Real bivariant K theory and index formulas

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For X compact space define $KO(X)$ like in the complex case

Equivalence classes of real vector bundles over X
$\rightarrow$ monoïd for Whitney sum

Add formal inverses
$\rightarrow KO(X)$

Extend to pairs (X, Y) and locally compact spaces
$\rightarrow KO(X, Y)$, $KO(X) = KO(X^+, \infty)$

Higher KO groups with suspensions $S^nX = \mathbb{R}^n \times X$
$\rightarrow KO_n(X) = KO(S^nX)$,

d	0	1	2	3	4	5	6	7
$KO_d(pt)$								

$KO(pt) = \mathbb{Z}$ given by dimension

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d	0	1	2	3	4	5	6	7
$KO_d(pt)$	$\mathbb{Z}$							

$KO_1(pt)$

d	0	1	2	3	4	5	6	7
$KO_d(pt)$	$\mathbb{Z}$							

$KO_1(pt)$ real vector bundles on the circle modulo trivial ones

d	0	1	2	3	4	5	6	7
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Clutching function at end point

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Clutching function at end point
$KO_1(pt) \simeq \pi_0(\mathcal{O}) \simeq \mathbb{Z}_2$

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$KO_1(pt)$ real vector bundles on the circle modulo trivial ones
Clutching function at end point
$KO_1(pt) \simeq \pi_0(\mathcal{O}) \simeq \mathbb{Z}_2$
$-1 \in \mathcal{O}_1$ gives the Möbius strip as the generator of $KO_1(pt)$

d	0	1	2	3	4	5	6	7
$KO_d(pt)$	$\mathbb{Z}$	$\mathbb{Z}_2$						

$KO_2(pt)$

d	0	1	2	3	4	5	6	7
$KO_d(pt)$	$\mathbb{Z}$	$\mathbb{Z}_2$						

$KO_2(pt)$ real vector bundles on the 2 sphere modulo trivial ones

d	0	1	2	3	4	5	6	7
$KO_d(pt)$	$\mathbb{Z}$	$\mathbb{Z}_2$						

$KO_2(pt)$ real vector bundles on the 2 sphere modulo trivial ones
Clutching function on the equator

d	0	1	2	3	4	5	6	7
$KO_d(pt)$	$\mathbb{Z}$	$\mathbb{Z}_2$						

$KO_2(pt)$ real vector bundles on the 2 sphere modulo trivial ones
Clutching function on the equator
$KO_2(pt) \simeq \pi_2(\mathcal{O}) \simeq \mathbb{Z}_2$

d	0	1	2	3	4	5	6	7
$KO_d(pt)$	$\mathbb{Z}$	$\mathbb{Z}_2$						

$KO_2(pt)$ real vector bundles on the 2 sphere modulo trivial ones
Clutching function on the equator
$KO_2(pt) \simeq \pi_2(\mathcal{O}) \simeq \mathbb{Z}_2$
The rotation $R_\theta \in \mathcal{O}_2$ gives the generator of $KO_2(pt)$

d	0	1	2	3	4	5	6	7
$KO_d(pt)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$					

$$KO_3(pt) = \pi_2(\mathcal{O}) = 0$$

d	0	1	2	3	4	5	6	7
$KO_d(pt)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0				

$$KO_4(pt) = \pi_3(\mathcal{O}) = \mathbb{Z}$$

d	0	1	2	3	4	5	6	7
$KO_d(pt)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$			

$KO_4(pt) = \pi_3(\mathcal{O}) = \mathbb{Z}$
generated by multiplication by elements of the unit sphere in $\mathbb{H}$

d	0	1	2	3	4	5	6	7
$KO_d(pt)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$			

$$KO_5(pt) = \pi_4(\mathcal{O}) = 0$$

d	0	1	2	3	4	5	6	7
$KO_d(pt)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0		

$$KO_6(pt) = \pi_5(\mathcal{O}) = 0$$

d	0	1	2	3	4	5	6	7
$KO_d(pt)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	

$$KO_7(pt) = \pi_6(\mathcal{O}) = 0$$

d	0	1	2	3	4	5	6	7
$KO_d(pt)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

Real elliptic operators

Let M a smooth manifold and D a real pseudodifferential operator. Its symbol is not real but verifies

$$\overline{\sigma(x,\xi)} = \sigma(x,-\xi)$$

KO theory for spaces with involution (X,τ) : complex vector bundles E with antilinear $J : E \to \tau^*E$: $J_x : E_x \to E_{\tau(x)}$.

$$KO(X, id) = KO(X)$$

$$KO(X \cup X, swap) = KU(X)$$

$$\mathbb{R}^{p,q} = (\mathbb{R}^p \times \mathbb{R}^q, (id, -id)),\ KO(\mathbb{R}^{1,0}) = \mathbb{Z}_2,\ KO(\mathbb{R}^{0,1}) = 0$$

$\mathbb{Z}_2$ index of skew-symmetric elliptic operators

The symbol of a pseudo-differential operator gives a class in $KO(T^*M, -id)$.
Let D be a real skew symmetric elliptic pseudo-differential operator on a smooth manifold $M \to Ind(D) = 0$

$$D_t = \cos(t)D + \sin(t)id \qquad 0 \leq x \leq \pi$$

The index class of the familly in $KO(S^1) \simeq \mathbb{Z}_2$ is given by $dimker(D) \mod 2$. By the Atiyah Singer index theorems for famillies of real elliptic this index is topological.
How to compute such an element ?

