Random walks on randomly oriented lattices: Three open problems

Nadine Guillotin-Plantard

Institut Camille Jordan - University Lyon I

Te Anau – January 2014

Georges Pólya (1887 – 1985)

Theorem (Pólya (1921))

There exists some constant $C = C(d)$ s.t. for n large enough

$$P[S_{2n} = 0] \sim C\ n^{-d/2}.$$

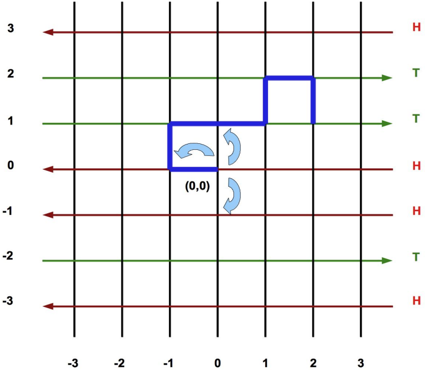
Main tool: Fourier Inversion Formula

$$P[S_{2n} = 0] = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} E(e^{i\Theta \cdot S_n})\, \mathrm{d}\Theta$$

Use that S_n is a sum of i.i.d. random vectors and for $||\Theta||$ small,

$$E[e^{i\Theta \cdot X_1}] = 1 - \frac{||\Theta||^2}{2d} + o(||\Theta||^2)$$

Take $\mathbb{Z}^2$ and for each horizontal line, (independently) throw a coin. If you get "Head" (resp. "Tail"), the whole line is directed to the left (resp. to the right). The vertical lines are not oriented.
Consider a random walker starting at the origin at time 0 and moving on this oriented version of $\mathbb{Z}^2$. At each step of time, he can move (uniformly) on its neighbors while respecting the orientations.



Is this random walk recurrent or transient ? Speed? FCLT? Local limit theorem ?

Theorem (Campanino – Pétritis (2003))

The random walk $(S_n)_n$ is transient for almost every realization of the orientations, with speed 0.

A more precise result can be proved.

Theorem (Castell – Guillotin-Plantard – Pène – Schapira (AOP, 2011))

For n large,

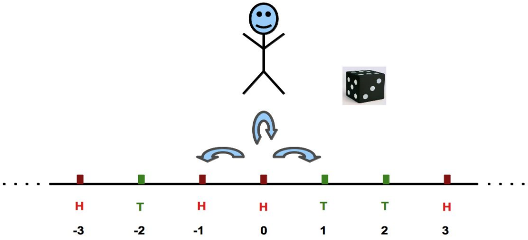
$$\mathbb{P}[S_{2n} = 0] \sim \frac{C}{n^{5/4}}.$$

$(S_n)_n$ has the same distribution as $(X_n, Y_n)_n$ where

- Y_n is the "lazy" random walk on $\mathbb{Z}$.
- X_n is "close" to the random walk (Y_n) in the random scenery ("H","T") :

$\xi_i = 1$ (resp. -1) if "Tail" (resp. "Head") at level $i \in \mathbb{Z}$,

$$X_n = \sum_{k=1}^{n} \xi_{Y_k} \mathbf{1}_{\{\Delta Y_k = 0\}} \sim \frac{1}{3} \sum_{k=1}^{n} \xi_{Y_k}$$



A functional limit theorem

Theorem (Kesten-Spitzer (1979), and Borodin (1979))

$$\left(\frac{X_{nt}}{n^{3/4}}\right)_{t\geq 0} \Rightarrow (\Delta_t)_{t\geq 0}$$

where $(\Delta_t)_{t\geq 0}$ *is not stable, self-similar with index* $3/4$*, with stationary increments.*
More precisely,

$$\Delta_t = \int_{\mathbb{R}} L_t(x)\,\mathrm{d}W(x)$$

where $(L_t(x))_{t\geq 0, x\in\mathbb{R}}$ *is the local time of a one-dimensional Brownian motion* $B := (B_t)_{t\geq 0}$*, and* $W := (W(x))_{x\in\mathbb{R}}$ *is a two-sided Brownian motion, independent of* B*.*

A local limit theorem

Theorem (Castell – Guillotin-Plantard – Pène – Schapira (AOP, 2011))

For n large,

$$\mathbb{P}[X_{2n} = 0] \sim \frac{C}{n^{3/4}}.$$

Heuristics

$$N_n(x) := \sum_{k=1}^{n} \mathbf{1}_{\{Y_k = x\}}$$ the local time of Y at x up to time n.

