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Examples of Non-rational Varieties of Adjoint Groups

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Let k be a field of characteristic $\neq 2$ and k_s a separable closure of k. We say that an algebraic variety X/k is stably k-rational if there exist two affine spaces $\mathbb{A}_k^m, \mathbb{A}_k^n$ and a k-rational map $\mathbb{A}^m \times X \approx \mathbb{A}^n$. Merkurjev [9] gave a criterion of stable k-rationality for the adjoint classical groups with absolute rank ≤ 3, which covers the case of the variety $\mathrm{PSO}(q)$ for any quadratic form q/k of rank ≤ 6. This criterion gives examples of field k and quadratic form q of rank 6 with non-trivial signed discriminant such that the variety $\mathrm{PSO}(q)$ is not stably k-rational. The main result of this paper is the following:

THEOREM. *There exist a field k of characteristic* 0 *with cohomological dimension* 3 *and a quadratic form q/k with rank* 8 *and trivial signed discriminant such that the variety* $\mathrm{PSO}(q)$ *is not stably k-rational.*

This is the first example of the quadratic form with trivial signed discriminant such that the variety $\mathrm{PSO}(q)$ is not stably k-rational and since [9], the 8-dimension is minimal. This example is an adjoint group which is an inner form of the split adjoint group of type D_4 [22] and it is the first example of an adjoint semisimple group which is an inner form and which is not a stably k-rational variety. In Section 3, we give another proof of the theorem with $\mathrm{cd}(k) = 6$ which is more elementary because we don't use the Index Reduction Theory.

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Notations. We denote by $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t, 1/t])$, $\mathbb{A}^n = \mathrm{Spec}(\mathbb{Z}[t_1, t_2, \ldots, t_n])$ and for any scheme X, we denote by $\mathbb{G}_{m,X} = \mathbb{G}_m \times_{\mathrm{Spec}(\mathbb{Z})} X$ and $\mathbb{A}_X^n = \mathbb{A}^n \times_{\mathrm{Spec}(\mathbb{Z})} X$ the affine space of rank n on X ($n \in \mathbb{N}$). Let

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$X' \to X$ be a finite locally free morphism of schemes. We can write [4] the exact sequence of X-tori $1 \to R^1_{X'/X}\mathbb{G}_m \to R_{X'/X}\mathbb{G}_m \overset{N_{X'/X}}{\to} \mathbb{G}_{m,X} \to 1$ where $R_{X'/X}\mathbb{G}_m$ is the restriction from X' to X of the X'-torus $\mathbb{G}_{m,X'}$.

Let X be a k-variety geometrically irreducible. We say that X is a k-rational variety if there exist an affine space $\mathbb{A}^n_k$ and a k-birational map $X \approx \mathbb{A}^n_k$. We say that X is a stably rational k-variety if there exist two affine spaces $\mathbb{A}^m, \mathbb{A}^n$ and a k-birational map $\mathbb{A}^m \times X \approx \mathbb{A}^n$. One defines the norm group of X which is denoted $N_X(k)$ as the subgroup of $k^\times$ generated by the $N_{L/k}(L^\times)$ for any finite field extension L/k such that $X(L)$ is not empty.

If A/k is a central simple algebra, there exists a division algebra T/k and an integer r (Wedderburn's theorem) such that $A \tilde{\to} M_r(T)$ and the integer r and T are well defined. Then we denote $\mathrm{ind}_k(D) = \sqrt{\dim_k(T)} \in \mathbb{Z}$ and $\deg(A) = \sqrt{\dim_k(A)} \in \mathbb{Z}$. If A/k, B/k are two central simple algebras, we say that A and B are similar and we denote $A \sim B$ if there exist some integers m, n such that $M_m(A) \simeq M_n(B)$. If $a, b \in k^\times$, we denote by $(a, b)_k$ the standard quaternion algebra. We assume that all quadratic forms will be regular. If q/k, q'/k are two quadratic forms, we denote by $q \perp q'$ their orthogonal sum, by $q \otimes q'$ their tensor product, and by $\mathrm{rk}(q)$ the rank of q. We denote by $C(q)$ the Clifford algebra of q and by $C_0(q)$ the even Clifford algebra of q. We denote by $W(k)$ the Witt ring of the field k, by $I(k)$ the fundamental ideal generated by forms with even rank, and by $\mathrm{disc}_\pm\colon I(k) \to k^\times/k^{\times 2}$ the morphism of signed discriminant. We will identify often a quadratic form q and its class $[q] \in W(k)$. If q is a k-quadratic form and E/k a field extension, we denote by q_E the quadratic form extended to E.

If $(a_i)_{=1,\dots,n}$ is a family of elements $k^\times$, we denote by $\langle a_1, a_2, \dots, a_n\rangle$ the quadratic form $\sum_{i=1,\dots,n} a_i X_i^2$ and by $\langle\langle a_1, a_2, \dots, a_n\rangle\rangle$ the n-fold Pfister form $\langle 1, a_1\rangle \otimes \langle 1, a_2\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$. We denote by $\mathbb{H} = \langle 1, -1\rangle$ the standard hyperbolic form of rank 2.

We recall that a central simple algebra D/k is a k-biquaternionic algebra if D/k is isomorphic to an algebra $(a, b) \otimes_k (c, d)$ with $a, b, c, d \in k^\times$ [7]. We can associate to this isomorphism the Albert form $\langle a, b, -ab, -c, -d, cd\rangle$. Although this Albert form $\langle a, b, -ab, -c, -d, cd\rangle$ is not canonical, its similarity class is well defined and depends only on D. We said that a quadratic form ψ is an Albert form for D if ψ is similar to $\langle a, b, -ab, -c, -d, cd\rangle$. We recall that D is a division algebra iff the form $\langle a, b, -ab, -c, -d, cd\rangle$ is anisotropic, and that a k-form φ with rank 6 and a trivial signed discriminant is an Albert form for some central simple algebra which is similar to $C(\varphi)$.

If q is a quadratic form with even rank, we denote by $\mathrm{SO}(q)$ (resp. $\mathrm{PSO}(q)$) the special orthogonal group of q (resp. projective special orthogonal) and by $G(q)$ the group of similarity factors of q, i.e., $G(q) = \{\alpha \in k^\times \mid \alpha q \cong q\}$. It is well known that $G(\langle 1, -a\rangle) = N_{k(\sqrt{a})/k}(k(\sqrt{a})^\times)$. If $a \in k^\times$, we will denote sometimes $N_k(a) = N_{k(\sqrt{a})}(k(\sqrt{a})^\times)$.

We denote by $\mathrm{cd}(k)$ the cohomological dimension of a field k [19] and by $u(k)$ the u-invariant of k, i.e., the supremum in $\mathbb{N} \cup \{\infty\}$ of the dimensions of anisotropic k-quadratic forms. If $P \subset k^\times$ is a subset of $k^\times$, we denote by $\mathbb{Z}\langle P\rangle$ the subgroup of $k^\times$ generated by P.

1. PRELIMINARIES

1.1. *Norm Groups and R-Equivalence* [9, 10]

For any quadratic space (q, V) of even rank n, we denote by $\mathrm{hyp}(q)$ the subgroup of $k^\times$ generated by the $N_{L/k}(L^\times)$ for any finite field extension L/k such that q_L is hyperbolic. This condition can be written in another way. Indeed, let X_q be the variety of totally isotropic subspaces of V with dimension $n/2$. It is known that X_q is a k-projective smooth variety which has a k-rational point iff $q = 0 \in W(k)$, i.e., q is an hyperbolic form. Then we have $\mathrm{hyp}(q) = N_{X_q}(k)$. This invariant is connected with the study of R-equivalence on the group $\mathrm{PSO}(q)$. Recall the definition of R-equivalence.

