Loop torsors: from Topology to Arithmetic

Philippe Gille

Institut Camille Jordan, Lyon and IMAR, Bucharest

Bucharest, Imar, October 1, 2025

A conference dedicated to Vasile Brînzănescu on the occasion of his 80th birthday

Principal bundles in Topology

- Let X be a topological space and let G be a topological group. A G-bundle Y over X is a topological space Y equipped with a continous right action of G together with a continous map $f : Y \to X$ which is G-invariant and such that X admits an open cover $(U_i)_{i \in I}$ and trivializations (which are G-equivariant)

$$\phi_i : U_i \times G \xrightarrow{\sim} Y \times_X U_i = f^{-1}(U_i).$$

- It follows that each

$$\phi_i^{-1} \circ \phi_j : (U_i \cap U_j) \times_X G \to (U_i \cap U_j) \times_X G$$

  is the left translation by a continuius function $g_{i,j} : U_i \cap U_j \to G$. Then $(g_{i,j})$ is the cocycle associated with the trivializations.

- A G-bundle is trivial if it is G-isomorphic to $X \times G$, equivalently the map $f : Y \to X$ admits a continous section.

Vector bundles

- If $G = \mathrm{GL}_n(\mathbb{R})$, a principal G-bundle over X is the same thing than a real vector bundle over X. In one way we associate to a $\mathrm{GL}_n(\mathbb{R})$-bundle Y the contracted product

$$Y \wedge^{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n = (Y \times \mathbb{R}^n)/\mathrm{GL}_n(\mathbb{R}).$$

Loop bundles

- Let Γ be a discrete group and let $p : Y \to X$ be a principal homogenous space for the group Γ.
- Given a homomorphism $u : \Gamma \to G$, we can form $u_*(Y) = Y \wedge^{\Gamma} G$ over X (where Γ acts on the right on Y and on the left on G through u).
- Then $u_*(Y)$ is a G-torsor and we say it a loop torsor.
- When X is connected, a special case is the construction of vector bundles over X from linear representations of the fundamental group $\pi_1(X, x)$.
- For example, the Moebius strip over the circle S^1 can be constructed with the 2-cover $p : S^1 \xrightarrow{\times 2} S^1$ and the map $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{R}^{\times}$.

Isotrivial bundles

- For the G-bundle $u_*(Y) = Y \wedge^{\Gamma} G$, the map $Y \to Y \times G$, $y \mapsto (y, 1_G)$, induces a commutative diagram

$$\begin{tikzcd} Y \arrow[rr] \arrow[drr, "p"] & & u_*(Y) = Y \wedge^{\Gamma} G \arrow[d] \\ & & X \end{tikzcd}$$

- In other words the pull-back $u_*(Y) \times_X Y$ over Y is a trivial G-bundle.
- We say that a principal G-bundle $E \to X$ is *isotrivial* if there exists a covering map $q : X' \to X$ such that the pull-back $q^*(E)$ is a trivial G-bundle over X'.
- This terminology comes from Grothendieck, another one is *Virtually trivial bundle*.
- Isotriviality is then a necessary condition for E to arise from some loop construction. What about the converse ?

Milnor's example

It seems that the simplest example of isotrivial principal G-bundle not arising from the loop construction is that of Milnor (1958).

- Let M be the compact oriented surface of genus $g \geq 1$. Its simply connected cover $\widetilde{M}$ is $\mathbb{C}$ of the Poincaré half plane, so it is contractible.
- Let G be a connected Lie group and let
$$1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$$
be the exact sequence of Lie groups arising from the simply connected cover of G.
- It induces a bijection
$$H^1(M, G) \xrightarrow{\sim} H^2(M, \pi_1(G))$$
where $H^1(M, G)$ stands for the set of isomorphism classes of principal G-bundles over M.
- We have
$$H^2(M, \pi_1(G)) \xrightarrow{\sim} Ext_{ab}(H_1(M, \mathbb{Z}), \pi_1(G)) \oplus \mathrm{Hom}\big(H_2(M, \mathbb{Z}), \pi_1(G)\big)$$

