (M^{\otimes d})^{\otimes d} = M^{\otimes d^2}.$$

It follows that $(M^{\otimes d},\theta_d)$ is isomorphic to (R,θ_0).

Next assume that $\phi(a) = [(R,\theta_a)]) = 0 \in A_d(R)$, that is, there exists an isomorphism $\phi : R \xrightarrow{\sim} R$ of R–modules such that $\phi_*\theta_0 = \theta_a$. The map ϕ is the multiplication by an unique $b \in R^\times$ and we have $b^d = a$. The injectivity of the first map is established.

Clearly the sequence $R^\times/(R^\times)^d \xrightarrow{\phi} A_d(R) \to \mathrm{Pic}(R)$ is a complex, let us prove its exactness. We are given (M,θ) such that $R \cong M$ so that we can deal with (R,θ). Then $\theta : R \xrightarrow{\sim} R^{\otimes d} = R$ is given by $a \in R^\times$. Therefore $(R,\theta) = (R,\theta_a)$. Finally the exactness at $\mathrm{Pic}(R)$ is obvious. □

We will see later that we can provide a cohomological meaning to the group $A_d(R)$ (Remark 4.12).

4. General definitions

Grothendieck-Serre's idea is to extend the notion of covers in algebraic geometry [53]. They did it originally with étale covers (to be discussed in §4.8) but it turns out that the flat cover setting is simpler in a first approach. This is the setting of the book by Demazure-Gabriel [18, §III], and there are variants.

4.1. Non-abelian Čech cohomology.

Definition 4.1. *A flat*[3] *cover of R is a finite collection $(S_i)_{i\in I}$ of R–rings satisfying*

(i) S_i is a flat R–algebra of finite presentation for $i \in I$;

(ii) $\mathrm{Spec}(R) = \bigcup_{i\in I} \mathrm{Im}\Big(\mathrm{Spec}(S_i) \to \mathrm{Spec}(R)\Big)$.

If we put $S = \prod_{i\in I} S_i$, the conditions rephrase by saying that S is a faithfully flat R–algebra of finite presentation. We can therefore always deal with a single ring.

[3] or fppf= fidèlement plat de présentation finie.

Remark 4.2. *For a partition $1 = f_1 + \cdots + f_n$, the family $(R_{f_j})_{j=1,\dots,n}$ is a flat cover of R and so is $R_{f_1} \times \cdots \times R_{f_n}$.*

We define now Čech non-abelian cohomology. Let S be a faithfully flat R-algebra of finite presentation. We denote by $p_i^* : S \to S \otimes_R S$ the coprojections $(i = 1, 2)$ and similarly $q_{i,j}^* : S \otimes_R S \to S \otimes_R S \otimes_R S$ the partial coprojections $(i < j)$.

Let G be an R–group scheme. A 1-*cocycle* for G and S/R is an element $g \in G\big(S \otimes_R S\big)$ satisfying

$$q_{1,2}^*(g)\, q_{2,3}^*(g) = q_{1,3}^*(g) \quad \in \quad G\big(S \otimes_R S \otimes_R S\big).$$

We denote by $\mathrm{Z}^1(S/R, G)$ the pointed set of 1-cocycles of S/R with values in G (it is pointed by the trivial 1–cocycle).

Two such cocycles $g, g' \in G(S)$ are *cohomologous* if there exists $h \in G(S)$ such that $g = p_1^*(h^{-1})\, g'\, p_2^*(h)$. We denote by $\check{\mathrm{H}}^1(S/R, G) = \mathrm{Z}^1(S/R, G)/\sim$ the pointed set of 1-cocycles up to cohomology equivalence.

Remark 4.3. In the case of a Zariski cover given by a partition of 1, the definition is the same as in §3.1. What lies behind this, is the fact that intersection of open subschemes is a special case of fiber product.

We can pass to the limit on all flat covers of $\mathrm{Spec}(R)$ and define $\check{\mathrm{H}}^1_{fppf}(R, G) = \varinjlim \check{\mathrm{H}}^1(S/R, G)$ [4]. This construction is functorial in R and in the group scheme G.

4.2. **Torsors.** A (right) G–torsor X (with respect to the flat topology) is an R-scheme equipped with a right action of G which satisfies the following properties:

(i) the action map $X \times_R G \to X \times_R X$, $(x, g) \mapsto (x, x.g)$, is an isomorphism;

(ii) There exists a flat cover R'/R such that $X(R') \neq \emptyset$.

The first condition reflects the simple transitivity of the action, i.e. $G(T)$ acts simply transitively on $X(T)$ for all R–rings T. The second condition is a local triviality condition. An example is $X = G$ with G acting by right translations, it is called the split G–torsor. A morphism of G-torsors $X \to Y$ is a G-equivariant morphism.

If $X(R) \neq \emptyset$, a point $x \in X(R)$ defines a morphism $G \to X$, $\phi_x : g \mapsto x.g$, which is G-equivariant (with respect to the right translation on G). The simply transitive property implies that ϕ_x is an isomorphism of R-schemes; we say that X is trivial and that ϕ_x is a trivialization.

Condition (ii) rephrases that an R–torsor X under G is locally trivial for the flat topology.

A morphism of G–torsors $X \to Y$ is a G–equivariant map.

Lemma 4.4. *A morphism of G-torsors $f : X \to Y$ is an isomorphism.*

[4]There are set-theoretic issues there allowing us to consider this limit, see [4, Remarque 1.4.3] and [63] for the fpqc setting.

Proof. Let R' be a flat cover of R which splits X. Let x' be an element of $X(R')$ and let y' be its image by f. Then we have a commutative square

$$\begin{array}{ccc} G_{R'} & \xrightarrow[\sim]{\phi_{x'}} & X_{R'} \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f_{R'}} \\ G_{R'} & \xrightarrow[\sim]{\phi_{y'}} & Y_{R'}. \end{array}$$

where the horizontal maps are orbit maps. As we have noticed before, the orbit maps are isomorphisms so that $f_{R'}$ is an isomorphism. We want to deduce that f is an isomorphism as well. Assume first by simplicity that X and Y are affine R–schemes (which holds if G is affine). We deal then with a morphism of R-rings $f^* : R[Y] \to R[X]$ such that $f^* \otimes_R R'$ is an isomorphism. It is then an isomorphism in view of a basic property of faithfully flat modules [6, I.3, §1, Proposition 2]. In the non affine case, we appeal to [20, 2.7.1.(viii)]. □

Thus the category of G–torsors is a groupoid and the above reasoning is a first step in descent arguments. The R–functor of automorphisms of the trivial G–torsor G is representable by G (acting by left translations).

We denote by $H^1_{fppf}(R, G)$ the set of isomorphism classes of G-torsors for the flat topology. If S is a flat cover of R, we denote by $H^1_{fppf}(S/R, G)$ the subset of isomorphism classes of G-torsors trivialized over S.

As in the vector bundle case, we shall construct a class map $\gamma : H^1_{fppf}(S/R, G) \to \check{H}^1_{fppf}(S/R, G)$ as follows.

Let X be a G–torsor over R equipped with a trivialization $\phi : G \times_R S \xrightarrow{\sim} X \times_R S$. Over $S \otimes_R S$, we then have two trivializations $p_1^*(\phi) : G \times_R (S \otimes_R S) \xrightarrow{\sim} X \times_R (S \otimes_R S)$ and $p_2^*(\phi)$. It follows that $p_1^*(\phi)^{-1} \circ p_2^*(\phi)$ is an automorphism of the trivial G–torsor over $S \otimes_R S$, so is the left translation by an element $g \in G(S \otimes_R S)$. A computation shows that g is a 1–cocycle [25, §2.2]; also changing ϕ changes g by a cohomologous cocycle. The class map is then well-defined. Its study involves a gluing technique in the flat setting.