Let G/k be a connected linear algebraic group. We recall that two rational points $g_0, g_1 \in G(k)$ are directly R-equivalent if there exists $g(t) \in G(k(t))$ such that $g(0) = g_0$ and $g(1) = 1$ and that the R-equivalence is the equivalence relation generated by this elementary relation. It is known [3] that the group $G(k)/R$ is trivial if the variety of the group G/k is stably k-rational. Merkurjev gave a formula which computes $G(k)/R$ for the adjoint classical groups. In the case of a group $\mathrm{PSO}(q)$, we have

$$\mathrm{PSO}(q)(k)/R \xrightarrow{\sim} G(q)/\mathrm{hyp}(q).k^{\times 2}.$$

Moreover, the invariant $G(k)/R$ (on suitable extensions of k) allows us to determine if an adjoint semisimple classical group with absolute rank ≤ 3 is (or is not) a stably k-rational variety. More precisely, in the case of $\mathrm{PSO}(q)$ with a quadratic form q of rank ≤ 6. Merkurjev's criterion is the following:

THEOREM 1 [9]. *Let q/k be a quadratic form with rank $2m$ ($m = 2$ or 3) and signed discriminant $(d) \in k^\times/k^{\times 2}$.*

(a) *If $d \in k^{\times 2}$, the variety $\mathrm{PSO}(q)$ is k-rational and one has $G(q)/\mathrm{hyp}(q).k^{\times 2} = 1$.*

(b) *If* $d \notin k^{\times 2}$, *we denote by* $L = k(\sqrt{d})$ *and* $C_0(q)$ *the even Clifford algebra of* q *which is a central simple algebra over* L. *One has the following alternative*:

(i) *If* $\mathrm{ind}_L(C_0(q)) = 1$ *or* 2, *then the variety* $\mathrm{PSO}(q)$ *is stably* k*-rational and* $G(q)/\mathrm{hyp}(q).k^{\times 2} = 1$.

(ii) *If* $\mathrm{Ind}_L(C_0(q)) = 4$, *then there exists a field extension* E/k *such that* $G(q_E)/\mathrm{hyp}(q_E).E^{\times 2} \neq 1$ *and the variety* $\mathrm{PSO}(q)$ *is not stably* k*-rational.*

Case (ii) can appear only if $\mathrm{rk}(q) = 6$. The proof of the theorem uses in a crucial way the Index Reduction theory (cf. [11, 18, 21]).

Remark 1. If k is a field ($car(k) \neq 2$) with cohomological dimension 1, it is well known that any group $\mathrm{PSO}(q)$ is a quasi-split group and a k-rational variety. For illustrating case (ii) of the theorem, it is necessary to assume $\mathrm{cd}(k) \geq 2$. We will show that $\mathrm{cd}(k) = 2$ is sufficient.

The construction by Merkurjev [12] for any integer n ($n \geq 2$) of a field with u-invariants (cf. Notations) equal to $2n$ from a division algebra D/k is functorial in k. More precisely, if D/k is isomorphic to $Q_1 \otimes_k Q_2 \cdots \otimes_k Q_{n-1}$ where the Q_i's are quaternion algebras, one associates a field $F(k, D)$ with cohomological dimension 2 satisfying $\mathrm{ind}(D_{F(k,D)}) = 2^{n-1}$ and $u(F(k, D)) = 2n$. Moreover, if k'/k is a field extension satisfying $\mathrm{ind}(D_{k'}) = \mathrm{ind}(D_k)$, one has a natural embedding $F(k, D) \hookrightarrow F(k', D_{k'})$. Let us apply this remark. We fix a field k of characteristic zero, D/k a division algebra which is a tensor product of 2 quaternion algebras, and a proper quadratic field extension $k' = k(\sqrt{d})$ satisfying $\mathrm{ind}(D_k) = \mathrm{ind}(D_{k'}) = 4$. For example, we can take $k = \mathbb{Q}(X_1, X_2, \ldots, X_{2n-1})$, $Q_i = (X_{2i}, X_{2i+1})_k$ for $i = 1, \ldots, n-1$ and $k' = k(\sqrt{X_1})$. Then we denote $F = F(k, D)$ and $F' = F(k', D_{k'})$. One has a natural embedding $F \hookrightarrow F'$ and since $\mathrm{ind}(D_{F'}) = 4$, one has $\mathrm{ind}(D_{F(\sqrt{d})}) = 4$. Denote $L = F(\sqrt{d})$. Let us fix an Albert form ψ for D which represents -1 and let us define the k-form q with rank 6 and signed discriminant d by $\langle 1, -d\rangle \perp \psi = q \perp \mathbb{H}$. Then $C_0(q)_L \sim D_L$, $\mathrm{cd}(F) = 2$, and q_F is an example of the quadratic form of case (ii) such that the variety $\mathrm{PSO}(q)$ is not stably F-rational.

1.2. *Norm Group of a Family of Quadratic Forms*

For any family of quadratic forms $(q_i)_{i=1,\ldots,m}$ with even rank, we denote by $\mathrm{hyp}(q_1, q_2, \ldots, q_m)$ the subgroup of $k^\times$ generated by the $N_{L/k}(L^\times)$ such that the forms $q_{i,L}$ are hyperbolic ($i = 1, \ldots, m$). Let X_i be the variety of totally isotropic subspaces of q_i with dimension $\dim(q_i)/2$. Then by definition, we have

$$\mathrm{hyp}(q_1, q_2, \ldots, q_m) = N_{X_1 \times X_2 \cdots \times X_m}(k) \subset k^\times.$$

LEMMA 1. *Let $(q_i/k)_{i=1,\ldots,m}$ be a family of quadratic forms with even rank and q/k a quadratic form with even rank.*

(a) $G(q \perp \mathbb{H}) = G(q)$.

(a′) $\mathrm{hyp}(q_1, q_2, \ldots, q_m, q \perp \mathbb{H}) = \mathrm{hyp}(q_1, q_2, \ldots, q_m, q)$.

(b) $\mathrm{hyp}(q_1, q_2, \ldots, q_m) \subset \bigcap_{i=1,\ldots,m} \mathrm{hyp}(q_i)$.

(c) *If L/k is a finite field extension, one has*

$$N_{L/k}\big(\mathrm{hyp}(q_{1,L}, q_{2,L}, \ldots, q_{m,L})\big) \subset \mathrm{hyp}(q_1, q_2, \ldots, q_m).$$

(d) *Let L/k be a finite splitting field extension for the forms $(q_i)_{i=1,\ldots,m}$. Then*

$$N_{L/k}\big(\mathrm{hyp}(q_L)\big) \subset \mathrm{hyp}(q_1, q_2, \ldots, q_m, q).$$

(e) (*respectively* [5, 9]). *Denote* $(d) = \mathrm{disc}_{\pm}(q) \in k^\times/k^{\times 2}$. *Then*

$$G(q) = G(\langle 1, -d\rangle) \cap G(q \perp \langle 1, -d\rangle)$$

and

$$\begin{aligned}\mathrm{hyp}(q_1, q_2, \ldots, q_m, q) &= \mathrm{hyp}\big(q_1, q_2, \ldots, q_m, (1, -d), q \perp \langle 1, -d\rangle\big)\\ &= N_{k(\sqrt{d})/k}\big(\mathrm{hyp}(q_{1,k(\sqrt{d})}, q_{2,k(\sqrt{d})}, \ldots, q_{m,k(\sqrt{d})}, q_{k(\sqrt{d})})\big).\end{aligned}$$

(f) *Let $\tilde{\mathsf{G}}$ be a subgroup of the profinite Galois group $\mathsf{Gal}(k_s/k)$ and $\tilde{k} = k_s^{\tilde{\mathsf{G}}}$. Then*

$$\mathrm{hyp}(q_{1,\tilde{k}}, q_{2,\tilde{k}}, \ldots, q_{m,\tilde{k}}) = \bigcup_{k' \subset \tilde{k}} \mathrm{hyp}(q_{1,k'}, q_{2,k'}, \ldots, q_{m,k'}),$$

where the union is taken on the extensions $k' \subset \tilde{k}$ of finite degree over k.

Remark 2. The main result of this paper is based on examples of quadratic forms for which the inclusion (b) is strict. For (e), Merkurjev's Theorem 1 shows that the inclusion $N_{k(\sqrt{d})/k}(G(q_{k(\sqrt{d})})) \subset G(q)$ is strict in general. For a quadratic form with rank 6 and signed discriminant d, one has indeed $G(q)/\mathrm{hyp}(q).k^{\times 2} = G(q)/N_{k(\sqrt{d})/k}(\mathrm{hyp}(q_{k(\sqrt{d})})).k^{\times 2} = G(q)/N_{k(\sqrt{d})/k}(G(q_{k(\sqrt{d})})).k^{\times 2}$ and this group is not trivial in general.