Heuristics

$$N_n(x) := \sum_{k=1}^{n} \mathbf{1}_{\{Y_k = x\}} \text{ the local time of } Y \text{ at } x \text{ up to time } n.$$

$$X_n = \sum_{k=1}^{n} \xi_{Y_k} = \sum_{x \in \mathbb{Z}} \xi_x N_n(x).$$

Heuristics

$$N_n(x) := \sum_{k=1}^{n} \mathbf{1}_{\{Y_k = x\}}$$ the local time of Y at x up to time n.

$$X_n = \sum_{k=1}^{n} \xi_{Y_k} = \sum_{x \in \mathbb{Z}} \xi_x N_n(x).$$

$$\mathbb{E}\left[X_n^2 | Y\right] = \textstyle\sum_{x \in \mathbb{Z}} N_n^2(x) = \sum_{1 \leq i,j \leq n} \mathbf{1}_{\{Y_i = Y_j\}} =: I_n$$

Heuristics

$$N_n(x) := \sum_{k=1}^{n} \mathbf{1}_{\{Y_k = x\}} \text{ the local time of } Y \text{ at } x \text{ up to time } n.$$

$$X_n = \sum_{k=1}^{n} \xi_{Y_k} = \sum_{x \in \mathbb{Z}} \xi_x N_n(x).$$

$$\mathbb{E}\left[X_n^2 | Y\right] = \sum_{x \in \mathbb{Z}} N_n^2(x) = \sum_{1 \le i,j \le n} \mathbf{1}_{\{Y_i = Y_j\}} =: I_n$$

I_n is the *self-intersection local time* of Y up to time n,

$$\mathbb{E}\left[I_n\right] \sim C\ n^{3/2} \text{ as } n \to +\infty.$$

Since $(S_n)_n$ is distributed as $(X_n, Y_n)_n$, we have

$$\begin{aligned} \mathbb{P}[S_{2n} = 0] &= \mathbb{P}[X_{2n} = 0; Y_{2n} = 0] \\ &\sim \mathbb{P}[X_{2n} = 0 | Y_{2n} = 0]\mathbb{P}[Y_{2n} = 0] \end{aligned}$$

We know that

$$\mathbb{P}[Y_{2n} = 0] \sim \frac{C}{n^{1/2}}$$

and (not easy !)

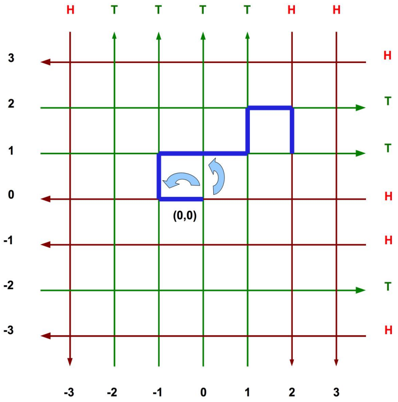
$$\mathbb{P}[X_{2n} = 0 | Y_{2n} = 0] \sim \frac{C}{n^{3/4}}$$

so

$$\mathbb{P}[S_{2n} = 0] \sim \frac{C}{n^{5/4}}$$

Riddle 1

Now the vertical lines are also oriented with a coin.



Can we prove that this random walk is transient? Speed? Functional limit theorem? Local limit theorem?

Riddle 2

In the first model where only the horizontal lines are directed, the continuous limit process is given (after correct normalizations) by

$$M := (\Delta_t, B_t)_{t\geq 0} = \left(\int_{\mathbb{R}} L_t(x)\,\mathrm{d}W(x), B_t\right)_{t\geq 0}.$$

For the planar Brownian motion, the winding number is known since Spitzer's result (1958).

Can we prove the same kind of result for M?

Riddle 3

Consider the first model where only the horizontal lines are directed. But now the orientations are generated by an irrational rotation on the circle. More precisely, take α your favorite irrational number. For any $k \in \mathbb{Z}$, if the fractional part of $k\alpha$ is less than $1/2$ (resp. in $(1/2, 1)$), the horizontal line at level k is directed to the right (resp. to the left).

Is the random walk on this oriented version of $\mathbb{Z}^2$ is transient? recurrent?

Two remarks:

- The orientations are now strongly correlated. The irrational rotation on the torus is ergodic but not weakly mixing !!
- In the case when α is rational, the orientations of the horizontal lines are periodic and the recurrence of the random walk can be proved.