4.3. **Interlude: Faithfully flat descent.** Let T be a faithfully flat extension of the ring R (not necessarily of finite presentation). We put $T^{\otimes d} = T \otimes_R T \cdots \otimes_R T$ (d times). One first important thing is that the Amitsur complex

$$0 \to M \to M \otimes_R T \xrightarrow{d_1} M \otimes_R T \otimes_R T \xrightarrow{d_2} M \otimes_R T^{\otimes 3} \dots$$

is exact for each R–module M [38, III.1] where

$$d_n(m \otimes t_1 \otimes \cdots \otimes t_n) = \sum_{i=0,\dots,n} (-1)^i \, m \otimes t_1 \otimes \cdots \otimes t_i \otimes 1 \otimes t_{i+1} \otimes \cdots \otimes t_n.$$

In the case $M = R$, this implies in particular that for any affine R-scheme X, we have an exact sequence

$$0 \to \mathrm{Hom}_R(R[X], R) \to \mathrm{Hom}_R(R[X], T) \xrightarrow{d_1} \mathrm{Hom}_R(R[X], T \otimes_R T)$$

In other words we have an identification

$$X(R) = \bigl\{ x \in X(T) \mid p_1^*(x) = p_2^*(x) \in X(T \otimes_R T) \bigr\}.$$

This holds actually for any R-scheme Y, in other words, the R-functor h_Y (defined by $h_Y(T) = Y(T)$ for each R-ring T) is a flat sheaf [62, Tag 023Q].

Given a T–module N we consider the $T \otimes_R T$–modules $p_1^*(N) = T \otimes_R N$ and $p_2^*(N) = N \otimes_R T$. A descent datum on N is an isomorphism $\varphi : p_1^*(N) \xrightarrow{\sim} p_2^*(N)$ of $T^{\otimes 2}$–modules such that the diagram

$$(4.1) \qquad \begin{array}{ccccc} T \otimes_R T \otimes_R N & & \xrightarrow{\quad\varphi_2\quad} & & N \otimes_R T \otimes_R T \\ & \searrow^{\varphi_1} & & \nearrow^{\varphi_3} & \\ & & T \otimes_R N \otimes_R T & & \end{array}$$

is commutative where φ_i is obtained by extending φ with $\otimes_R T$ at the position i. For example we have $\varphi_1(t_1 \otimes t_2 \otimes n) = t_1 \otimes \varphi(t_2 \otimes n)$.

There is an obvious notion of morphisms for T-modules equipped with a descent datum from T to R. If M is an R–module, the identity of M gives rises to a canonical isomorphism $can_M : p_1^*(M \otimes_R T) \xrightarrow{\sim} p_2^*(M \otimes_R T)$, this is a descent datum.

Theorem 4.5. *(Faithfully flat descent, see* [38, III, Theorem 2.1.2] *)*

(1) The functor $M \to (M \otimes_R T, can_M)$ is an equivalence of categories between the category of R–modules and that of T–modules with descent datum. An inverse functor (the descent functor) is $(N, \varphi) \mapsto \{ n \in N \mid n \otimes 1 = \varphi(1 \otimes n) \}$.

(2) The functor above induces an equivalence of categories between the category of R–algebras (commutative, unital) and that of T-algebras (commutative, unital) with descent datum.

For (2), we need to explain what we mean by a descent datum on a T-algebra B. This is an isomorphism of $T^{\otimes 2}$-*algebras* $\varphi : T \otimes_R B \xrightarrow{\sim} B \otimes_R T$ which satisfies the analogous rule of (4.1). For an exhaustive view, we recommend [62, Tag 023F]. We shall later see examples of descent beyond the case of Zariski covers (e.g. 4.16).

4.4. **The linear case.** An important example is the extension of Swan-Serre's correspondence. A consequence of the faithfully flat descent theorem (and of the fact that the property of being locally free of rank r is local for the flat topology [62, Tag 05B2], [38, III.2.8]) is the following.

Theorem 4.6. *Let $r \geq 0$ be an integer.*

(1) Let M be a locally free R–module of rank r. Then the R–functor $S \mapsto \mathrm{Isom}_{S-mod}(S^r, M \otimes_R S)$ is representable by a GL_r–torsor X^M over $\mathrm{Spec}(R)$.

(2) The functor $M \mapsto X^M$ induces an equivalence of categories between the groupoid of locally free R–modules of rank r and the category of GL_r–torsors over $\mathrm{Spec}(R)$.

Proof. See [12, 2.4.3.1]. This reference is for Zariski topology and étale topology but works for the flat topology in view of the upcoming Proposition 4.14. □

This implies that the GL_r–torsors are the same with flat topology or with Zariski topology.

Corollary 4.7. *(Hilbert-Grothendieck 90) We have $H^1_{\mathrm{Zar}}(R, \mathrm{GL}_r) = H^1_{fppf}(R, \mathrm{GL}_r)$. In particular, if R is a local (or semilocal) ring, we have $H^1_{fppf}(R, \mathrm{GL}_r) = 1$.*

This is a special case of a more general statement which holds for $\mathrm{GL}_1(\mathcal{B})$ where $\mathcal{B}$ is an Azumaya R–algebra see [27, §4.2]. More generally, it holds for a separable R–algebra (for example Azumaya or finite étale) which is a locally free R–module of finite rank.

4.5. **Torsors, cocycles, and twists.**

Lemma 4.8. *The map $\gamma : H^1_{fppf}(S/R, G) \to \check{H}^1_{fppf}(S/R, G)$ is injective.*

Proof. Once again we limit ourselves to the kernel for simplicity (for the general argument, see [25, §2.2]). If (X, ϕ) gives rise to a cocycle which is cohomologous to the trivial cocycle, it means that there exists a trivialization $\phi' : G \times_R S \xrightarrow{\sim} X \times_R S$ such that the associated cocycle is trivial. We put $x = \phi'(1) \in X(S)$. Then $p_1^*(x) = p_2^*(x) = 1$. Since $X(R)$ identifies with $\{x \in X(S) \mid p_1^*(x) = p_2^*(x)\}$, we conclude that $X(R)$ is non-empty. □

Theorem 4.9. *If G is affine, the class map $H^1_{fppf}(S/R, G) \to \check{H}^1_{fppf}(S/R, G)$ is bijective.*

Note that by passing to the limit on the flat covers, we get a bijection $H^1_{fppf}(R, G) \to \check{H}^1_{fppf}(R, G)$. The fact that we can descend torsors under an affine group scheme is a consequence of the faithfully flat descent theorem. The sketch is as follows where we denote by $R[G]$ the coordinate ring of G. We are given a cocycle $g \in G(S \otimes_R S)$, recall that it satisfies $q_{1,2,*}(g)\, q_{2,3,*}(g) = q_{1,3,*}(g) \in G(S \otimes_R S \otimes_R S)$. We consider the map $L_g^* : (S \otimes_R S)[G] \xrightarrow{\sim} (S \otimes_R S)[G]$ (where $L_g : G_{S \otimes_R S} \to G_{S \otimes_R S}$ is the left multiplication by g). Define φ_g by the diagram

$$\begin{array}{ccc} S \otimes_R S[G] & \xrightarrow[\sim]{\varphi_g} & S[G] \otimes_R S \\ \cong \downarrow \alpha & & \cong \downarrow \beta \\ (S \otimes_R S)[G] & \xrightarrow[\sim]{L_g^*} & (S \otimes_R S)[G] \end{array}$$

where $\alpha(s_1 \otimes f) = (s_1 \otimes 1) p_2^*(f)$ and $\beta(f \otimes s_2) = p_1^*(f)(1 \otimes s_2)$. The cocycle condition implies that φ_g is a descent datum for the S-algebra $S[G]$. Theorem 4.5 defines an R–algebra $R[X]$ and X is actually a G-torsor denoted by E_g.