Proof. The assertion (a) is a straightforward result of Witt's theorem. The assertions (b), (c), and (d) are direct consequences of the definition and of the functoriality of norm maps for a tower of field extensions. Let us show the assertion (e). It is clear that we can assume $d \in k^{\times} \setminus k^{\times 2}$. First, the inclusion $G(\langle 1, -d\rangle) \cap G(q \perp \langle 1, -d\rangle) \subset G(q)$ is obvious. Conversely, if $a \in G(q)$, one has $\langle 1, -a\rangle \otimes q = 0 \in W(k)$ and since $q = \langle 1, -d\rangle \bmod I^2(k)$, one has $\langle 1, -a\rangle \otimes \langle 1, -d\rangle = 0 \bmod I^3(k)$ and it is known [17, p. 88, Theorem 14.3] that $\langle 1, -a\rangle \otimes \langle 1, -d\rangle = 0 \in W(k)$. Hence $\langle 1, -a\rangle \otimes (q \perp \langle 1, -d\rangle) = 0 \in W(k)$ and $a \in G(\langle 1, -d\rangle) \cap G(q \perp \langle 1, -d\rangle)$.

The second formula of (e) is simpler and results from the following fact: any field extension L/k such that q_L is hyperbolic satisfies $d \in L^{\times 2}$ and then contains a subfield isomorphic to $k(\sqrt{d})$.

(f) This identity is formal. There exists a variety X/k such that $\mathrm{hyp}(q_{1,k}, q_{2,k}, \ldots, q_{m,k}) = N_X(k)$ and it is not difficult to show that $N_Y(\tilde{k}) = \bigcup_{k' \subset \tilde{k}} N_Y(k')$ for any variety Y/k. Then one has the formula. ∎

Let us give an application of Scharlau's transfer map [17, Sect. 5] which will be useful for showing Proposition 1.

LEMMA 2. *Let q, q' be k-quadratic forms and $k' = k(x)/k$ a finite field extension with degree $[k' : k]$. Assume that $q = \langle 1, x\rangle \otimes q' \in W(k')$.*

(a) *If $[k' : k]$ is even, then one has $\langle 1, -N_{k'/k}(x)\rangle \otimes q = \langle 1, -N_{k'/k}(x)\rangle \otimes q' = 0 \in W(k)$, i.e., $N_{k'/k}K(x) \in G(q) \cap G(q')$.*

(b) *If $[k' : k]$ is odd, then one has $q = \langle 1, N_{k'/k}(x)\rangle \otimes q' \in W(k)$.*

Proof. Denote $r = [k' : k]$. In the two cases, we apply Scharlau's transfer $s_* : W(k(x)) \to W(k)$ associated with the linear form $s : k(x) \to k$ defined by $s(1) = 1$, $s(x) = s(x^2) = \cdots = s(x^{r-1}) = 0$. One has a projection formula $s_*(\varphi_{k'} \otimes \psi) = \varphi \otimes s_*(\psi)$ for any $\varphi \in W(k), \psi \in W(k')$ which reduces the calculation to $s_*(\langle 1\rangle)$ and $s_*(\langle x\rangle)$.

(a) If r is even, one has $s_*(\langle 1\rangle) = \langle 1, -N_{k'/k}(x)\rangle$ and $s_*(\langle x\rangle) = 0$. Applying s_* to q, one has $\langle 1, -N_{k'/k}(x)\rangle \otimes q = \langle 1, -N_{k'/k}(x)\rangle \otimes q' \in W(k)$. Moreover, since $\langle x\rangle \otimes q = \langle 1, x\rangle \otimes q' \in W(k')$, it follows $\langle 1, N_{k',k}(x)\rangle \otimes q' = 0 \in W(k)$ and $\langle 1, -N_{k'/k}(x)\rangle \otimes q = 0 \in W(k)$.

(b) If r is odd, one has $s_*(\langle 1\rangle) = \langle 1\rangle$ and $s_*(\langle x\rangle) = \langle N_{k'/k}(x)\rangle$. Applying s_* to q, we obtain $q = \langle 1, N_{k',k}(x)\rangle \otimes q' \in W(k)$. ∎

1.3. *Milnor's Residue Maps* (*cf*. [17, *p*. 207])

We denote by $K = k((t))$ the field of formal series with valuation ring $O = k[[t]]$. Recall that there exists an exact sequence of groups

$$0 \to W(k) \xrightarrow{i} W(K) \xrightarrow{\partial_t} W(k) \to 0.$$

The map i is the restriction of k to K and let us describe the map ∂_t. A K-quadratic form q can be diagonalized in $\langle u_1, \dots, u_m, tv_1, \dots, tv_n \rangle$ where $u_i, v_j \in O^\times$. Then $\partial_t(q) = \langle \bar{v}_1, \dots, \bar{v}_n \rangle$ where $\bar{v}_i \in k^\times = (O/t)^\times$. Let us give an application for similarity factors.

LEMMA 3. *Let γ be a k-quadratic form.*

(a) *If γ is not hyperbolic, then $G(\gamma_K) = G(\gamma).K^{\times 2}$.*

(b) *One has $G(\langle\langle t \rangle\rangle \otimes \gamma) = \mathbb{Z}\langle t \rangle.G(\gamma).K^{\times 2}$.*

Proof. (a) The inclusion $G(\gamma).K^{\times 2} \subset G(\gamma_K)$ is obvious. Conversely, let x be in $G(\gamma_K)$. Then $x = t^d a^2 \alpha$ with $a \in K^\times$, $\alpha \in k^\times$, and $d = 0$ or 1. If $d = 1$, one has $0 = \partial(\langle 1, -x \rangle \otimes \gamma) = \langle -\alpha \rangle \otimes \gamma \in W(k)$ then γ is hyperbolic and $d = 0$. Hence $\alpha \in G(\gamma_K) \cap k^\times$. It follows $0 = (\langle 1, -\alpha \rangle \otimes \gamma)_K = i(\langle 1, -\alpha \rangle \otimes \gamma)$. Hence $0 = \langle 1, -\alpha \rangle \otimes \gamma \in W(k)$, $\alpha \in G(\gamma)$, and $x \in G(\gamma).K^{\times 2}$.

(b) If the form γ is hyperbolic, then the assertion is obvious. We can assume that γ is not hyperbolic. The inclusion $\mathbb{Z}\langle t \rangle.G(\gamma).K^{\times 2} \subset G(\varphi_K)$ is obvious. Conversely, let x be in $G(\varphi_K)$. Then $x = t^{v(x)} a^2 \alpha$ with $a \in O^\times$ and $\alpha \in G(\varphi_K) \cap k^\times$. Applying the residue map $\partial : W(K) \to W(k)$, it yields $0 = \partial(\langle 1, -\alpha \rangle \otimes \varphi) = \partial(\langle 1, -\alpha \rangle \otimes \langle 1, t \rangle \otimes \gamma) = \langle 1, -\alpha \rangle \otimes \gamma \in W(k)$. Hence $\alpha \in G(\gamma)$ and $x \in \mathbb{Z}\langle t \rangle.G(\gamma).K^{\times 2}$. ∎

2. PROOF OF THE MAIN RESULT

The main result is a direct consequence of the following proposition and Merkurjev's Theorem 1.

PROPOSITION 1. *Let k be a field of characteristic zero. Let $(q_i)_{i=1,\dots,m}$ be a family of k-quadratic forms, $a \in k^\times \setminus k^{\times 2}$ and ψ/k a quadratic form satisfying the following condition*

(C) *For any $b \in k^\times$, the form $\langle\langle -a, b \rangle\rangle \perp \psi$ is not hyperbolic.*

We denote by $K = k((t))$ the field of formal series power with valuation ring $O = k[[t]]$ and

$$q = \langle\langle -a, t \rangle\rangle \perp \psi.$$

Then

$$G(q_K) = (G(\langle 1, -a \rangle) \cap G(\psi)).K^{\times 2}$$

and

$$\operatorname{hyp}\big((q_{i,K})_{i=1,\dots,m},q_K\big).K^{\times 2}=\operatorname{hyp}\big((q_i),\langle 1,-a\rangle,\psi\big).K^{\times 2}.$$

Proof. First, we observe that the condition (C) implies that the form ψ is not hyperbolic.