KKO for real $\mathcal{C}^\star$algebras

Real spaces are commuative real $\mathcal{C}^\star$algebras.

$$\mathrm{C}(X,\tau) = \{f : x \to \mathbb{C} | \forall x \in X, f(\tau(x)) = \overline{f(x)}\}$$

Let A and B be two graded real $\mathcal{C}^\star$algebras.

Definition

An $A - B$ Kasparov bimodule is a triple $(E, \varepsilon, \phi, F)$ where (E, ε) is a finitely generated graded Hilbert $B-$module, $\phi : A \to \mathfrak{L}_B(E)$ is a (non necessarily unital) graded $\mathcal{C}^\star$algebra morphism and $F \in \mathfrak{L}_B(E)$ is of degree 1 and verifies :

$$(F^* - F)\phi(A) \subset K(E), \quad (F^2 - 1)\phi(A) \subset K(E), \quad [F, \phi(A)] \subset K(E)$$

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$KKO(A, B)$ is the set of homotopy equivalence classes of Kasparov bimodules.

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$\to$ Kasparov pairing.

KKO and KKU

For A and B two real $\mathcal{C}^\star$algebras. $A_\mathbb{C}$ and $B_\mathbb{C}$ are complex $\mathcal{C}^\star$algebras.

$$KKO(A_\mathbb{C}, B) \simeq KKU(A_\mathbb{C}, B_\mathbb{C}) \simeq KKO(A, B_\mathbb{C})$$

related to the construction of Atiyah and his $KO(A_\mathbb{C}) \simeq KU(A_\mathbb{C})$

Higher KKO groups using suspensions
$\rightarrow KKO_n(A,B) = KKO(A, S^n B)$

$$KO_i(A) = KKO(S^i A, \mathbb{R}) \qquad KO^i(A) = KKO(\mathbb{R}, S^i A)$$

Theorem (Bott-Kasparov)

We have a KK equivalence between $\mathcal{C}^{p,q}$ and $\mathrm{C}^0(\mathbb{R}^{p,q})$ induced by

$$\boldsymbol{\alpha_{p,q}} = \left[L^2(\mathbb{R}^{p+q}, \Lambda^\star \mathbb{R}^{p+q}), \not{D}/\sqrt{1+\not{D}^2}\right] \in KK(\mathrm{C}^0(\mathbb{R}^{p,q}), \mathcal{C}^{q,p})$$

$$\boldsymbol{\beta_{p,q}} = \left[\mathrm{C}_0(\mathbb{R}^{p+q}, \Lambda^\star \mathbb{R}^{p+q}), \not{x}/\sqrt{1+||x||^2}\right] \in KK(\mathcal{C}^{q,p}, \mathrm{C}^0(\mathbb{R}^{p,q}))$$

$$KKO_{i+8}(A,B) = KKO_i(A,B)$$

Bott periodicity

A real structure of degree $d \in \mathbb{Z}_8$ on (E, ϕ, F) is the data of a unitary antilinear $J : E \to E$ such that $\forall a \in A_{\mathbb{C}}, \forall b \in B_{\mathbb{C}}$:

$$\bar{b} = JbJ^*, \qquad \phi(\bar{a}) = J\phi(a)J^*$$

$$J^2 = \alpha, \qquad FJ = \alpha' JF, \qquad J\varepsilon = \alpha'' \varepsilon J$$

d	0	1	2	3	4	5	6	7
α	1	1	-1	-1	-1	-1	1	1
α'	1	-1	1	1	1	-1	1	1
α''	1		-1		1		-1	

Complexification morpism :

$$c: \quad KKO(A,B) \to KKU(A_{\mathbb{C}}, B_{\mathbb{C}})$$
$$(E,\phi,F) \mapsto (E\otimes_{\mathbb{R}}\mathbb{C}, \phi\otimes 1, F \quad \otimes id)$$

Realification morpism :

$$r: \quad KKU(A_{\mathbb{C}}, B_{\mathbb{C}}) \to KKO(A,B)$$
$$(E,\phi,F) \mapsto (E,\phi,F)$$