This construction is a special case of *twisting*. Again we are given a cocycle $g \in G(S \otimes_R S)$, it satisfies $q_{1,2,*}(g)\, q_{2,3,*}(g) = q_{1,3,*}(g) \in G(S \otimes_R S \otimes_R S)$. More generally, we consider an affine R-scheme Y equipped with a left action of G. We denote by $\underline{\mathrm{Aut}}(Y)$ the fppf R-sheaf (in groups) of automorphisms of Y defined by $T \mapsto \underline{\mathrm{Aut}}_T(Y_T)$ and the action is nothing but a homomorphism of R-sheaves in groups $a : G \to \underline{\mathrm{Aut}}(Y)$. We can deal then with the automorphism $\varphi_g = a(g) \in \underline{\mathrm{Aut}}_{S \otimes_R S}(Y_{S \otimes_R S})$ and the point is that $\varphi_g : Y \times_R (S \otimes_R S) \xrightarrow{\sim} Y \times_R (S \otimes_R S)$ defines a descent datum. This gives rise to the twist of Y_g of Y by the 1-cocycle g. The scheme Y_g is affine over R.

A special case is the action of G on itself by inner automorphisms, twisted R-group scheme G_g carries a natural structure of R-group schemes and it called the twisted R-group scheme by the 1-cocycle g. The R–group scheme G_g acts on Y_g for Y as above. In particular it acts on E_g so that we have a map of R-sheaves in groups

$$\rho_g : G_g \to \underline{\mathrm{Aut}}_G(E_g)$$

where the right handside stands for the fppf sheaf of G-equivariant isomorphisms of E_g. In the case of the trivial cocycle, this is the map $\rho_0 : G \to \underline{\mathrm{Aut}}_G(E_0)$ arising from the left translation on the trivial G–torsor $E_0 = G$. Since ρ_0 is an isomorphism, ρ_g is locally (for fppf) an isomorphism so is an isomorphism. It implies that $\underline{\mathrm{Aut}}_G(E_g)$ is representable by an affine R–group scheme and that it could be taken as an alternative definition for the twisted R-group scheme G_g.

Remarks 4.10. (a) The above construction does not depend on choices of trivializations. It can be abstracted as follows. We can define for a G–torsor E the twist ${}^E Y$ and ${}^E G$ by means of contracted products as in [32, III.1, III.2.3]. For short the contracted product $E \wedge^G Y$ is the fppf quotient sheaf of $E \times_R Y$ by the left (free) action of G given by $g.(e, y) = (e.g^{-1}, g.y)$; if Y is affine over R, this sheaf is representable by an R-scheme which is nothing but ${}^E Y$.

(b) In practice, the affineness assumption in Theorem 4.5 is too strong. More generally we can twist G–schemes equipped with an ample invertible G-linearized bundle, see [5, §6, 7 and §10, lemma 6] for details. Another case due to Gabber is that of ind-quasi-affine schemes [62, Tag 0APK].

4.6. **Examples.** *(a) Vector group schemes.* Let M be a locally free R–module of finite rank, we claim that $\check{H}^1(R, \mathbf{W}(M)) = 0$ so that each $\mathbf{W}(M)$–torsor is trivial. We are given a flat cover S/R. Since the complex

$$M \otimes_R S \xrightarrow{p_1^* - p_2^*} M \otimes_R S \otimes_R S \to M \otimes_R S \otimes_R S \otimes_R S$$

is exact, each cocycle $g \in \mathbf{W}(M)(S \otimes_R S) = M \otimes_R S \otimes_R S$ is a coboundary. Thus $\check{H}^1(S/R, \mathbf{W}(M)) = 0$ and $\check{H}^1(R, \mathbf{W}(M)) = 0$.

(b) An important case is $G = \Gamma_R$, that is, the *finite constant group scheme* attached to an abstract finite group Γ. Recall that $G(S)$ is the group of locally constant functions $\mathrm{Spec}(S) \to \Gamma$. In other words, $G = \coprod_{\gamma \in \Gamma} \mathrm{Spec}(R)_\gamma$ so that its coordinate ring identifies with $R^{(\Gamma)}$.

In this case a Γ_R–torsor $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is the same thing as a Galois Γ–algebra S and is called often a Galois cover[5]. A special case is that of a finite Galois extension L/k of fields of Galois group Γ.

(c) As with GL_r, a special nice case is the case of *forms*, that is when G is the automorphism group of some algebraic structure, see [12, §2.2.3] for an exhaustive discussion.

For example, the orthogonal group scheme O_{2n} is the automorphism group of the hyperbolic quadratic form attached to R^{2n}. As regular quadratic forms of rank $2n$ are locally isomorphic to the hyperbolic form in the flat topology, descent theory provides an equivalence of categories between the groupoid of regular quadratic forms of rank $2n$ and the category of O_{2n}-torsors. This is what we wanted in §3, that is, $H^1(R, O_{2n})$ classifies the isomorphism classes of regular quadratic R–forms of rank $2n$ [18, III.5.2].

(d) Another important example is that of the symmetric group S_n. For any R–algebra S, the group $S_n(S)$ is the automorphism group of the S–algebra $S^n = S \times \cdots \times S$ (n–times). Since finite étale algebras of degree n are locally isomorphic to R^n for the étale topology, the same yoga shows that there is an equivalence of categories between the category of S_n–torsors and that of finite étale R–algebras of rank n.

The functor which associates to a finite étale R–algebra of rank n a S_n–torsor is defined by descent but can be described explicitely. This is the Galois closure construction done by Serre in [53, §1.5], see also [3].

4.7. **Functoriality issues.** Let $G \to H$ be a monomorphism of R–group schemes. We say that an R-scheme X equipped with a map $f : H \to X$ is a *flat quotient* of H by G if for each R–algebra S the map $H(S) \to X(S)$ induces an injective map $H(S)/G(S) \hookrightarrow X(S)$ and if for each $x \in X(S)$, there exists a flat cover S' of S such that $x_{S'}$ belongs to the image of $H(S') \to X(S')$ (we say that f is "couvrant" in French). If it exists, a flat quotient is unique (up to unique isomorphism); furthermore, if G is normal in H, then X carries a natural structure of R–group schemes, we say in this case that $1 \to G \to H \to X \to 1$ is an exact sequence of R–group schemes (for the flat topology).

[5]This is our convention for Galois covers which has the advantage to be stable for base change. In [51], one requires furthermore R, S to be connected.

Lemma 4.11. *Assume that the R–scheme X is the flat quotient of H by G.*

(1) The map $H \to X$ is a G–torsor.

(2) There is an exact sequence of pointed sets

$$1 \to G(R) \to H(R) \to X(R) \xrightarrow{\varphi} H^1_{fppf}(R,G) \to H^1_{fppf}(R,H)$$

where $\varphi(x) = [f^{-1}(x)]$.

For the proof, see [18, III.4.2, Corollary 1.8 and III.4.4].

Remark 4.12. (a) Assume that X is affine (or is equipped with an ample G-linearized invertible sheaf, see [5, §6.1, Theorem 7 and §10, Lemma 6] for details). Then the category of G–torsors over $\mathrm{Spec}(R)$ is equivalent to the category of couples (F,x) where F is an H-torsor and $x \in ({}^F X)(R)$ (where ${}^F X$ is the twist of X by the H–torsor F).

(b) If G is normal in H, then X has natural structure of an R-group scheme. In this case (a) rephrases by saying that the category of G–torsors over $\mathrm{Spec}(R)$ is equivalent to the category of couples (F,ϕ) where F is an H–torsor and ϕ a trivialization of the X–torsor ${}^F X$.

(c) The sequence $1 \to \mathrm{SL}_r \to \mathrm{GL}_r \to \mathbb{G}_m \to 1$ is an exact sequence of R–group schemes. Using the extended Swan-Serre correspondence 4.6, an example of (b) is that the category of SL_r-torsors is equivalent to the category of pairs (M,θ) where M is a locally free R–module of rank r and $\theta : R \xrightarrow{\sim} \Lambda^r(M)$ is a trivialization of the determinant of M.

(d) For an integer d, we have the Kummer exact sequence $1 \to \mu_d \to \mathbb{G}_m \xrightarrow{\times d} \mathbb{G}_m \to 1$. Similarly the category of μ_d–torsors is equivalent to the category of pairs (M,θ) where M is an invertible R–module and $\theta : R \xrightarrow{\sim} M^{\otimes r}$ a trivialization. This is related with §3.2.