1st Step. *The first equality*. The inclusion $(G(\langle 1,-a\rangle)\cap G(\psi)).K^{\times 2}\subset G(q_k)$ is obvious. Conversely, let $x\in G(q_K)$. Then $x=t^d\beta^2 b$ with $\beta\in K^\times$, $d=0$ or 1, and $b\in k^\times$. If $d=1$, applying the residue map $\partial: W(K)\to W(k)$, one has $0=\partial(\langle 1,-bt\rangle\otimes q)=\partial(\langle\langle -bt,t,-a\rangle\rangle\perp\langle 1,-bt\rangle\otimes\psi)=\partial(\langle\langle -b,t,-a\rangle\rangle\perp\langle 1,-bt\rangle\otimes\psi)=\langle\langle -b,-a\rangle\rangle\perp\langle -b\rangle\otimes\psi\in W(k)$. Since $-b\in G(\langle\langle -b,-a\rangle\rangle)$, it yields $\langle\langle -b,-a\rangle\rangle\perp\psi=0\in W(k)$, which is a contradiction for the hypothesis (C). It follows that $d=0$ and $b\in G(q_K)\cap k^\times$. Applying again the map ∂_t, one can see easily that $b\in G(\langle 1,-a\rangle)$ and since $q=\langle\langle t\rangle\rangle\otimes\langle 1,-a\rangle\perp\psi$, one has $b\in G(\langle 1,-a\rangle)\cap G(\psi)$ and $x\in(G(\langle 1,-a\rangle)\cap G(\psi)).K^{\times 2}$.

2nd Step. *Reduction to the case where the base field k has no proper odd extension*. For the second equality, we will show that we can assume that the base field k has no proper odd extension. First, let us check that the condition (C) stays when we extend the scalars with an odd field extension. If k'/k is a finite odd extension and if there exists $b'\in k'^\times$ such that $\langle\langle -a,b'\rangle\rangle\perp\psi=0\in W(k')$, since $[k':k(b')]$ is odd, Springer's theorem for odd extensions [17, p. 62] yields $\langle\langle -a,b'\rangle\rangle\perp\psi=0\in W(k(b'))$ and Lemma 2 implies $\langle\langle -a,N_{k(b')/k}(b')\rangle\rangle\perp\psi=0\in W(k)$, which is a contradiction for the hypothesis (C).

Let $\tilde{\mathcal{G}}\subset \mathcal{G}al(k_s/k)$ be a 2-Sylow subgroup of the profinite Galois group $\mathcal{G}al(k_s/k)$, $\tilde{k}=k_s^{\tilde{\mathcal{G}}}$, and $\tilde{K}=K\otimes_k\tilde{k}$ and let us assume that

$$\operatorname{hyp}\big((q_{i,\tilde{K}}),q_{\tilde{K}}\big).\tilde{K}^{\otimes 2}=\operatorname{hyp}\big((q_{i,\tilde{k}}),\langle 1,-a\rangle_{\tilde{k}},\psi_{\tilde{k}}\big).\tilde{K}^{\times 2}.$$

Due to Lemma 1(f), one has

$$\operatorname{hyp}\big((q_{i,\tilde{k}}),\langle 1,-a\rangle_{\tilde{k}},\psi_{\tilde{k}}\big).\tilde{K}^{\times 2}=\bigcup_{k'\subset\tilde{k}}\operatorname{hyp}\big((q_{i,k'}),\langle 1,-a\rangle_{k'},\psi_{k'}\big).(K\otimes_k k')^{\times 2},$$

where the reunion is taken on the subextensions $k'\subset\tilde{k}$ finite over k. Now, we can show the equality

$$\operatorname{hyp}\big((q_{i,K})_{i=1,\dots,m},q_K\big).K^{\times 2}=\operatorname{hyp}\big((q_i),\langle 1,-a\rangle,\psi\big).K^{\times 2},$$

where the inclusion $\supset$ is obvious. For the inverse inclusion, let x be in

$\mathrm{hyp}((q_{i,K})_{i=1,\dots,m}, q_K).K^{\times 2}$. Since the inclusion

$$\mathrm{hyp}\big((q_{i,K})_{i=1,\dots,m}, q_K\big).K^{\times 2} \subset \mathrm{hyp}\big((q_{i,\tilde{K}}), q_{\tilde{K}}\big).\tilde{K}^{\times 2},$$

there exists a finite odd extension k'/k such that

$$x \in \mathrm{hyp}\big((q_{i,k'}), \langle 1, -a\rangle_{k'}, \psi_{k'}\big).(K \otimes_k k')^{\times 2}.$$

Hensel's lemma allows us to assume that $x \in k^\times$. If $[k' : k] = 2p + 1$, one has $N_{k'/k}(x) = x.x^{2p}$ and Lemma 1(c) yields $x \in \mathrm{Hyp}((q_i), \langle 1, -a\rangle, \psi).K^{\times 2}$.

3rd Step. *The Second Equality*. We can assume that the field k has no proper odd extensions. The inclusion

$$\mathrm{hyp}\big((q_i), \langle 1, -a\rangle, \psi\big).K^{\times 2} \subset \mathrm{hyp}\big((q_{i,K})_{i=1,\dots,m}, q_K\big).K^{\times 2}$$

is obvious. For the inverse inclusion, we have to show for any finite extension L/K splitting q and the q_i's that $N_{L/K}(L^\times) \subset \mathrm{hyp}((q_i), \langle 1, -a\rangle, \psi).K^{\times 2}$. Let L/K be such a finite extension with valuation ring O_L, residue field k', ramification index e, and residual index f. Let us denote by K'/K the maximal non-ramified extension of K with valuation ring O'. Since k has characteristic zero, the field K' is k-isomorphic to $k'((t))$. Therefore we can assume that $K' = k'((t))$.

$$\begin{array}{c} L \\ |\, e \\ K' \\ |\, f \\ K \end{array}$$

We recall that there exists an uniformizing parameter π of L/K such that $\pi^e t^{-1} \in k'$. If π is an uniformizing parameter of L, then $\pi^e t^{-1}$ has valuation 1 and since $O'^\times/O'^{\times e} \simeq k'^\times/k'^{\times e}$, there exists $a \in O'^\times$ such that $(a\pi)^e g^{-1} \in k'$. Therefore we can take an uniforming parameter π of L such that $\pi^e = ut$ with $u \in k'$. With Hensel's lemma, we can compute easily the norm group $N_{L/K}(L^\times)$ up to $U_1 = \mathrm{Ker}(O^\times \to k^\times)$, which is sufficient because one has $U_1 \subset K^{\times 2}$.

LEMMA 4. $N_{L/K}(L^\times) = \mathbb{Z}\Big\langle N_{k'/k}\big((-1)^{e+1}u\big)t^f, \big(N_{k'/k}(k'^\times)\big)^e\Big\rangle$ mod U_1.

In order to use the hypothesis q_L hyperbolic, we write the functoriality of Milnor's residue maps for the extensions $K \subset K' \subset L$.

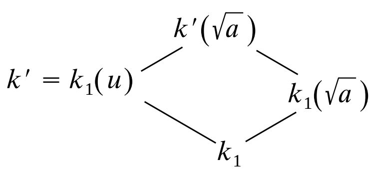
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{W(k)} & \xrightarrow{i} & \mathrm{W(K)} & \xrightarrow{\partial_t} & \mathrm{W(k)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Res}_k^{k'}} & & \downarrow{\scriptstyle \mathrm{Res}_K^{K'}} & & \downarrow{\scriptstyle \mathrm{Res}_k^{k'}} & & \\
0 & \longrightarrow & \mathrm{W(k')} & \xrightarrow{i_1} & \mathrm{W(K')} & \xrightarrow{\partial_t} & \mathrm{W(k')} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \mathrm{Res}_{K'}^{L}} & & \downarrow{\scriptstyle \rho} & & \\
0 & \longrightarrow & \mathrm{W(k')} & \xrightarrow{j} & \mathrm{W(L)} & \xrightarrow{\partial_\pi} & \mathrm{W(k')} & \longrightarrow & 0,
\end{array} \qquad (*)$$

where $\rho = 0$ if e is even and $\rho = \mathrm{id}_{W(k')}$ if e is odd. Since L/K splits the q_i's, the diagram shows that the $q_{i,k}$'s are hyperbolic forms.

