Corrigé :

Ex 1 (1) (a) Soit ψ borélienne, bornée, positive

$$E[\psi(Y)] = E[\psi(|X|)]$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(|x|)\, e^{-\frac{x^2}{2}} dx$$
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \psi(x)\, e^{-x^2/2} dx \quad \text{(Fct° paire)}$$
$$\Rightarrow \boxed{dP_Y(y) = \sqrt{\frac{2}{\pi}}\, e^{-y^2/2}\, \mathbb{1}_{\mathbb{R}_+}(y)\, dy.}$$

(b). $E[Y] = \sqrt{\frac{2}{\pi}} \int_0^\infty y\, e^{-y^2/2} dy = \sqrt{\frac{2}{\pi}} \left[-e^{-y^2/2}\right]_0^\infty$

$$\Rightarrow \boxed{E[Y] = \sqrt{\frac{2}{\pi}}.}$$

- $Var(Y) = E[Y^2] - E[Y]^2$

Or $E[Y^2] = E[X^2] = Var(X) = 1$.

$$\Rightarrow \boxed{Var(Y) = 1 - \frac{2}{\pi}.}$$

(2) (a). $\Gamma(a) = \int_0^{+\infty} x^{a-1} e^{-x} dx$

. En 0, $x^{a-1} e^{-x} \sim x^{a-1} = \frac{1}{x^{1-a}}$

avec $1-a<1$ donc intégrable en 0.

. En $+\infty$, $x^{a+1} e^{-x} \underset{x\to+\infty}{\longrightarrow} 0$

donc $x^{a-1}e^{-x} \leq \frac{C}{x^2}$ intégrable en $+\infty$

$\Rightarrow \underline{\forall a>0,\ \Gamma(a) < +\infty}.$

- $\Gamma(a+1) = \int_0^\infty x^a e^{-x} dx$

$$= \cancel{\left[-x^a e^{-x}\right]_0^\infty} + a \int_0^\infty x^{a-1} e^{-x} dx$$
$$= a\,\Gamma(a).$$

- $\Gamma(n+1) = n\,\Gamma(n)$

$$= n(n-1)\Gamma(n-1) = \dots = n!$$

car $\Gamma(1) = 1$ donc $\boxed{\Gamma(n) = (n-1)!}$

(b) * Soit X, Y v.a.i. de loi $\Gamma(a,\lambda)$ et $\Gamma(b,\lambda)$.

. $E[X] = \frac{\lambda^a}{\Gamma(a)} \int_0^\infty x^a e^{-\lambda x} dx$

$u = \lambda x$, $x = \frac{u}{\lambda}$:
$$= \frac{\lambda^{a-1}}{\Gamma(a)} \int_0^\infty \frac{u^a}{\lambda^a} e^{-u} du$$
$$= \frac{\Gamma(a+1)}{\lambda\,\Gamma(a)} = \boxed{\frac{a}{\lambda}}.$$

- $E[X^2] = \frac{\lambda^a}{\Gamma(a)} \int_0^\infty x^{a+1} e^{-\lambda x} dx$

$u = \lambda x$:
$$= \frac{\lambda^a}{\Gamma(a)} \int_0^\infty \frac{u^{a+1}}{\lambda^{a+1}} e^{-u} \frac{du}{\lambda}$$
$$= \frac{\Gamma(a+2)}{\Gamma(a)} \frac{1}{\lambda^2} = \frac{a(a+1)}{\lambda^2}$$
$$\Rightarrow Var(X) = \frac{a(a+1)}{\lambda^2} - \frac{a^2}{\lambda^2} = \boxed{\frac{a}{\lambda^2}}.$$

* Soit ψ borélienne, bornée, positive.

$$E[\psi(X+Y)] = \frac{\lambda^{a+b}}{\Gamma(a)\Gamma(b)} \int_0^\infty\!\!\int_0^\infty \psi(x+y)\, e^{-\lambda(x+y)} x^{a-1} y^{b-1} dx\,dy$$
$$= \frac{\lambda^{a+b}}{\Gamma(a)\Gamma(b)} \int_0^\infty \left(\int_0^\infty \psi(x+y)\, e^{-\lambda(x+y)} x^{a-1} dx\right) y^{b-1} dy$$

$u = x+y$:
$$= \frac{\lambda^{a+b}}{\Gamma(a)\Gamma(b)} \int_0^\infty \left(\int_y^{+\infty} \psi(u)\, e^{-\lambda u} (u-y)^{a-1} du\right) y^{b-1} dy$$
$$= \frac{\lambda^{a+b}}{\Gamma(a)\Gamma(b)} \int_0^\infty \psi(u)\, e^{-\lambda u} u^{a+b-2} \left(\int_0^u \left(1-\frac{y}{u}\right)^{a-1} \left(\frac{y}{u}\right)^{b-1} dy\right) du$$