Examples 4.7.1. $\mathbb{G}_m$ is the flat quotient of GL_r by SL_r and $\mathbb{G}_m$ is the flat quotient of $\mathbb{G}_m$ by μ_d.

There are of course many more functorial properties for example when G is commutative. In this case, $H^1(R,G)$ is equipped with a natural structure of an abelian group arising from the product morphism $G \times_R G \to G$.

4.8. **Étale covers.** We remind to the reader that an étale morphism of rings $R \to S$ is a smooth morphism of relative dimension zero [41, §I.3]. There are several alternative definitions, for example, S is a flat R–algebra of finite presentation such that for each R–field F, then $S \otimes_R F$ is an étale F–algebra (i.e. a finite geometrically reduced F–algebra).

Examples 4.13. *(a) A localization morphism $R \to R_f$ is étale.*

(b) If d is invertible in R, the Kummer morphism $\mathbb{G}_m \to \mathbb{G}_m$, $t \mapsto t^d$ is étale.

(c) More generally, if d is invertible in R and $r \in R^\times$, then $S = R[x]/(x^d - r)$ is a finite étale R–algebra.

Let G be an R-group scheme. We denote by $\check{H}^1_{fppf}(R,G)$ the Čech non-abelian cohomology set defined by means of cocycles in §4.1.

We define the subset $\check{H}^1_{ét}(R,G)$ of $\check{H}^1_{fppf}(R,G)$ of classes which arises from cocycles supported by étale covers. We define similarly $H^1_{ét}(R,G)$ from $H^1_{ét}(R,G)$.

Proposition 4.14. *If G is affine smooth, then we have $H^1_{ét}(R,G) = H^1_{fppf}(R,G) \xrightarrow{\sim} \check{H}^1_{fppf}(R,G)$.*

Proof. The right handside bijection is Theorem 4.9. Smoothness is a local property with respect to the flat topology [20, Corollaire 17.7.3.(ii)] so that any G-torsor E is smooth over R. According to the existence of quasi-sections [20, 17.16.3], E admits locally sections with respect to the étale topology. □

4.9. **Isotrivial torsors and Galois cohomology.** We are given a Galois R–algebra S of group Γ. The action isomorphism $\mathrm{Spec}(S) \times_R \Gamma_R \xrightarrow{\sim} \mathrm{Spec}(S) \times_R \mathrm{Spec}(S)$ can be viewed as an isomorphism $S \otimes_R S \xrightarrow{\sim} S \otimes_R R^{(\Gamma)} = S^{(\Gamma)}$. A 1-cocycle is then an element $z = (z_\gamma)_{\gamma \in \Gamma} \in G(S \otimes_R S) = G(S)^{(\Gamma)}$ satisfying a certain relation.

Since Γ acts on the left on S, it acts as well on the left on $G(S)$.

Lemma 4.15. *(see [25, lemme 2.2.3]) A Γ-tuple $z = (z_\sigma)_{\sigma \in \Gamma} \in G(S^{(\Gamma)}) = G(S)^{(\Gamma)} = \mathrm{Hom}_{sets}(\Gamma, G(S))$ is a 1–cocycle for S/R if and only if*

$$z_{\sigma\tau} = z_\sigma \, \sigma(z_\tau)$$

for all $\sigma, \tau \in \Gamma$.

We find that $Z^1(S/R,G)$ is the set of Galois cocycles $Z^1(\Gamma, G(S))$ and that $\check{H}^1(S/R,G)$ is the set of non-abelian Galois cohomology $H^1(\Gamma, G(S)) = Z^1(\Gamma, G(S))/\sim$ where two cocycles z, z' are cohomologous if $z_\gamma = g^{-1} \, z'_\gamma \, \gamma(g)$ for some $g \in G(S)$.

An interesting case is that of a constant group scheme G associated to an abstract group Θ and S is connected. In this case, we have $Z^1(S/R,G) = \mathrm{Hom}_{gp}(\Gamma, \Theta)$ and $\check{H}^1(S/R,G) = \mathrm{Hom}_{gp}(\Gamma,\Theta)/\Theta$.

Remark 4.16. Galois descent is therefore a special case of faithfully flat descent. The reader can check that the category of R-modules is equivalent to the category of couples (N,ρ) where N is an S-module equipped with a semilinear action of Γ (i.e. $\rho(\sigma)(\lambda \,.\, n) = \sigma(\lambda) \,.\, \rho(\sigma)(n)$). A reference is [29, §14.20].

We say that a torsor E under an R–group scheme G is isotrivial if it is split by a finite étale cover (which can be assumed Galois by taking the Galois closure). This is subclass of torsors which can be explicitly studied by Galois cohomology computations. It is often a preliminary question to decide whether a given torsor is isotrivial. For example, for the ring of Laurent polynomials in characteristic zero and a reductive group scheme, this is the case [27].

5. Torsors over affine curves

5.1. **The Dedekind case.** Let R be a Dedekind ring with fraction field K. Let $f \in R$ and put $\Sigma = \mathrm{Spec}(R) \setminus \mathrm{Spec}(R_f) = \{\mathbf{p}_1, \dots, \mathbf{p}_c\}$, and use the notation of the proof of Corollary 2.10. Let G be an affine flat R–group scheme. As in the proof of 2.10 we have a class map

$$\begin{aligned}(5.1)\qquad &\ker\Big(H^1_{fppf}(R,G) \to H^1_{fppf}(R_f,G) \times \prod_{i=1,\dots,c} H^1_{fppf}(\widehat{R}_{\mathbf{p}_i},G)\Big)\\ &\qquad\longrightarrow \quad c_\Sigma(R,G) = G(R_f)\backslash \prod_{i=1,\dots,c} G(\widehat{K}_{\mathbf{p}_i})/G(\widehat{R}_{\mathbf{p}_i}).\end{aligned}$$

This map is injective [35, §2.3], this generalizes the GL_n case established in 2.11.

Remark 5.1. As already mentioned in Remark 2.12.(b), the surjectivity of the class is a general fact obtained by using patching techniques (see [5, §6.2, D.4]). We will not use that in the sequel.

The next results are due to Harder [35, Corollary 2.3.2 and Satz 3.3].

Corollary 5.2. *If $c_\Sigma(R,G)=1$ (in particular if $G(R_f)$ is dense in $\prod_{j=1,\dots,c} G(\widehat{K}_{\mathbf{p}_i})$), we have $\ker\Big(H^1_{fppf}(R,G) \to H^1_{fppf}(R_f,G) \times \prod_{i=1,\dots,c} H^1_{fppf}(\widehat{R}_{\mathbf{p}_i},G)\Big) = 1$.*

We examine now more closely the reductive case.

Proposition 5.3. *Assume that G is a reductive split R–group scheme and let (B,T) be a Killing couple, i.e. T is a maximal split R-torus of G and B an R–Borel subgroup scheme containing it.*

(1) The sequence of pointed sets

$$H^1_{fppf}(R,T) \to H^1_{fppf}(R,G) \to H^1_{fppf}(K,G)$$

is exact.

(2) We have $H^1_{\mathrm{Zar}}(R,G) = \ker\Big(H^1_{fppf}(R,G) \to H^1_{fppf}(K,G)\Big)$.

(3) If G is simply connected, then $\ker\Big(H^1_{fppf}(R,G) \to H^1_{fppf}(K,G)\Big) = 1$ and $H^1_{\mathrm{Zar}}(R,G)=1$.

At this stage we need to explain the vocabulary for reductive (resp. semisimple) algebraic groups and also for group schemes. A reference is [16, §1.5 and Exercise 6.5.2].

- A smooth connected affine algebraic group G defined over an algebraically closed field k is reductive (resp. semisimple) if 1 is the only smooth connected k-subgroup which is normal and unipotent (resp. normal and solvable). Semisimple simply connected here is more complicated; in characteristic zero this is equivalent to say that G

is semisimple simply connected for Grothendieck's theory [51] of finite étale covers[6]. Examples of semisimple simply connected algebraic groups are SL_n, Sp_{2n}, Spin_n.