Not inverse one of each other : $r \circ c = 2id$

Karoubi Wood exact sequence for bivariant K theory

Inclusion of realspace $\{0\} \hookrightarrow \mathbb{R}^{0,1}$ gives

$$0 \to \mathrm{C}^0(\mathbb{R})_{\mathbb{C}} \to \mathrm{C}^0(\mathbb{R}^{0,1}) \to \mathbb{R} \to 0$$

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$$KKO_1(A,B) \to KKO(A, SB_{\mathbb{C}}) \to KKO(A, S^{0,1}B) \to KKO(A,B)$$

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$$KKO_1(A,B) \xrightarrow{x} KKU_1(A_{\mathbb{C}},B_{\mathbb{C}}) \xrightarrow{y} KKO_{-1}(A,B) \xrightarrow{\eta} KKO_0(A,B)$$

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$$z \quad \in KKO(S^{0,1}, \mathbb{R})$$

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$$z \quad \in KKO_1(\mathbb{R}, \mathbb{R})$$

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η the generator of $KO_1(\mathbb{R})$

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$x \in KKO(\mathbb{R}, \mathbb{C})$

Inclusion of realspace $\{0\} \hookrightarrow \mathbb{R}^{0,1}$ gives

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$$KKO_1(A,B) \xrightarrow{c} KKU_1(A_{\mathbb{C}},B_{\mathbb{C}}) \xrightarrow{r\beta^{-1}} KKO_{-1}(A,B) \xrightarrow{\eta} KKO_0(A,B)$$

- η the generator of $KO_1(\mathbb{R})$
- c the complexification morphism

Karoubi Wood exact sequence for bivariant K theory

Inclusion of realspace $\{0\} \hookrightarrow \mathbb{R}^{0,1}$ gives

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- $y \in KKO(S\mathbb{C}, S^{0,1})$

Inclusion of realspace $\{0\} \hookrightarrow \mathbb{R}^{0,1}$ gives

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- η the generator of $KO_1(\mathbb{R})$
- c the complexification morphism
- y $\in KKO_{-2}(\mathbb{C},\mathbb{R})$

Karoubi Wood exact sequence for bivariant K theory

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η the generator of $KO_1(\mathbb{R})$

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$r\beta^{-1}$ Bott periodicity composed with the realification morphism

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- η the generator of $KO_1(\mathbb{R})$
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Compatible with Kasparov product

Corollaries

Inverting 2 we obtain 4 periodicity and a decomposition of KKU :

$$KKO_i(A,B)[\frac{1}{2}] \simeq KKO_{i+4}(A,B)[\frac{1}{2}]$$

$$KKO_i(A,B)[\frac{1}{2}] \oplus KKO_{i+2}(A,B)[\frac{1}{2}] \simeq KKU_i(A_{\mathbb{C}},B_{\mathbb{C}})[\frac{1}{2}]$$

$KKO_i(A,B) = 0$ for every i if and only if $KKU_i(A_{\mathbb{C}},B_{\mathbb{C}}) = 0$ for every i

For $\alpha \in KKO(A,B)$, α is an isomorphism if and only if $\alpha_{\mathbb{C}}$ is (Shick)

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & KU_n(A_{\mathbb{C}}) & \longrightarrow & KO_n(A) & \longrightarrow & KO_{n+1}(A) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \boldsymbol{\alpha}_{\mathbb{C}}} & & \downarrow{\scriptstyle \boldsymbol{\alpha}} & & \downarrow{\scriptstyle \boldsymbol{\alpha}} & & \\
\cdots & \to & KU_{n+k}(B_{\mathbb{C}}) & \to & KO_{n+k}(B) & \to & KO_{n+k+1}(B) & \to & \cdots
\end{array}
$$

For $\boldsymbol{\alpha} \in KKO_k(A, B)$

$$
\begin{array}{ccccccccc}
\cdots & \to & KU_n(A_{\mathbb{C}}) & \to & KO_n(A) & \to & KO_{n+1}(A) & \to & \cdots \\
 & & \Big\downarrow \boldsymbol{\alpha}_{\mathbb{C}} & & \Big\downarrow \boldsymbol{\alpha} & & \Big\downarrow \boldsymbol{\alpha} & & \\
\cdots & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \cdots
\end{array}
$$

For $\boldsymbol{\alpha} \in KO^{n-2}(A)$

mod 2 pairing

Connes formulas gives the $\mathbb{Z}_2$ value pairing between $[p] \in KO_0(A)$ and a $2n$ summable Fredohlm module (H, ε, F) is given by :

$$ic_n \int_0^1 \sum_k (-1)^k \operatorname{Tr}((u_t - 1)[F_0, u_t^*][F_0, u_t] \cdots u_t' \cdots [F_0, u_t][F_0, u_t^*])dt \quad \text{mo}$$

Where u_t is any smooth homotopy of unitaries linking $2p - 1$ to the identity Johannes

Thank you for your attention !

Geometric η K homology, over manifold in $ker\eta$