Milnor's example, II

- It induces a bijection

$$H^1(M, G) \xrightarrow{\simeq} H^2(M, \pi_1(G)) = \pi_1(G).$$

- We take $G = \mathrm{GL}_2(\mathbb{R})^+$ so that $\pi_1(G) = \mathbb{Z}$ and

$$H^1(M, G) = \mathbb{Z}.$$

- Milnor has shown that loop principal G-bundles gives rise to positive indices.
- Thus the class $-1 \in \mathbb{Z}$ defines a principal G–bundle over M which does not arise from the loop construction.

Torsors in Algebraic Geometry

Let k be an algebraically closed field of characteristic zero (e.g. $\mathbb{C}$).

- *Torsor* is the name of principal bundle in Algebraic Geometry, this generalization is due to Grothendieck and Serre in 1958.
- It went together with the notion of *étale* morphisms, that is, of smooth morphisms of algebraic varieties of relative dimension zero. An étale cover of an algebraic variety X is a finite collection of étale maps $f_1 : U_1 \to X, \ldots, f_n : U_n \to X$ such that $f_1(U_1) \cup \cdots \cup f_n(U_n) = X$.
- Let X be a k-variety and let G be a linear algebraic group. A G-torsor Y over X is an algebraic k-variety Y equipped with a right action of G together with map $f : Y \to X$ which is G-invariant and such that X admits an étale cover $(U_i)_{i \in I}$ and trivializations (which are G-equivariant)

$$\phi_i : U_i \times G \xrightarrow{\sim} Y \times_X U_i.$$

Torsors in Algebraic Geometry,II

- In the case of GL_n, a GL_n–torsor is the same thing than an algebraic vector bundle of rank n. In that case, Zariski topology is enough.
- We consider the rank one torus $\mathbb{G}_{m,k} = \mathrm{GL}_1 = \mathrm{Spec}(k[t^{\pm 1}])$. The map $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$, $x \mapsto x^2$ is a $\mathbb{Z}/2\mathbb{Z}$–torsor. It is not locally split for the Zariski topology.
- If Γ is a finite group, it is also an algebraic k–group. In that case Γ-torsors $Y \to X$ with X, Y connected are nothing but Galois covers of Grothendieck theory of fundamental group in algebraic geometry.

Loop torsors in Algebraic Geometry

- Given a Γ-torsor $p : Y \to X$ and a homomorphism $u : \Gamma \to G$, we can associate the G-torsor $u_*(Y) = Y \wedge^\Gamma G$ by taking the quotient in the algebraic sense (that is using flat sheaves and descent theory).
- This is a loop torsor and as in the topological case, the base change $u_*(Y) \times_X Y$ is a trivial G–torsor.
- We say that a G-torsor E over X is *isotrivial* if there exists a finite étale cover $X' \to X$ (right notion of covering map in Algebraic Geometry) such that $E \times_X X'$ is a trivial G-torsor.
- Loop torsors are then isotrivial but what about the converse ?

Loop/Isotrivial I

- We denote by $H^1(X,G)$ the pointed set of isomorphism classes of G–torsors over X.
- Example : $G=\mathbb{G}_m$. We have $H^1(X,\mathbb{G}_m)=\mathrm{Pic}(X)$. The isotrivial torsors correspond to the torsion elements of $\mathrm{Pic}(X)$ and they are loop torsors by using the Kummer exact sequence $1\to\mu_n\to\mathbb{G}_m\xrightarrow{\times n}\mathbb{G}_m\to 1$.
- We assume that X is a smooth connected projective variety. In this case, isotrivial G-torsors (and especially isotrivial vector bundles) have been studied by M. Nori (1976). Also there are important classification results by M. Brion in the case of abelian varieties.
- In this case, isotrivial G-torsors are indeed loop torsors. Let E be a G-torsor over X which is trivialized by a finite étale cover Y. Without loss of generality, we can assume that Y is connected and Grothendieck's theory tells us that $Y\to X$ is dominated by a Galois cover $Z\to X$ of finite group Γ.