(i) 1*st Case*. *e Is Even*. Lemma 4 shows that $N_{L/K}(L^\times) \subset \mathbb{Z}\langle N_{k'/k}(-u)t^f\rangle.K^{\times 2}$. It is sufficient to show that f is even and that $N_{k'/k}(-u) \in \mathrm{hyp}(q_i, \langle 1, -a\rangle, \psi).k^{\times 2}$. One has $q_L = \langle\langle t, -a\rangle\rangle \perp \psi = \langle\langle u\pi^e, -a\rangle\rangle \perp \psi = \langle\langle u, -a\rangle\rangle \perp \psi = j(\langle\langle u, -a\rangle\rangle_{k'} \perp \psi_{k'})$. Then $q_L = j(\langle\langle u, -a\rangle\rangle_{k'} \perp \psi_{k'})$ and since $q_L = 0 \in W(L)$, it follows

$$0 = \langle\langle u, -a\rangle\rangle_{k'} \perp \psi_{k'} \in W(k'). \qquad (**)$$

The hypothesis (C) implies that $f = [k' : k] = 2^s > 1$ and f is even. It remains to show that $N_{k'/k}(-u) \in \mathrm{hyp}((q_i), \langle 1, -a\rangle, \psi).k^{\times 2}$. If $[k' : k(u)] = 2^r > 1$, one has $N_{k'/k}(-u) \in k^{\times 2}$ and there is nothing to do. We can assume that $k' = k(u)$. Let us denote $k_1 = k(u^2) \subset k' = k_1(u)$ which is a quadratic extension and let us consider the following diagram of quadratic extensions:



Lemma 2 applied to the extension $k'/k_1 = k_1(u)/k_1$ and the identity $\langle 1, u\rangle \otimes \langle 1, -a\rangle_{k'} = \langle -1\rangle \otimes \psi_{k'}$ $(**)$ yields

$$N_{k'/k_1}(u) \in G(\langle 1, a\rangle_{k_1}) \cap G(\psi_{k_1}).$$

Then $N_{k'/k_1}(u) = N_{k'/k_1}(-u) \in N_{k_1(\sqrt{a})/k_1}(k_1(\sqrt{a})^\times)$. On the other hand, since $k_1(u) = k'$ and $k_1(\sqrt{a})$ are two quadratic extensions of k_1, it is known (Lemma 1.4 of [7]) that

$$N_{k'/k_1}(k'^{\times}) \cap N_{k_1(\sqrt{a})/k_1}\Big(k_1(\sqrt{a})^{\times}\Big) = N_{k'(\sqrt{a})/k_1}\Big(k'(\sqrt{a})^{\times}\Big).k_1^{\times 2}.$$

The extension $k'(\sqrt{a})$ splits the forms $\langle 1, -a\rangle$, $\psi_{k'} = \langle -1, -u\rangle \langle 1, -a\rangle_{k'}$ and the q_i's. Therefore one has $N_{k'/k_1}(-u) \in \text{hyp}((q_i, k_1), \langle 1, -a\rangle_{k_1}, \psi_{k_1}).k_1^{\times 2}$. Applying Lemma 1(c) to the extension k_1/k, it follows that $N_{k'/k}(u) = N_{k_1/k}(N_{k'/k_1}(-u)) \in \text{hyp}((q_i), \langle 1, -a\rangle, \psi).k^{\times 2}$. We showed this case.

(ii) 2*nd Case*. *e Is Odd*. With the diagram of Milnor's residue maps, we see that the form $\langle 1, -a\rangle_{k'} = \partial_\pi(q_L)$ is hyperbolic. Moreover, $0 = q_L = j(\langle 1, -a\rangle_{k'} \perp \psi_{k'})$. Then the form $\psi_{k'}$ is hyperbolic. Since ψ is not hyperbolic, according to Springer's theorem for odd extensions [17, p. 62], the integer $f = [k' : k]$ is even. Hence, one has $N_{L/K}(L^{\times}) \subset N_{k'/k}(k'^{\times}).K^{\times 2}$. The forms $q_{i,k'}, \psi_{k'}$ and $\langle 1, -a\rangle_{k'}$ are hyperbolic and then it yields $N_{k'/k}(k'^{\times}) \subset \text{hyp}((q_i), \psi, \langle 1, -a\rangle)$ and $N_{L/K}(L^{\times}) \subset \text{hyp}((q_i), \langle 1, -a\rangle, \psi)).K^{\times 2}$. ∎

THEOREM 2. *Let $a \in k^{\times} \setminus k^{\times 2}$ and D/k be a biquaternion algebra and let ψ/k be an Albert form associated with D which represents* -1. *Denote $K = k((t))$. Let us define the k-form q_0 and the K-form q by*

$$\langle 1, -a\rangle \perp \psi = q_0 \perp \mathbb{H}$$

and

$$\langle\langle -a, t\rangle\rangle \perp \psi = q \perp \mathbb{H}.$$

One has $\text{rk}_k(q_0) = 6$, $\text{disc}_{\pm}(q_0) = (a)$, $\text{rk}_K(q_0) = 8$, *and* $\text{disc}_{\pm}(q) = 1$.

(a) *If* $\text{ind}(D_{k(\sqrt{a})}) \neq 1$, *there exists a natural isomorphism*

$$G(q_0)/\text{hyp}(q_0).k^{\times 2} \xrightarrow{\sim} G(q_K)/\text{hyp}(q_K).K^{\times 2}.$$

(b) *If* $\text{ind}(D_{k(\sqrt{a})}) = 4$, *there exists a field extension E/k such that* $G(q_{K\otimes_k E})/\text{hyp}(q_{K\otimes_k E}).(K \otimes_k E)^{\times 2} \neq 1$ *and such that the variety* PSO(q) *is not stably K-rational.*

Proof. Due to Lemma 1(a), we can do the proof with $q_0 = \langle 1, -a\rangle \perp \psi$ and $q = \langle\langle -a, t\rangle\rangle \perp \psi$.

(a) We will apply the preceding proposition to the form q and we have to check hypothesis (C). Let $b \in k^{\times}$ be such that $\langle\langle -a, b\rangle\rangle \perp \psi = 0 \in W(k)$. Then the form $\psi_{k(\sqrt{a})}$ is hyperbolic and the algebra $D_{k(\sqrt{a})}$ is split, which is a contradiction for the hypothesis $\text{ind}(D_{k(\sqrt{a})}) > 1$. The hypothesis (C) is checked and the proposition yields

$$G(q_K) = (G(\langle 1, -a\rangle) \cap G(\psi)).K^{\times 2}$$

and

$$\mathrm{hyp}(q_K).K^{\times 2} = \mathrm{hyp}(\langle 1, -a\rangle, \psi).K^{\times 2}.$$

Due to Lemma 1(e), one has $G(q_0) = G(\langle 1, -a\rangle) \cap G(\psi)$ and $\mathrm{hyp}(q_0).k^{\times 2} = N_{k(\sqrt{a})/k}(\mathrm{hyp}(\psi_{k(\sqrt{a})})).k^{\times 2}$. Then we have an isomorphism

$$G(q_0)/\mathrm{hyp}(q_0).k^{\times 2} \xrightarrow{\sim} G(q_K)/\mathrm{hyp}(q_K).K^{\times 2}.$$

(b) Since $\mathrm{ind}(D_{k(\sqrt{a})}) = 4$, Theorem 1 shows the existence of a field extension E/k such that $a \notin E^{\times 2}$, $\mathrm{ind}(D_{E(\sqrt{a})}) = 4$, and $G(q_{0,E})/\mathrm{hyp}(q_{0,E}).E^{\times 2} \neq 1$. Hence $G(q_{K\otimes_k E})/\mathrm{hyp}(q_{K\otimes_k E}).(K \otimes_k E)^{\times 2} \neq 1$ and the variety $\mathrm{PSO}(q_K)$ is not stably K-rational. ∎

Remark 1 yields a field k with cohomological dimension 2, a quadratic field extension $L = k(\sqrt{a})$, and an Albert form ψ which represents -1 and satisfies $\mathrm{ind}(C_0(\psi)_L) = 4$. We showed the result claimed in the Introduction.