• A smooth affine group scheme G over a ring R is *reductive* (resp. semisimple, resp. semisimple simply connected) if each geometric fiber $G_{\overline{s}}$ is reductive (resp. semisimple, resp. semisimple simply connected).

• [16, 5.1.1] Let G be a reductive group scheme over a connected ring R. It is *split* if there exists a maximal torus $T \cong \mathbb{G}_m^r$ such that each root space $\mathrm{Lie}(G)_a$ for $a \in \widehat{T}$ is free of rank 1 over R. It admits a Borel R-subgroup scheme (i.e. a closed smooth R-subgroup whose geometric fibers are Borel subgroups) containing T [52, XXII.5.1.1].

We proceed now to the proof of Proposition 5.3.

Proof. (1) Since $H^1_{fppf}(K,T) = 1$ (Hilbert 90), the sequence $H^1_{fppf}(R,T) \to H^1_{fppf}(R,G) \to H^1_{fppf}(K,G)$ is a complex of pointed sets. In order to establish the exactness, we claim first that the map

$$H^1(R,B) \to \ker\big(H^1_{fppf}(R,G) \to H^1_{fppf}(K,G)\big)$$

is onto. Let E be an R-torsor under G which becomes trivial over K. We admit that the fppf sheaf G/B is representable by a projective R-scheme [52, XXVI.1.2]. The idea is to introduce the twisted R-scheme $Y = {}^E(G/B)$ (it is the scheme of Borel subgroups of the twisted R–group scheme ${}^EG$ so is projective over R [52, XXVI, Théorème 3.3]). Since E_K is trivial we have $Y(K) \neq \emptyset$. Next we have $Y(R) = Y(K)$ in view of the valuative criterion of properness. It follows that Y has an R–point (equivalently E^G carries an R-Borel subgroup scheme). According to Remark 4.12.(a), it follows that $[E]$ belongs to the image of $H^1(R,B) \to H^1(R,G)$.

We have $B = U \rtimes T$ where U admits a T–equivariant filtration $U_0 = 1 \subset \cdots \subset U_r = U$ such that U_{i+1}/U_i is isomorphic to the commutative unipotent R–group $(\mathbb{G}_a)^{l_i}$. Since $H^1_{fppf}(R,\mathbb{G}_a) = 1$ (Example 4.6.(a)), a dévissage argument shows that the map $H^1_{fppf}(R,T) \to H^1_{fppf}(R,B)$ is bijective[7]. We conclude that $[E]$ belongs to the image of $H^1_{fppf}(R,T) \to H^1_{fppf}(R,G)$.

(2) Taking an isomorphism $T \cong \mathbb{G}_m^r$, we have $H^1_{fppf}(R,T) \cong \mathrm{Pic}(R)^r$. In view of (1), we have $\ker\big(H^1_{fppf}(R,G) \subset H^1_{fppf}(K,G)\big) \subseteq H^1_{\mathrm{Zar}}(R,G)$. The converse inclusion is obvious.

(3) We assume now that G is semisimple simply connected. We are given $[E] \in H^1_{\mathrm{Zar}}(R,G) = \ker\big(H^1_{fppf}(R,G) \to H^1_{fppf}(K,G)\big)$. From (2), there exists $f \in R$ such that E_{R_f} is trivial as well as $E_{R_{\mathfrak{p}_i}}$ for the maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_c \in \Sigma = \mathrm{Spec}(R) \setminus \mathrm{Spec}(R_f)$. It makes then sense to consider the class map of $[E]$ in $c_\Sigma(R,G)$.

Claim 5.1.1. $c_\Sigma(R,G) = 1$.

[6] In particular, for the field of complex numbers, this notion coincide with the topological one.

[7] this is a general fact, see [52, XXVI.2.3].

The Claim and Corollary 5.2 implies that $[E] = 1 \in H^1_{fppf}(R,G)$ as desired. To establish the Claim, we consider an opposite Borel R–subgroup B^- to B, i.e. $T = B \cap B^-$ [16, Proposition 5.2.12]. We denote by U^- its unipotent radical. Since each $G(\widehat{K}_{p_i})$ is generated by $U^+(\widehat{K}_i)$ and $U^-(\widehat{K}_i)$ [59, lemma 64] and since U^+ (resp. U^-) is isomorphic as R–scheme to $\mathbb{A}^n$, we have that $U^+(R_f)$ is dense in $\prod_{i=1} U^+(\widehat{K}_{\mathfrak{p}_i})$ and similarly for U^-. It follows that $G(R_f)$ is dense in $\prod_{i=1} G(\widehat{K}_{\mathfrak{p}_i})$ whence the Claim. □

We find then one more time that $H^1_{\mathrm{Zar}}(R,\mathrm{SL}_n) = 1$ but get for example also that $H^1_{\mathrm{Zar}}(R,E_8) = 1$ where E_8 stands for the split group of type E_8. Since $\mathrm{Pic}(k[t]) = 0$ for a field k, it follows that $H^1_{\mathrm{Zar}}(k[t],G) = 1$ for a semisimple split simply connected k-group G.

Remark 5.4. Proposition 5.3.(2) holds for an arbitrary reductive R-group scheme G, this is a result of Nisnevich [44], see also [34]. Furthermore by taking into account Remark 5.1 on the surjectivity of the class map, we get that the class map (5.1) induces a bijection $H^1_{\mathrm{Zar}}(R,G) \xrightarrow{\sim} c_\Sigma(R,G)$.

5.2. **Affine curves over an algebraically closed field.**

Theorem 5.5. *Let G be a semisimple algebraic k–group where k is an algebraically closed field. Let C be a smooth connected affine curve. Then $H^1_{fppf}(C,G) = 1$.*

Note that such a G is necessarily split. A slightly more general version is available in [15, §3]. One first ingredient is Steinberg's theorem.

Theorem 5.6. [58, Theorem 11.1] *Let F be a field and let H be a semisimple algebraic F–group which is quasi-split (i.e. admits a Borel F–subgroup). Then the map*

$$\bigsqcup_{T \subset H} H^1(F,T) \to H^1(F,H)$$

is onto where T runs over the maximal F–tori of H.

For the field $k(C)$, we have that $\mathrm{Br}(k(C)) = 0$ and more generally that $\mathrm{cd}(k(C)) = 1$, this is a consequence of Tsen's theorem stating that $k(C)$ has the C_1 property [54, II.3.3]. A classical dévissage yields that $H^1(k(C),T) = 1$ for each $k(C)$-torus T[8]. Combining with Theorem 5.6 yields that $H^1(k(C),G) = 1$ for each semisimple (split) k–group G. A special case is that of PGL_n which can be rephrased by saying that the central simple algebras over $k(C)$ are matrix algebras.

A second ingredient is the fact that the Picard group $\mathrm{Pic}(C)$ is divisible which follows from the structure of $\mathrm{Pic}(C^c)$ where C^c is a smooth compactification of C. We have an exact sequence

$$0 \to J_{C^c}(k) \to \mathrm{Pic}(C^c) \to \mathbb{Z} \to 0$$

[8]Hint: Let n be the degree of a splitting field of T, show that $nH^1(k,T) = 1$ and consider the exact sequence $1 \to {}_nT \to T \xrightarrow{\times n} T \to 1$.

where J_{C^c} is the Jacobian variety of C^c [42, §1] (or [62, Tag 03RN]). If $C = C^c \setminus \{x_1, \dots, x_s\}$ the surjective map $\mathrm{Pic}(C^c) \to \mathrm{Pic}(C)$ induces an epimorphism $J_{C^c}(k) \twoheadrightarrow \mathrm{Pic}(C)$. Thus $\mathrm{Pic}(C)$ is a divisible group.

We proceed now to the proof of Theorem 5.5.

Proof. We assume first that G is simply connected. Proposition 5.3 shows that $\ker\big(H^1_{fppf}(C,G) \to H^1_{fppf}(k(C),G)\big) = 1$. Since $H^1(k(C),G) = 1$, it follows that $H^1_{fppf}(C,G) = 1$.