$v = y/u$:
$$= \lambda^{a+b} \int_0^\infty \psi(u)\, e^{-\lambda u} u^{a+b-1} \frac{\left(\int_0^1 (1-v)^{a-1} v^{b-1} dv\right)}{\Gamma(a)\Gamma(b)} du$$
$$= \int_{\mathbb{R}} \psi(u) \left(\frac{\lambda^{a+b}}{\Gamma(a+b)} e^{-\lambda u} u^{a+b-1} \mathbb{1}_{\mathbb{R}_+}(u)\right) du$$
$$\Rightarrow \boxed{X+Y \sim \Gamma(a+b, \lambda).}$$

(c) $X \sim \mathcal{N}(0,1)$. Soit ψ borél., bornée, pos.

$$E[\psi(X^2)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(x^2)\, e^{-x^2/2} dx$$
$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \psi(x^2)\, e^{-x^2/2} dx \quad \text{(Fct° paire)}$$

$u = x^2$, $x = \sqrt{u}$:
$$= \int_{\mathbb{R}} \psi(u) \frac{e^{-u/2}}{\sqrt{2\pi u}} \mathbb{1}_{\mathbb{R}_+}(u)\, du$$
$$\Rightarrow \boxed{X^2 \sim \Gamma\left(\tfrac{1}{2}, \tfrac{1}{2}\right)} \text{ avec } \boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}.}$$

(d) $X_1, \dots, X_n$ v.a.i. $\sim \mathcal{N}(0,1)$

$X_1^2 \sim \Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$

(2)(b) $\Rightarrow Z = X_1^2 + \dots + X_n^2 \sim \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$

et d'après (2)(b), $\boxed{E[Z] = n}$

et $\boxed{Var(Z) = 2n}$

(3) $Y = \begin{cases} \frac{X_1}{X_2} & \text{si } X_2 \neq 0 \\ 0 & \text{sinon.} \end{cases}$

Loi de Y ? Soit ψ borél., bornée positive.

$$E\left[\psi\left(\frac{X_1}{X_2}\right)\right] = \frac{1}{2\pi} \int_{\mathbb{R}^2} \psi\left(\frac{x}{y}\right) e^{-\frac{1}{2}(x^2+y^2)} dx\,dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0}\left(\int_{\mathbb{R}} \varphi\left(\frac{x}{y}\right) e^{-x^2/2}\, dx\right) e^{-y^2/2}\, dy$$

$$+ \frac{1}{2\pi}\int_{0}^{+\infty}\left(\int_{\mathbb{R}} \varphi\left(\frac{x}{y}\right) e^{-x^2/2}\, dx\right) e^{-y^2/2}\, dy$$

$$\underset{u=\frac{x}{y}}{=} \frac{1}{2\pi}\int_{-\infty}^{0}\left(\int_{+\infty}^{-\infty} \varphi(u) e^{-u^2y^2/2}\, y\, du\right) e^{-y^2/2}\, dy$$

$$+ \frac{1}{2\pi}\int_{0}^{+\infty}\left(\int_{\mathbb{R}} \varphi(u) e^{-u^2y^2/2}\, y\, du\right) e^{-y^2/2}\, dy$$

$$\underset{y'=-y}{=} \frac{1}{\pi}\int_{\mathbb{R}} \varphi(u)\left(\int_{0}^{+\infty} y\, e^{-(u^2+1)y^2/2}\, dy\right) du$$

$$= \frac{1}{\pi}\int_{\mathbb{R}} \varphi(u)\left[\frac{-e^{-(u^2+1)y^2/2}}{1+u^2}\right]_0^{+\infty} du$$

$$= \int_{\mathbb{R}} \varphi(u)\left(\frac{1}{\pi}\cdot\frac{1}{1+u^2}\right) du$$

$$\Rightarrow \boxed{dP_Y(y) = \frac{1}{\pi}\cdot\frac{1}{1+y^2}\, dy}$$

<u>Ex 2.</u>

$(U_i)_{i\geq 1}$ v.a.i.i.d. et f mesurable donc $(f(U_i))_{i\geq 1}$ est une suite de v.a.i.i.d.

Par hypothèse,

$\mathbb{E}[|f(U_1)|] = \int_0^1 |f(x)|\, dx < \infty$

D'après la LFGN,

$$\boxed{I_n \xrightarrow[n\to+\infty]{p.s.,\ \mathbb{P}} I = \int_0^1 f(x)\, dx}$$

The End