THEOREM 3. *There exist a field k of characteristic* 0, *with cohomological dimension* 3 *and a quadratic form q with rank* 8 *and signed discriminant* 1 *such that the variety* $\mathrm{PSO}(q)/k$ *is not stably k-rational.*

Due to Theorem 1, the dimension 8 is minimal for such an example with trivial signed discriminant. On the other hand, we don't know if there exists such an example with $\mathrm{cd}(k) = 2$. The method used here brings nothing if $\mathrm{cd}(k) = 2$. In this case, due to the Merkurjev–Suslin theorem, the Galois symbol yields an isomorphism $I^2(k') \xrightarrow{\sim} Br_2(k')$ for any finite extension k'/k (cf. [1]). For any quadratic form q with trivial signed discriminant and Clifford algebra $C(q)/k$, one has $k^{\times} = Nrd(C(q)^{\times}).k^{\times 2} = \mathrm{hyp}(q).k^{\times 2} = G(q)$ and the invariant $G(q_k)/\mathrm{hyp}(q_k).k^{\times 2}$ is trivial on k.

We have to underline that we used the Index Reduction Theory (through [9]) for giving proof of our result. We shall see that with cohomological dimension 6 instead 3, we can show the same result without the Index Reduction Theory and thus we can produce explicit elementary examples of non-rational adjoint groups built from an iteration of Proposition 1 with a field of iterated formal power series. This method contains some analogies with Platonov's counterexample [15] to the Kneser–Tits conjecture, showing the existence of simply connected semisimple groups defined over a field k, which are not k-rational varieties.

3. SUMS OF QUATERNIONIC FORMS

First, we introduce an invariant related to the multiquadratic extensions. This invariant will be used for computing some group $\mathrm{PSO}(q)/R$.

DEFINITION 1. Let $A = (a_i)_{i=1,\dots,m}$ be a family of elements of $k^\times$. Denote $k_i = k[t]/(t^2 - a_i)$ for $i = 1,\dots,m$ and $M = k_1.\otimes k_2 \otimes_k \cdots \otimes_k k_m$. One defines the group

$$\Lambda(A/k) = \Big(\bigcap_{i=1,\dots,m} N_k(a_i) \Big) / N_{M/k}(M^\times).k^{\times 2}.$$

PROPOSITION 2. *Let $A = (a_i)_{i=1,\dots,m}$ be a family of elements of $k^\times$ and M/k as in the definition. The following assertions hold.*

(a) $\mathrm{hyp}((\langle 1, -a_i \rangle)_{i=1,\dots,m}).k^{\times 2} = N_{M/k}(M^\times).k^{\times 2}$.

(b) *Let T be the k-torus defined by the equations*

$$N_{k_1/k}(y_1) = N_{k_2/k}(y_2) = \cdots = N_{k_m/k}(y_m) \neq 0.$$

Then, we have a natural isomorphism $T(k)/R \xrightarrow{\sim} \Lambda(A/k)$.

Proof. We denote by $G = \mathrm{Gal}(M/k)$ the Galois group of $k_1.k_2 \cdots k_m/k$ and by $G_i \subset G$ the subgroup which fixes $\sqrt{a_i}$ $(i = 1,\dots,m)$. One can assume that $a_i \notin k^{\times 2}$ for $i = 1,\dots,m$ and let us denote by σ_i the generator of $G/G_i = \mathrm{Gal}(k_i/k)$. One has an injective morphism $j : T \subset \prod_{i=1,\dots,m} R_{k_i/k}\mathbb{G}_m$ and a morphism $q = N_{k_1/k} \circ j : T \to \mathbb{G}_m$ whose kernel is denoted by $T' = \prod_{i=1,\dots,m} R^1_{k_i/k}\mathbb{G}_m$. We define a surjective morphism of k-tori

$$p\colon R_{M/k}\mathbb{G}_m \times \mathbb{G}_m \times \prod_{i=1,\dots,m} R^1_{k_i/k}\mathbb{G}_m \to T \subset \prod_{i=1,\dots,m} R_{k_i/k}\mathbb{G}_m,$$

where

$$[p(y, x, y_1, \dots, y_m)]_i = N_{M/k_i}(y).x.y_i/\sigma_i(y_i) \quad \text{for } i = 1,\dots,m.$$

Let us denote by $E = R_{M/k}\mathbb{G}_m \times \mathbb{G}_m \times \prod_{i=1,\dots,m} R_{k_i/k}\mathbb{G}_m$, by $S = \mathrm{Ker}(p)$ the torus kernel of p, and by $\hat{S}^0$ the Galois module of cocharacters of S, i.e., $\hat{S}^0 = \mathrm{Hom}_{gr}(\mathbb{G}_m, S)$. The following lemma is easy to show.

LEMMA 5. $H^1(H, \hat{S}^0) = 0$ *for any subgroup $H \subset G$.*

In other words, the morphism p defines an exact sequence of k-tori

$$1 \to S \to E \xrightarrow{p} T \to 1,$$

which is a flasque resolution of the torus T (cf. [3]) and then the boundary map ∂: $T(k) \to H^1(k,S)$ induces an isomorphism $T(k)/R \simeq H^1(k,S)$. Since $H^1(k,E) = 1$ [20, chap. X], one has an isomorphism $T(k)/p(E(k)) \simeq H^1(k,S)$. We consider the following commutative exact diagram

$$\begin{array}{ccccccc} & & \mathrm{T}'(\mathrm{k}) & & & & \\ & & \downarrow & & & & \\ \mathrm{E}(\mathrm{k}) & \xrightarrow{p} & \mathrm{T}(\mathrm{k}) & \longrightarrow & \mathrm{T}(\mathrm{k})/\mathrm{R} & \longrightarrow & 1. \\ {\scriptstyle q\circ p}\downarrow & & {\scriptstyle q}\downarrow & & & & \\ \mathrm{k}^\times & = & \mathrm{k}^\times. & & & & \end{array}$$

Since the torus $T' = \prod_{i=1,\dots,m} R^1_{k_i/k}\mathbb{G}_m$ is a rational variety, the map $T(k) \to T(k)/R$ factorizes by p and then one has an isomorphism $T(k)/R = T(k)/p(E(k)) \simeq q(T(k))/q \circ p(E(k)) = \Lambda(A/k)$.

Remark 3. Following [8], if k is a number field, the invariant $\Lambda(A/k)$ is always trivial, and Colliot-Thélène and Sansuc showed that the group $T(k)/R$ is finite for any torus defined over a field of finite type over the prime field [3]. Therefore, if the field k is of finite type over the prime field, the group $\Lambda(A/k)$ is finite.

We know that $n = 2$ yields $\Lambda(A/k) = 1$ [7, Lemma 1.4]. We can show this with the proposition (b). The torus T is indeed an open subset of a quadric having a rational point which is a rational variety, hence $1 = T(k)/R = \Lambda(A/k)$.

For $n = 3$, we can deduce the non-triviality of the invariant Λ of Proposition 2.4 of [8]. More precisely, one has the following nice result of Tignol which connects the invariant N_1 of a triquadratic extensions and Λ.