For the general case, let $f : G^{sc} \to G$ be the simply connected cover of G (e.g. $\mathrm{SL}_n \to \mathrm{PGL}_n$, $\mathrm{Spin}_n \to \mathrm{SO}_n$) and put $\mu = \ker(f)$. Let T^{sc} be a maximal torus of G^{sc}, then $T = T^{sc}/\mu$ is a maximal torus of G. We consider the commutative diagram

$$\begin{array}{ccc} H^1_{fppf}(C,T^{sc}) & \xrightarrow{\quad f_* \quad} & H^1_{fppf}(C,T) \\ \downarrow & & \downarrow\!\!\!\downarrow \\ 1 = H^1_{fppf}(C,G^{sc}) & \longrightarrow & H^1_{fppf}(C,G). \end{array} \tag{5.2}$$

The surjectivity of the right vertical map follows from $H^1(k(C),G) = 1$ and of Proposition 5.3.(1). We use now the exact sequence $1 \to \mu \to T^{sc} \xrightarrow{f} T \to 1$. We choose isomorphisms $T^{sc} \cong \mathbb{G}_m^r$ and $T \cong \mathbb{G}_m^r$, f is given by a map $A : \mathbb{Z}^r \to \mathbb{Z}^r$ (on the cocharacters) such that $\det(A) \in \mathbb{Q}^\times$. It follows that f_* reads

$$A : \mathrm{Pic}(C)^r \to \mathrm{Pic}(C)^r.$$

Since $\det(A) \in \mathbb{Q}^\times$ and $\mathrm{Pic}(C)$ is divisible, the map f_* is then onto. Diagram chase in the diagram (5.2) enables us to conclude that $H^1(C,G) = 1$. □

Remark 5.7. The reductive case is of the same vein. Let $S = G/DG$ be the coradical torus of G. One can show that the map $H^1(C,G) \to H^1(C,S)$ is bijective. This generalizes the bijection $H^1(C,\mathrm{GL}_r) \xrightarrow{\sim} H^1(C,\mathbb{G}_m) = \mathrm{Pic}(C)$ seen in Theorem 2.8.

5.3. **The case of the affine line.**

Theorem 5.8. *(Raghunathan-Ramanathan* [49]*) Let G be a reductive k–group over a field k. Then we have a bijection*

$$H^1(k,G) \xrightarrow{\sim} \ker\big(H^1(k[t],G) \to H^1(k_s[t],G)\big).$$

If k is perfect or if the characteristic of p is "good" for G, we have $H^1(k_s[t],G) = 1$ so that $H^1(k,G) = H^1(k[t],G)$. When it happens, we say that G–torsors over $k[t]$ are constant. There are a few exotic cases when it does not hold.

Example 5.3.1. Assume that k is not perfect of characteristic $p > 0$ and pick $a \in k \setminus k^p$. We consider the $k[t]$–algebra $\mathcal{A}$ (unital, associative) generated by X, Y submitted to the relations

$$X^p - X = t, Y^p = a, YXY^{-1} = X + 1.$$

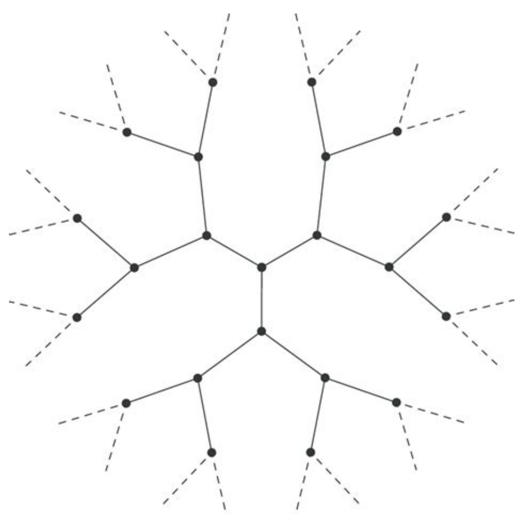
It is an Azumaya $k[t]$-algebra of degree p so defines a class $[\mathcal{A}] \in H^1(k[t], \mathrm{PGL}_p)$. It not trivial over $k_s[t]$ because it is not trivial over $k_s((\frac{1}{t}))$ (use for example [28, Corollary 4.7.4]). Cohomologically speaking this class arises from the k-subgroup $\mu_p \times \mathbb{Z}/p\mathbb{Z}$ of PGL_p. We have $H^1_{fppf}(k[t], \mu_p) = k[t]^\times / k[t]^{\times p} = k^\times/(k^\times)^p$ and $H^1_{fppf}(k[t], \mathbb{Z}/p\mathbb{Z}) = k[t]/\mathcal{P}(k[t])$ where $\mathcal{P} = (x) = x^p - x$ is the Artin Schreier cover. In the same manner as in the field case [28, Proposition 4.7.3], one can check that the class $[\mathcal{A}]$ is the image of $\big((a), [T]\big)$ in $H^1(k[t], \mathrm{PGL}_p)$. There are also examples for simply connected groups [23, §2.4].

There are variations of the original proof [24, 14, 1]; all involve Bruhat-Tits theory, which is the theory of reductive algebraic groups over a complete (or henselian) discretly valued field [9, 10] and their integral models. Note that Kaletha and Prasad wrote recently a wonderful book on this theory [37].

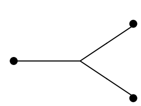
We will sketch the recent proof of [1] when the k-group G is split semisimple simply connected and almost simple (i.e. its Dynkin diagram is connected), e.g. $G = \mathrm{SL}_n, \mathrm{Spin}_{2n}, G_2$. There is no need to deal with the projective line but only with the completion $K = k((\frac{1}{t}))$ of the function field $k(t)$ with respect to the point ∞.

We consider the Bruhat-Tits building $\mathcal{B} = \mathcal{B}(G_K)$ of G_K [37, §7.6]. This is a contractible simplicial complex equipped with a metric. There is a “strongly transitive" action of $G(K)$ on $\mathcal{B}$. Each maximal k-split torus T of G gives rises to an apartment $\mathcal{A}(T) \subset \mathcal{B}$ which is an euclidean affine space.

Example 5.3.2. If $G = \mathrm{SL}_2$, $\mathcal{B}$ is the Bruhat-Tits tree. If $k = \mathbb{F}_2$, it looks as follows.



In this case, the apartments are the infinite lines. In this geometry, a triangle has the shape



In particular the triangles are thin compared to euclidean geometry. This is an occurence of non-positive curvature, see the reference book [8, II, Appendix].

Let (B,T) be a Killing couple for G and consider the root system $\Phi = \Phi(G,T)$ and its base Δ. We call $\mathcal{A}(T)$ the standard apartment of $\mathcal{B}$. The center ϕ of $\mathcal{B}$ is defined in [9, 9.1.19.(c)]. It belongs to $\mathcal{A}(T)$ and is characterized as the unique point fixed by $G(k[[\frac{1}{t}]])$. We have a decomposition of the euclidean space

$$\mathcal{A}(T) = \phi + \widehat{T}^0 \otimes_{\mathbb{Z}} \mathbb{R}$$

where $\widehat{T}^0 = \mathrm{Hom}_{k-gp}(\mathbb{G}_m, T)$ stands for the group of cocharacters of T. We define the cone

$$\mathcal{Q} = \phi + \big\{ v \in \widehat{T}^0 \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \alpha, v \rangle \geq 0 \;\; \forall \;\; \alpha \in \Delta \big\}.$$

Theorem 5.9. *(Soulé* [56, Théorème 1]*) The cone $\mathcal{Q}$ is a fundamental simplicial domain for the action of $G(k[t])$ on $\mathcal{B}$. In other words any simplex of $\mathcal{B}$ is $G(k[t])$-conjugated to a unique simplex of $\mathcal{Q}$.*

Remark 5.10. Using the precise shape of stabilizers, Soulé provided a presentation of $G(k[t])$ generalizing Nagao's presentation $\mathrm{SL}_2(k[t]) = \mathrm{SL}_2(k) \star_{B(k)} B(k[t])$. For G not split, it has been generalized by Margaux [40].