Loop/Isotrivial II

- Let E be a G-torsor over X which is trivialized by a finite étale cover Y. We have seen that we can assume Z connected and to be a Galois cover of X of group Γ.
- We appeal to the exact sequence of pointed sets

$$1 \to H^1(\Gamma, G(Z)) \to H^1(X, G) \to H^1(Z, G)$$

where $H^1(\Gamma, G(Z))$ stands for the set of non-abelian group cohomology defined by 1-cocycles.

- Since Z is projective, we have $G(k) = G(Z)$ so that $H^1(\Gamma, G(Z)) = \mathrm{Hom}(\Gamma, G(k))/G(k)$.
- The class of E gives rise then to some $u : \Gamma \to G$ and it is routine to check that $u_*(Z) \cong E$.

Affine case

- In the proceding reasonning, there was no need of the projective assumption until the analysis of the sequence

$$1 \to H^1(\Gamma, G(Z)) \to H^1(X, G) \to H^1(Z, G).$$

- When Z is affine, there is no reason a priori why the map $H^1(\Gamma, G(k)) \to H^1(\Gamma, G(Z))$ should be onto.
- With my collegues Chernousov and Pianzola, we were interested in the case of the torus

$$(\mathbb{G}_m)^n = \mathrm{Spec}(k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]).$$

- A first result is that all G–torsors over $(\mathbb{G}_m)^n$ are isotrivial, this uses the strong vanishing fact $H^1(\mathbb{A}^n_k, G) = 1$ (Quillen-Suslin, Raghunathan).

A counterexample

- We know the algebraic fundamental group of $(\mathbb{G}_m)^n$, it is $\widehat{\mathbb{Z}}^n$ and a basis of finite covers is $(\mathbb{G}_m)^n \xrightarrow{\times d} (\mathbb{G}_m)^n$ is by taking n copies of the Kummer cover of degree d.
- If $n = 1$ and G is connected, all torsors are loop (2012).
- The simplest example of (isotrivial) non loop torsor is for $G = \mathrm{PGL}_2$ and $n = 2$. In this case $H^1((\mathbb{G}_m)^2, \mathrm{PGL}_2)$ classifies quaternion Azumaya algebras over the Laurent polynomial ring $R = k[t_1^{\pm 1}, t_2^{\pm 1}]$.
- The example is explicit. We consider first the loop object $\mathscr{Q}$ defined by the presentation

$$T_1^2 = t_1,\; T_2^2 = t_2, ,\; T_1 T_2 + T_2 T_1 = 0.$$

It is an R-quaternion algebra arising from the map $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathrm{PGL}_2(k)$ applying $(1,0)$ on $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $(0,1)$ on $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Counterexample, II

The R-quaternion algebra $\mathscr{Q}$ is defined by

$$T_1^2 = t_1,\ T_2^2 = t_2,,\ T_1 T_2 + T_2 T_1 = 0.$$

- The idea (due to Ojanguren-Sridharan) is to take $\mathscr{Q}' = \mathrm{End}_{\mathscr{Q}}(\mathcal{L})$ for a non-free invertible right $\mathscr{Q}$-module $\mathcal{L}$.
- We define $\mathcal{L}$ as the kernel of the *split* map of right $\mathscr{Q}$-modules

$$\Phi : \mathscr{Q}^2 \to \mathscr{Q},\ (p, q) \mapsto (1 + T_1)p - (1 + T_2)q.$$

- It works because

$$\Phi(1 + T_2, 1 + T_1) = (1 + T_1)(1 + T_2) - (1 + T_2)(1 + T_1)$$

-

$$= T_1 T_2 - T_2 T_1 = 2T_1 T_2 \in \mathscr{Q}^\times.$$

- We have $\mathcal{L} \oplus \mathscr{Q} \xrightarrow{\sim} \mathscr{Q}^2$ and by computations, we can show that $\mathcal{L}$ is not free.