PROPOSITION 3 (Tignol, unpublished). *Let $A = \{a, b, c\}$ be a family $k^\times$. Denote $M = k(\sqrt{a}, \sqrt{b}, \sqrt{c})$ and $E = k(\sqrt{c})$. Then there exists an isomorphism of groups*

$$N_1(a,b,c) = \frac{k^\times \cap N_E(a).N_E(b)}{\big(k^\times \cap N_E(a)\big).\big(k^\times \cap N_E(b)\big)} \xrightarrow{\sim} \frac{N_k(a) \cap N_k(b) \cap N_k(c)}{N_{M/k}(M^\times).k^{\times 2}}.$$

Proof. If the extension E/k is not proper, the two groups are trivial. We can assume that E/k is a proper extension and we denote by $h \to \overline{h}$ the action of $\mathrm{Gal}(E/k)$ on E. One defines the map between the two quotients with the following map θ. If $f = N_{E(\sqrt{a})/E}(x).N_{E(\sqrt{b})/E}(y) \in k^\times$, we define

$$\theta(f) = \left[N_{E(\sqrt{a})/k}(x)\right] = \left[f^2 N_{E(\sqrt{b})/k}(y)\right]$$
$$\in N_k(a) \cap N_k(b) \cap N_k(c) \qquad \text{mod } N_{M/k}(M^\times).k^{\times 2}.$$

Let us show that the element $\theta(f)$ is well defined. Indeed, if $f = N_{E(\sqrt{a})/E}(x')N_{E(\sqrt{b})/E}(y')$, one has

$$N_{E(\sqrt{a})/E}(xx'^{-1}) = N_{E(\sqrt{b})/E}(yy'^{-1})$$
$$\in N_E(a) \cap N_E(b) = N_{M/E}(M^\times).E^{\times 2}$$

using again Lemma 1.4 of [7]. Hence $N_{E(\sqrt{a})/k}(xx'^{-1}) \in N_{M/k}(M^\times).k^{\times 2}$. On the other hand, if $f \in (k^\times \cap N_E(a)).(k^\times \cap N_E(b))$, then we can assume $N_{E(\sqrt{a})/E}(x) \in k^\times$ and hence $N_{E(\sqrt{a})/k}(x) \in k^{\times 2}$. Denoting again the quotient map by θ, we define a morphism of groups

$$\theta\colon \frac{k^\times \cap N_E(a).N_E(b)}{\left(k^\times \cap N_E(a)\right).\left(k^\times \cap N_E(b)\right)} \to \frac{N_k(a) \cap N_k(b) \cap N_k(c)}{N_{M/k}(MU\times).k^{\times 2}}.$$

Let us show the injectivity of θ. If $N_{E(\sqrt{a})/k}(x) = g^2 N_{M/k}(z)$ with $g \in k^\times$, $z \in M^\times$, then

$$N_{E/k}\left(N_{E(\sqrt{a})/E}(x)\right) = N_{E/k}\left(gN_{M/E}(z)\right).$$

Hence by [20, chap. X], $N_{E(\sqrt{a})/E}(x) = gN_{M/k}(z)h\bar{h}^{-1}$ with $h \in E$. Then $N_{E(\sqrt{a})/E}(x) = (gh\bar{h})(\bar{h}^{-2}N_{M/E}(z))$. One has

$$\bar{h}^{-2}N_{M/E}(z) \in E^{\times 2}.N_{M/E}(M^\times) = N_E(a) \cap N_E(b).$$

Then the preceding equality shows that $gh\bar{h} \in k^\times \cap N_E(a)$. On the other hand, since $f = N_{E(\sqrt{a})/E}(x)N_{E(\sqrt{b})/E}(y)$, one has

$$f = \left(gh\bar{h}\right).\left(\bar{h}^{-2}N_{M/E}(z)N_{E(\sqrt{b})/E}(y)\right).$$

The second term is an element of $N_E(b)$ but has to be also an element of $k^\times$, then $f \in (k^\times \cap N_E(a)).(k^\times \cap N_E(b))$.

Let us show the surjectivity of θ for finishing the proof. If $t \in N_k(a) \cap N_k(b) \cap N_k(c)$, we can choose $u \in E^\times$ such that $t = N_{E/k}(u)$. Since $t \in N_k(a)$, one has $u \in k^\times.N_E(a)$; in the same way, one has $u \in k^\times.N_E(b)$ because $t \in N_k(b)$. Then

$$u = gN_{E(\sqrt{a})/E}(x) = hN_{E(\sqrt{a})/E}(y^{-1}),$$

and $g^{-1}h = N_{E(\sqrt{a})/E}(x).N_{E(\sqrt{a})/E}(y) \in k^{\times} \cap N_E(a)N_E(b)$ has for image by θ, $N_{E(\sqrt{a})/k}(x) = g^{-2}N_{E/k}(u) = t \bmod k^{\times 2}$. ∎

Remark 4. We denote by $\mathbb{Q}_2$ the 2-adic completion of $\mathbb{Q}$. If $k \in \mathbb{Q}_2(x)$ (or $\mathbb{Q}(x)$), it is shown in [16, Sect. 5.4] that $N_1(x+4, x+1, x) \neq 1$. Then for $A = \{x+4, x+1, x\}$, the group $\Lambda(A/\mathbb{Q}_2(x))$ is not trivial. Let us give an explicit element of $\Lambda(A/\mathbb{Q}_2(x))$. Due to Theorem 5.1 of [16], we know that the class of 2 in $N_1(x+4, x+1, x)$ is not trivial. If θ denotes the isomorphism $N_1(x+4, x+1, x) \simeq \Lambda(A/\mathbb{Q}_2(x))$ given by the proposition, one computes easily $\theta(2) = -x$. Hence the class of $-x$ is not trivial in $\Lambda(A/\mathbb{Q}_2(x))$. There exists an example of non-trivial invariant Λ with the base field $\mathbb{C}(t_1, t_2)$ which has cohomological dimension 2 [16].

THEOREM 4. *Assume that the base field k has characteristic* 0. *Let m be an integer, $m \geq 2$, and $A = (a_i)_{i=1,\dots,m}$ a family of elements in $k^{\times} \setminus k^{\times 2}$ such that $a_i/a_{i-1} \notin k^{\times 2}$ for $i = 2,\dots,m$. Denote $k_i = k(\sqrt{a_i})$ for $i = 1,\dots,m$ and $M = k_1.k_2 \cdots k_m$. Let $(c_i)_{i=1,\dots,m}$ be a family of elements of $k^{\times}$ and $(X_i)_{i=1,\dots,m}$ a family of indeterminates on k. Denote $F_0 = k$, $F_i = k((X_1))((X_2)) \cdots ((X_i))$ $(i = 1,\dots,m)$, $F = F_m$, and*

$$\Phi = \langle c_1\rangle \otimes \langle\langle -a_1, X_1\rangle\rangle \perp \langle c_2\rangle \otimes \langle\langle -a_2, X_2\rangle\rangle \cdots \perp \langle c_m\rangle \otimes \langle\langle -a_m, X_m\rangle\rangle.$$

Then one has

$$G(\Phi_F) = \Big(\bigcap_{i=1,\dots,m} N_k(a_i)\Big).F^{\times 2}, \qquad \mathrm{hyp}(\Phi_F).F^{\times 2} = N_{M/k}(M^{\times}).F^{\times 2},$$

and

$$\Lambda(A/k) \xrightarrow{\sim} G(\Phi_F)/\mathrm{hyp}(\Phi_F).F^{\times 2}.$$

In order to apply Proposition 1, we have to check the validity of condition (C).

LEMMA 6. *Let $m, \Phi, \dots$ as in Theorem* 4. *Denote*

$$\Phi^{m-1} = \langle c_1\rangle \otimes \langle\langle -a_1, X_1\rangle\rangle \perp \langle c_2\rangle \otimes \langle\langle -a_2, X_2\rangle\rangle \perp \cdots \perp \langle c_{m-1}\rangle \otimes \langle\langle -a_{m-1}, X_{m-1}\rangle\rangle.$$

Then for any $b \in F_{m-1}^{\times}$, one has

$$\langle\langle -a_m, b\rangle\rangle \perp \langle c_m^{-1}\rangle \Phi^{m-1} \neq 0 \in W(F_{m-1}).$$

Proof of the Lemma. We denote by $v_{X_{m-1}}\colon F_{m-1}^{\times} \to \mathbb{Z}$ the valuation associated to the uniformizing parameter X_{m-1}. We apply the residue map $\partial_{X_{m-1}}\colon W(F_{m-1}) \to W(F_{m-2})$ to a relation $\langle\langle -a_m, b\rangle\rangle \perp \langle c_m^{-1}\rangle \otimes \Phi^{m-1} = 0 \in W(F_{m-1})$ where $b \in F_{m-1}^{\times}$. If $v_{X_{m-1}}(b)$ is even, then $\langle c_{m-1}\rangle \otimes \langle 1, -a_{m-1}\rangle = 0 \in W(F_{m-2})$ and $a_{m-1} \in k^{\times} \cap F_{m-2}^{\times 2} = k^{\times 2}$, which is wrong by hypothesis. Then $v_{X_{m-1}}(b)$ is odd and the map $\partial_{X_{m-1}}$ yields $\langle c_m\rangle \otimes \langle 1, -a_m\rangle \perp \langle c_{m-1}\rangle \otimes \langle 1, -a_{m-1}\rangle = 0 \in W(F_{m-2})$. Taking the signed discriminant, we have $a_m/a_{m-1} \in k^{\times} \cap F_{m-2}^{\times 2} = k^{\times 2}$, which is wrong by hypothesis. We showed the lemma. ∎

With this lemma, we can apply Proposition 1. Let us show by induction on $m \geq 2$ the equalities

(1) $G(\Phi_F) = \Big(\bigcap_{i=1,\dots,m} N_k(a_i)\Big).F^{\times 2}$ and

(2) $\mathrm{hyp}((q_{j,F}), \Phi_F).F^{\times 2} = \mathrm{hyp}\big((q_j), \langle 1, -a_1\rangle, \dots, \langle 1, -a_m\rangle\big).F^{\times 2}$

for any finite family (q_j) of k-forms.