We proceed now to the proof of the above special case of Theorem 5.8.

Proof. We are given a G-torsor X over $k[t]$ which is trivialized after an extension $l[t]$ where l/k is a finite Galois extension of group Γ. In other words X is given by a 1–cocycle $z : \Gamma \to G(l[t])$. We put $L = l((1/t))$ and consider the Bruhat-Tits building $\mathcal{B}_l$ of G_L. We denote by $\mathcal{A}_l(T)$ the standard apartment of $\mathcal{B}_l$, by ϕ_l the center of $\mathcal{B}_l$ and by $\mathcal{Q}_l \subset \mathcal{A}_l(T)$ the cone associated to the positive roots of $\Phi(G_l, T_l)$. We have a natural embedding $\iota : \mathcal{B} \to \mathcal{B}_l$ which applies ϕ to ϕ_l and $\mathcal{A}(T)$ to $\mathcal{A}(T_l)$ [9, Example 9.1.19.(c)] (or [37, §7.9.2]). Clearly we have $\iota(\mathcal{Q}) = \mathcal{Q}_l$. The group $G(L) \rtimes \Gamma$ acts[9] on $\mathcal{B}_l$ so that we get a twisted action of Γ on $\mathcal{B}_l$ defined by

$$\sigma \star x = z_\sigma \, . \, \sigma(x).$$

The Bruhat-Tits fixed point theorem implies the existence of a fixed point; another way to see that is to take the barycenter (as defined in [36, Definition 3.1]) of a Galois orbit. Let $x \in \mathcal{B}_l$ be a fixed point by the twisted Galois action. Soulé's result 5.9 applied over l provides a point $x_0 \in \mathcal{Q}_l$ such that $x = g\, x_0$ for some $g \in G(l[t])$. Since $x_0 \in \mathcal{Q}_l = \iota(\mathcal{Q})$, x_0 is fixed by Γ.

Since $x = \sigma \star x = z_\sigma \, \sigma(x)$, it follows that $g \, . \, x_0 = z_\sigma \, \sigma(g \, . \, x_0) = z_\sigma \, \sigma(g) \, . \, x_0$ so that

$$z'_\sigma . \, x_0 = x_0.$$

[9]The unramified descent theorem [37, Theorem 9.2.7] states that ι induces a bijection $\mathcal{B} \xrightarrow{\sim} (\mathcal{B}_l)^\Gamma$ and we do not use it.

where $z'_\sigma = g^{-1}\, z_\sigma\, \sigma(g)$ is an equivalent cocycle. It follows tha z' takes values in the stabilizer $G(l[t])_{x_0}$. According to [56, §1.1], there exists a subset $I \subset \Delta$ and a split k–unipotent k–group U_{x_0} such that $G(l[t])_{x_0} = U_{x_0}(l) \rtimes L_I(l)$ where L_I is the standard Levi subgroup of the standard parabolic k–subgroup P_I of G.

We use now that the map $H^1\big(\Gamma, L_I(l)\big) \to H^1\big(\Gamma, U_{x_0}(l) \rtimes L_I(l)\big)$ is bijective [26, lemme 7.3] so that $[z] = [z'] \in H^1\big(\Gamma, G(l[t])\big)$ belongs to the image of $H^1(\Gamma, G(l)) \to H^1\big(\Gamma, G(l[t])\big)$. The proof is completed. □

5.4. **The case of the punctured affine line.** This case is more complicated than the affine line.

Theorem 5.11. *(see [14]) Let G be a reductive k–group over a field k of characteristic zero. The map*

$$H^1(k[t^{\pm 1}], G) \xrightarrow{\sim} H^1(k((t)), G)$$

is bijective.

The surjectivity is easy and comes by reduction to a finite subgroup. The hard part is the injectivity where one crucial step is to show an existence of a maximal torus for the relevant twisted group scheme. This involves Bruhat-Tits theory and twin buildings. Note that Bruhat-Tits theory also provides a description of $H^1(k((t)), G)$ [11].

6. What is next?

6.1. **Dimension one.** Fedorov constructed exotic examples of non constant G–torsors over $R[t]$ with R a local ring (henselian if we want) [21]. The way to detect that the constructed torsors are not constant is to establish that the torsors do not extend to the projective line $\mathbb{P}^1_R$. That method is related with the work on the Grothendieck-Serre's conjecture [13, §5].

6.2. **Higher dimensions.** Theorem 5.8 does not extend in dimension 2. The first example is that of Ojanguren-Sridharan [48] with the field $\mathbb{R}$ of real numbers and the unit group $G = \mathrm{GL}_1(\mathbf{H})$ of the Hamilton quaternion algebra $\mathbf{H}$. They show that $1 = H^1(\mathbb{R}, G) \subsetneq H^1(\mathbb{R}[x,y], G)$. In other words, there is an invertible (right) $\mathbf{H}[x,y]$–module which is not free.

On the other hand, positive results start with Quillen-Suslin's theorem rephrased in $H^1(k[x_1,\dots,x_n], \mathrm{GL}_n) = 1$. For an enough isotropic reductive k–group G, we have $H^1(k, G) = H^1(k[x_1,\dots,x_n], G)$ (Raghunathan's results [50]). Note also the related Stavrova's results on higher Laurent polynomial rings [57].

Over polynomial rings over a nice ring, we have Lindel's theorem [39] and generalizations by Asok-Hoyois-Wendt [2] which are essential in $\mathbf{A}^1$-homotopy theory.

7. Exercices (T.A. Margot Bruneaux)

Let R be a commutative (unital) ring.

Exercise 1. *Let M be an R–module of finite presentation. Let $\mathbf{W}(M)$ be the R–functor defined by $S \mapsto M \otimes_R S$. Show that $\mathbf{W}(M)$ is representable if and only if M is a locally free R–module of finite type.*

[Hint: To show that if $\mathbf{W}(M)$ is representable by a G–scheme then M is a locally free R–module of finite type, one can show that G is smooth and then consider the tangent vector bundle.]

Exercise 2. *Let M be a locally free R–module of rank $2n \geq 2$ equipped with a regular quadratic form q. Show that, locally for the flat topology, (M, q) is hyperbolic. [Hint: One can deal first with the case of a local ring where 2 is invertible.]*

Exercise 3. *Let B be standard Borel R–subgroup of upper triangular matrices of* $\mathrm{GL}_{2,R}$.

(1) *Show that the flat quotient of $\mathrm{GL}_{2,R}$ by B exists in the category of R–schemes and is isomorphic to the projective line.*
(2) *Deduce an exact sequence of pointed sets*
$$1 \to B(R) \to \mathrm{GL}_2(R) \to \mathbb{P}^1(R) \to H^1_{fppf}(R, B) \to H^1_{fppf}(R, \mathrm{GL}_2).$$
(3) *For R local, show that $H^1_{fppf}(R, B) = 1$ and that $H^1_{fppf}(R, \mathbb{G}_a) = 1$.*

Exercise 4. *Let G, G' be affine group schemes over $\operatorname{Spec} R$, T be a G–torsor and $\phi : G \to G'$ be a homomorphism. We denote by $T \wedge^G G'$ the fppf-sheaf associated to the presheaf $S \mapsto T(S) \times_{\operatorname{Spec}(R)} G'(S)/\{(t, g') \sim (t \cdot g^{-1}, \phi(g)g')\}$.*

(1) *Show that $T \wedge^G G'$ is a G'–torsor. We obtain a map $H^1(\phi) : H^1(X, G) \to H^1(X, G')$ of pointed sets.*
(2) *Show that the following diagram is commutative*
$$\begin{array}{ccc} H^1(X,G) & \xrightarrow{H^1(\phi)} & H^1(X,G) \\ \downarrow{\scriptstyle c_G} & & \downarrow{\scriptstyle c_{G'}} \\ \check{H}^1(X,G) & \xrightarrow{\check{H}^1(\phi)} & \check{H}^1(X,G'). \end{array}$$