Loop brothers

- Since invertible modules are locally free, the counterexample coincides with a loop torsor locally for the Zariski topology. This is a general phenomenon.
- **Theorem** (CGP, 2017) Assume that G is reductive. Let E be a G-torsor over $(\mathbb{G}_m)^n$. Then E admits a unique loop brother E_0, that is, a loop G-torsor E_0 such that E and E_0 coincide locally for the Zariski topology.
- Before that we conjectured that $E \cong E_0$ under an additional assumption of reductibility.
- Let E be a G-torsor over $(\mathbb{G}_m)^n$ and consider its twisted associated group scheme ${}^E G$ over $(\mathbb{G}_m)^n$ so gives rise to an algebraic group $H(E)$ over $k(t_1, ..., t_n)$.
- **Conjecture.** Assume that G is semisimple and an almost simple, and that $H(E)$ carries a proper parabolic subgroup. Then E is a loop G-torsor.

Stavrova's theorem.

- **CGP Conjecture.** Assume that DG is almost simple, and that the algebraic group $H(E)$ carries a proper parabolic subgroup. Then E is a loop G-torsor.
- A. Stavrova has proven the conjecture in 2019.
- This uses crucially subgroups of $G(k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}])$ generated by unipotent elements and a trick called *doubling variables*.

Finally Arithmetic

Let $\mathbb{Z}_p$ be the ring of p-adic integers for a prime p; it can be replace by any complete DVR e.g. $k[[t]]$ for k a field.

- Torsors are of interest over rings as $\mathbb{Z}_p[[t]][\frac{1}{p}, \frac{1}{t}]$ which have some similarity with $k[t_1^{\pm 1}, t_2^{\pm 2}]$.
- The ring $\mathbb{Z}_p[[t]]$ is a regular complete local ring whose fraction field is an overfield of the field $\mathbb{Q}_p(t)$. This field is called *semi-global* and the arithmetic of semi-global fields is a very active area (Harbater, Parimala,....).
- The similarity is due to the shape of covers which relates to the Abhyankar's lemma.

Abhyankar's lemma

The setting is the following. Let A be a complete (or henselian) local ring of dimension $r \geq 1$, $f_1, \ldots, f_r$ a family of regular parameters.

- We denote by κ its residue field and by p its characteristic exponent.
- We consider the divisor $D = \mathrm{div}(f_1) + \cdots + \mathrm{div}(f_r)$, the localization $A_D = A[f_1^{-1}, \ldots, f_r^{-1}]$ and the fraction field $K = Frac(A)$.
- Example 1 : $A = k[[x_1, \ldots, x_r]]$, $f_i = x_i$, $A_D = k[[x_1, \ldots, x_r]][x_1^{-1}, \ldots, x_r^{-1}]$. In this case $\kappa = k$.
- Example 2 : $A = \mathbb{Z}_p[[x_1, \ldots, x_{r-1}]]$, $f_1 = p$, $f_i = x_i$ for $i = 1, \ldots, r-1$ and A_D is a subring of $\mathbb{Q}_p[[x_1, \ldots, x_{r-1}]][x_1^{-1}, \ldots, x_{r-1}^{-1}]$. In this case $\kappa = \mathbb{F}_p$.

Abhyankar's lemma II

The ring A is a complete (or henselian) local ring, A_D a localization with respect with a sequence of regular parameters.

- The Abyankhar's lemma is about a special kind of Galois covers of $\mathrm{Spec}(A_D)$ which are called *tame* (or tamely ramified). This is a technical condition which excludes for example to take $\mathbb{Z}_p[\sqrt[p]{p}]$; if the Galois group is of prime to p order, the tame condition is always satisfied.
- **Abyankhar's lemma.** Let $Y \to \mathrm{Spec}(A_D)$ be a Galois tame cover. Then it is dominated by a Galois tame cover of the shape
$$B_m = B[f_1^{-m}, \ldots, f_r^{-m}]$$
where $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a Galois cover and m is a positive prime to p integer such that $\mu_m(B)$ is of cardinal m.
- The Galois group is $(\mu_m(B))^r \rtimes \mathrm{Aut}(B/A)$.
- If κ is algebraically closed, we have $A = B$ so that the Galois theory is the same than for Laurent polynomial rings.