$m = 2$. Due to Proposition 1 applied to the base field of F_2 and forms $\langle\langle a_2, X_2\rangle\rangle$, $\psi = \Phi^1 = \langle c_1\rangle \otimes \langle\langle -a_1, X_1\rangle\rangle$ and the uniformizing parameter X_2, one has

$$G(\Phi_{F_2}) = \big(G(\langle\langle -a_1, X_1\rangle\rangle_{F_1}) \cap G(\langle 1, -a_2\rangle_{F_1})\big).F_2^{\times 2}.$$

Applying Lemma 3 with the uniformizing parameter X_1, it produces

$$G(\langle 1, -a_2\rangle_{F_1}) = G(\langle 1, -a_2\rangle).F_1^{\times 2}.$$

Since $k^{\times} \cap G(\langle 1, -a_2\rangle).F_1^{\times 2} = G(\langle 1, -a_2\rangle)$, one has

$$G(\Phi_{F_2}) = \Big(\bigcap_{i=1,2} G(\langle 1, -a_i\rangle)\Big).F_2^{\times 2} = \Big(\bigcap_{i=1,2} N_k(a_i)\Big).F_2^{\times 2}.$$

For the other equality, Proposition 1 shows that

$$\begin{aligned}
&\mathrm{hyp}\big((q_{j,F_2}), \Phi_{F_2}\big).F_2^{\times 2}\\
&\quad = \mathrm{hyp}\big((q_{j,F_2}), \langle c_1\rangle \otimes \langle\langle -a_1, X_1\rangle\rangle_{F_2} \perp \langle c_2\rangle \otimes \langle\langle -a_2, X_2\rangle\rangle\big).F_2^{\times 2}\\
&\quad = \mathrm{hyp}\big((q_{j,F_1}), \langle 1, -a_1\rangle_{F_1}, \langle 1, -a_2\rangle_{F_1}\big).F_2^{\times 2}\\
&\quad = \mathrm{hyp}\big((q_j), \langle 1, -a_1\rangle, \langle 1, -a_2\rangle\big).F_2^{\times 2} \qquad \text{(Lemma 3)}.
\end{aligned}$$

$m \geq 3$. Let us denote $\Phi^{m-1} = \langle c_1\rangle \otimes \langle\langle -a_1, X_1\rangle\rangle \perp \langle c_2\rangle \otimes \langle\langle -a_2, X_2\rangle\rangle \perp \cdots \perp \langle c_{m-1}\rangle \otimes \langle\langle -a_{m-1}, X_{m-1}\rangle\rangle$. Lemma 6 allows us to apply Proposition 1 with the uniformizing parameter X_m and it yields

$$G(\Phi_{F_m}) = \Big(G\big(\Phi^{m-1}_{F_{m-1}}\big) \cap G\big(\langle 1, -a_m\rangle_{F_{m-1}}\big)\Big).F_m^{\times 2}.$$

The induction hypothesis yields

$$G\big(\Phi^{m-1}_{F_{m-1}}\big) = \Bigg(\bigcap_{i=1,\dots,m-1} G(\langle 1, -a_i\rangle)\Bigg).F_{m-1}^{\times 2}$$

and with an iteration of Lemma 3(b), one has

$$G\big(\langle 1, -a_m\rangle_{F_{m-1}}\big) = G\big(\langle 1, -a_m\rangle\big).F_{m-1}^{\times 2}.$$

Since $k^\times \cap G(\langle 1, -a_m\rangle).F_{m-1}^{\times 2} = G(\langle 1, -a_m\rangle)$, we have

$$\begin{aligned} G(\Phi_{F_m}) &= \Bigg(G(\langle 1, -a_m\rangle) \cap \bigcap_{i=1,\dots,m-1} G(\langle 1, -a_i\rangle)\Bigg).F_m^{\times 2} \\ &= \Bigg(\bigcap_{i=1,\dots,m} N_k(a_i)\Bigg).F_m^{\times 2}. \end{aligned}$$

For the equality (2), Proposition 1 shows that

$$\begin{aligned} &\operatorname{hyp}\big((q_{j,F_m}), \Phi_{F_m}\big).F_m^{\times 2} \\ &\quad = \operatorname{hyp}\big((q_{j,F_{m-1}}), \Phi^{m-1}_{F_{m-1}}, \langle 1, -a_m\rangle_{F_{m-1}}\big).F_m^{\times 2} \\ &\quad = \operatorname{hyp}\big((q_j), \langle 1, -a_1\rangle, \langle 1, -a_2\rangle, \dots, \langle 1, -a_m\rangle\big).F_m^{\times 2} \end{aligned}$$

due to the induction hypothesis applied with $m-1$ and the set of k-forms $((q_j), \langle 1, -a_m\rangle)$. We showed by induction the two equalities. Taking $q_j = 0$ in the equality (2), we have

$$\operatorname{hyp}(\Phi_{F_m}).F_m^{\times 2} = \operatorname{hyp}(\langle 1, -a_1\rangle, \dots, \langle 1, -a_m\rangle).F_m^{\times 2} = N_{M/k}(M^\times).F_m^{\times 2}.$$

Since $k^\times \cap F^{\times 2} = k^{\times 2}$, it is easy to check that one has an isomorphism

$$\Lambda(A/k) \xrightarrow{\sim} G(\Phi_F)/\operatorname{hyp}(\Phi_F).F^{\times 2}. \quad \blacksquare$$

Application. Let a_1, a_2, a_3 be in $k^\times$ such that $a_1/a_2, a_2/a_3 \notin k^{\times 2}$ and let us denote $F = k((X_1))((X_2)))((X_3))$ and $M = k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$. Let us apply Theorem 4 to the quadratic form

$$\begin{aligned} \Phi &= \langle\langle -a_1, X_1\rangle\rangle \perp \langle -1\rangle \otimes \langle\langle -a_2, X_2\rangle\rangle \perp \langle a_1\rangle \otimes \langle\langle -a_3, X_3\rangle\rangle \\ &= q \perp \mathbb{H} \perp \mathbb{H}. \end{aligned}$$

The form q has rank 8 and signed discriminant 1, and with the notations of the theorem above, one has

$$\Lambda(A/k) \tilde{\to} G(\Phi_F)/\mathrm{hyp}(\Phi_F).F^{\times 2} \tilde{\to} G(q_F)/\mathrm{hyp}(q_F).F^{\times 2}.$$

Then, for the field $F = \mathbb{Q}_2(x)((X_1))((X_2))((X_3))$, which has cohomological dimension 6, or for the field $F = \mathbb{Q}(x)((X_1))((X_2))((X_3))$, following Remark 4 and taking $a_1 = x + 4$, $a_2 = x + 1$, $a_3 = x$, we have $G(q_F)/\mathrm{hyp}(q_F).F^{\times 2} \neq 1$ and the variety $\mathrm{PSO}(q)/F$ is not F-stably rational. More precisely, in this case we have

$$\begin{aligned}\Phi &= \langle\langle -(x+4), X_1\rangle\rangle \perp \langle -1\rangle \otimes \langle\langle -(x+1), X_2\rangle\rangle \\ &\perp \langle x+4\rangle \otimes \langle\langle -x, X_3\rangle\rangle \\ &= q \perp \mathbb{H} \perp \mathbb{H},\end{aligned}$$

and $-x$ is a similarity factor of q_F such that $-x \notin \mathrm{hyp}(q_F).F^{\times 2}$.

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