Exercise 5. *Let R' be a finite locally free R–algebra. Let $r \geq 0$ be an integer. Let f denote the map from $\operatorname{Spec} R'$ to $\operatorname{Spec} R$.*

(1) *Show that the R–functor $S \mapsto \operatorname{End}_{S \otimes_R R'}\Big((S \otimes_R R')^r\Big)^*$ is representable by an affine R–group scheme. We denote it by $\widetilde{G} = R_{R'/R}(\mathrm{GL}_r)$ (the Weil restriction).*

(2) *Show that a* $\mathrm{GL}_{r,R'}$*–torsor is locally trivialised by an open of the form* $f^{-1}(U)$ *where* U *is an open of* $\operatorname{Spec} R$*.*
(3) *Show that the category of* $\widetilde{G}$*–torsors is equivalent to the category of locally free* R'*–modules of rank* r*.*
(4) *Give an interpretation of the map* $H^1(R, \mathrm{GL}_r) \to H^1(R, \widetilde{G})$ *and show that this map is not in general injective nor surjective.*

Exercise 6. *Let* $d \geq 1$ *be an integer and let* R' *be a* $\mathbb{Z}/d\mathbb{Z}$*–Galois extension, i.e.,* $\operatorname{Spec} R'$ *is a* $\mathbb{Z}/d\mathbb{Z}$*–torsor over* $\operatorname{Spec} R$*. We denote by* σ *the canonical generator of* $\mathbb{Z}/d\mathbb{Z}$*.*

(1) *Show that the formula* $N(y) = y\,\sigma(y)\cdots\sigma^{r-1}(y)$ *defines a group scheme homomorphism* $N : R_{R'/R}(\mathbb{G}_m) \to \mathbb{G}_m$*.*
(2) *Show that* $1 \to \ker(N) \to R_{R'/R}(\mathbb{G}_m) \to \mathbb{G}_m \to 1$ *is an exact sequence of* R*–group schemes.*
(3) *Deduce an exact sequence involving* $H^1(R, \ker(N))$*.*
(4) *Show that the flat quotient of* $R_{R'/R}(\mathbb{G}_m)$ *by* $\mathbb{G}_m$ *exists in the category of schemes and is isomorphic to* $\ker(N)$*.*
(5) *Construct an exact sequence*

$$R^\times \to (R')^\times \xrightarrow{\sigma-1} \ker(N)(R) \to \ker\left(\mathrm{Pic}(R) \to \mathrm{Pic}(R')\right).$$

(6) *Discuss the case of the coordinate ring* $A = R[\ker(N)]$ *of* $\ker(N)$*.*
(7) *For* $R = \mathbb{R}$ *and* $S = \mathbb{C}$*, is the* $\mathbb{G}_m$*–torsor* $R_{S/R}(\mathbb{G}_m) \to R_{S/R}(\mathbb{G}_m)/\mathbb{G}_m$ *trivial?*

Exercise 7. *(Lang isogeny) Let* k *be a field of caracteristic* p*. We denote by* F *the Frobenius morphism.*

Let G *be a smooth algebraic group. Show that the map* $g \mapsto g.F(g^{-1})$ *is an étale isogeny, i.e., it is surjective and finite étale. Deduce that* $SL_{n,\overline{\mathbb{F}_p}}$ *is not simply-connected in the sense of* [51].

Exercise 8. *(Weil restriction) Let* R' *be a finite locally free* R*–algebra. Let* $r \geq 0$ *be an integer. Let* f *denote the map from* $\operatorname{Spec} R'$ *to* $\operatorname{Spec} R$*.*

(1) *Show that the* R*–functor* $S \mapsto \mathrm{End}_{S\otimes_R R'}\Big((S\otimes_R R')^r\Big)^*$ *is representable by an affine* R*–group scheme. We denote it by* $\widetilde{G} = R_{R'/R}(\mathrm{GL}_r)$ *(the Weil restriction).*
(2) *Show that a* $\mathrm{GL}_{r,R'}$*–torsor is locally trivialized by an open of the form* $f^{-1}(U)$ *where* U *is an open of* $\operatorname{Spec}(R)$*.*
(3) *Show that the category of* $\widetilde{G}$*–torsors is equivalent to the category of locally free* R'*–modules of rank* r*.*
(4) *Give an interpretation of the map* $H^1(R, \mathrm{GL}_r) \to H^1(R, \widetilde{G})$ *and show that this map is not in general injective nor surjective.*

Exercise 9. *(after Ojanguren and Sridharan [48]). Let $\mathbb{H}$ be the Hamilton quaternion algebra. Show that $A = \mathbb{H}[x, y]$ admits an invertible (right) $\mathbb{H}$–module which is not free.*

[Hint : One can consider the exact sequence

$$0 \to P \to A^2 \xrightarrow{f} A \to 0$$

where $f : (\gamma, \mu) \mapsto (X + i)\gamma - (Y + j)\mu$.

Then one can find two solutions (γ_1, μ_1) and (γ_2, μ_2) of degree two and deduce a contradiction.]

Exercise 10. *Let R be an unitary commutative ring.*

Show that $\mathrm{Pic}(R[t]) = \mathrm{Pic}(R)$ when R is a normal ring of finite type over a field k of dimension n.

[Hint: One can use that, for every X–scheme and every effective Cartier divisor $\mathcal{L}$ of X, there is a Gysin-map $\mathcal{L}\cdot$ from $CH_k(X)$ to $CH_{k-1}(|\mathcal{L}|)$ defined by $\mathcal{L} \cdot [Y] = [i^*\mathcal{L}]$ where $i : Y \hookrightarrow X$ is an irreducible subscheme of X of dimension k. (See [22, §2.6].)]

Remark 7.0.1. See [64, §I.3] for examples in which $\mathrm{Pic}(R[t]) \neq \mathrm{Pic}(R)$.

Exercise 11. *(Patching) Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_0$ be three categories and $\alpha_i : \mathcal{C}_i \to \mathcal{C}_0$ for $i = 1, 2$. We define the category $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ whose objects are the triple $(\mathcal{C}_1, \mathcal{C}_2, \phi)$ where $\phi : \alpha_1(\mathcal{C}_1) \to \alpha_2(\mathcal{C}_2)$ is an isomorphism and a morphism from $(\mathcal{C}_1, \mathcal{C}_2, \phi)$ to $(\mathcal{C}_1', \mathcal{C}_2', \phi')$ consists of two morphisms $f_i : \mathcal{C}_i \to \mathcal{C}_i'$ for $i = 1, 2$ such that $\phi' \circ \alpha_1(f_1) = \alpha_2(f_2) \circ \phi$.*

If F is a field, we denote by $\mathrm{Vect}_n(F)$ the category of F–vector spaces of dimension n and if R is a ring, we denote by $Mod_n(R)$ the category of R–modules locally free of rank n.

(1) *Let F_1, F_2 be two fields, F_0 be an extension of this two fields, and let $F = F_1 \cap F_2$.*

Show that if for every n, $\mathrm{GL}_n(F_0) = \mathrm{GL}_n(F_1)\ \mathrm{GL}_n(F_2)$, then the map:

$$\begin{array}{rccc} \beta : & \mathrm{Vect}_n(F) & \to & \mathrm{Vect}_n(F_1) \times_{\mathrm{Vect}_n(F_0)} \mathrm{Vect}_n(F_2) \\ & V & \mapsto & (V \otimes_F F_1, V \otimes_F F_2, \phi : V \otimes_F F_1 \otimes_{F_1} F_0 \xrightarrow{\sim} V \otimes_F F_2 \otimes_{F_2} F_0) \end{array}$$

is an equivalence of categories.

(2) *If R is a DVR and K is its field of fractions, one can define in the same way an application*

$$\gamma : Mod_n(R) \to \mathrm{Vect}_n(K) \times_{\mathrm{Vect}_n(\hat{K})} Mod_n(\hat{R}).$$

First show that $GL_n(\hat{K}) = GL_n(K) GL_n(\hat{R})$ and then that γ is an equivalence of categories.

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