Torsors over Spec(A_D)

- Parimala's insight was that the results on torsors for Laurent polynomials should have suitable extensions for A_D.
- Let G be an affine A–group scheme (e.g. the orthogonal group of some quadratic A–form).
- A G-torsor E over Spec(A_D) is *tame loop* if its class is the image of the composite

$$\begin{tikzcd}
H^1\big(\mathrm{Aut}(B_m/B), G(B)\big) \arrow[d] \arrow[drr] & & \\
H^1\big(\mathrm{Aut}(B_m/A_D), G(B_m)\big) \arrow[rr, hook] & & H^1(\mathrm{Spec}(A_D), G).
\end{tikzcd}$$

  for some $B_m = B[f_1^{-m}, \dots, f_r^{-m}]$ as above.
- It means that it arises from a Galois cocycle of a very nice shape (in particular with no denominators) with respect to an Abhyankar's cover.

The injectivity result

- We denote by $H^1_{\substack{tame\\loop}}(A_D, G) \subset H^1(A_D, G)$ the subset consisting of classes of tame loop torsors over $\mathrm{Spec}(A_D)$.
- I will give an example of a generalized result to the Abyankhar's setting and an example of an open question.
- We denote by $\widehat{X} \to X = \mathrm{Spec}(A)$ the blow-up of $\mathrm{Spec}(A)$ as its closed point. The exceptional divisor defines a valuation v on the fraction field K of A. We denote by K_v its completion.
- Example 1 : For $A = k[[x_1, \dots, x_r]]$ and $r \geq 2$, then $K_v \cong k\big(\frac{x_1}{x_r}, \dots, \frac{x_{r-1}}{x_r}\big)((x_r))$.
- Example 2 : $A = \mathbb{Z}_p[[x_1, \dots, x_{r-1}]]$ K_v is a complete field for a discrete valuation of unequal characteristic and of residue field isomorphic to $\mathbb{F}_p(t_1, \dots, t_{r-1})$.
- **Theorem** (G., 2024). Assume that G is a reductive A–group scheme. The map

$$H^1_{\substack{tame\\loop}}(A_D, G) \subset H^1(A_D, G) \to H^1(K_v, G)$$

is injective.

The injectivity result, II

We assume that G is a reductive A-group scheme.

- **Theorem** (G., 2024). The map

  $$H^1_{\substack{tame\\loop}}(A_D, G) \subset H^1(A_D, G) \to H^1(K_v, G)$$

  is injective.

- The interest is that $H^1(K_v, G)$ is a Galois cohomology set which is well-understood by means of the Bruhat-Tits' theory.

- In Example 1, it says for G/k reductive that the map

  $$H^1_{\substack{tame\\loop}}\big(k[[x_1, \dots, x_r]][x_1^{-1}, \dots, x_r^{-1}], G\big) \to H^1\big(k(\frac{x_1}{x_r}, \dots, \frac{x_{r-1}}{x_r})((x_r)), G\big)$$

  is injective.

- This is quite close of the previous injectivity result

  $$H^1_{loop}\big(k[x_1^{\pm 1}, \dots, x_r^{\pm 1}], G\big) \to H^1\big(k((x_1))\dots((x_r)), G\big).$$

- The last map is bijective but it is not intrinsecal.

Open questions

We come back to the case of A_D and of a reductive A-group scheme G.

- Let E be a G-torsor over $\mathrm{Spec}(A_D)$ which is split by an Abhyankar's cover.
- **Question 1.** Does E admits a tame loop brother ?
- That is, does it exists a tame loop G-torsor E_0 such that E and E_0 coincide locally for the Zariski topology ?
- Good news are that unicity follows from the injectivity theorem.
- **Question 2.** Under some reducibility condition, is E a tame loop G–torsor ?