Operads and Rewritings (Lyon 02.11.2011– 04.11. 2011)

Identities for skew-symmetric n-algebras

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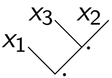
Algebra

- *Vector space*
- *Multiplication*

Multiplication (in a free magma) = binary tree



$$= \quad x_1 \cdot (x_3 \cdot x_2).$$

Polynomial identity

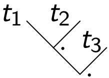
Let $f = f(t_1, \ldots, t_k)$ be an element of free magma (non-commutative non-associative polynomial)

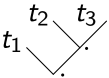
$f = 0$ is a polynomial identity of A if

$$f(a_1, \ldots, a_k) = 0$$

for any substitution $t_i := a_i \in A$.

Example. Associative algebra is an algebra with identity $ass = 0$, where

$ass =$ 

$-$ 

$= (t_1 t_2) t_3 - t_1 (t_2 t_3)$

Commutator

If $A = (A, \circ)$ *is an associative algebra and*

$$[a, b] = a \circ b - b \circ a$$

then $(A^{-} = (A, [\ ,\ ])$ *is Lie,*

$$[a, b] = -[b, a]$$

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$$

Lie commutator

Let $Diff_n$ be an associative algebra of differential operators on $K[x_1, \ldots, x_n]$ endowed by a composition operation. Let $Vect_n = Diff_n^{[1]}$ be its subspace of differential operators of first order (vector fields on n-dimensional manifold).

(Lie) $Vect(n)$ is not close under composition, but under commutator $Vect(n)$ is a Lie algebra.

Lie commutator in a coordinate form

$$[u\partial_i, v\partial_j] = u\partial_i(v)\partial_j - v\partial_j(u)\partial_i.$$

Poisson commutator

Poisson commutator is defined on $K[x_1, \ldots, x_{2n}]$ by

$$\{a, b\} = \sum_{i=1}^{n} \partial_i(a)\partial_{i+n}(b) - \partial_{i+n}(a)\partial_i(b)$$

Lie and Poisson commutators are compatible.

Let $D_a = \sum_{i=1}^{n} \partial_i(a)\partial_{i+n} - \partial_{i+n}(a)\partial_i$. *Then*

$$[D_a, D_b] = D_{\{a,b\}},$$

for any $a, b \in K[x_1, \ldots, x_{2n}]$.

Our aim: Construct three kinds of n-ary generalisations of these commutators.

n-Algebra

- *Vector space*
- *n-Multiplication*

A n-Multiplication is a n-ary multilinear map $\omega : A \times \cdots \times A \to A$
A n-Multiplication ω is skew-symmetric if

$$\omega(a_1, \dots, a_n) = sign\, \sigma\, \omega(a_{\sigma(1)}, \dots, a_{\sigma(n)}), \quad \forall a_1, \dots, a_n \in A.$$

A n-Multiplication ω is symmetric if

$$\omega(a_1, \dots, a_n) = \omega(a_{\sigma(1)}, \dots, a_{\sigma(n)}), \quad \forall a_1, \dots, a_n \in A.$$

For n-ary monomial given by n-ary tree say that it has ω-*degree* k if number of inner vertices is k.
For $f = f(t_1, t_2, \ldots) \in Magma(t_1, t_2, \ldots)$ say that it has ω-degree k if f is a linear combination of monomials of ω-degree k.

Definition. $f = 0$ is an identity on n-ary algebra $A = (A, \alpha)$ if $f(a_1, a_2, \ldots) = 0$ for any substitutions $t_i := a_i \in A$ where $\omega := \alpha$.
Example.

$$nlie = \omega(t_1, \ldots, t_{n-1}, \omega(t_n, \ldots, t_{2n-1})) -$$

$$\sum_{i=1}^{n} (-1)^{i+n} \omega(t_n, \ldots, \widehat{t_{n+i-1}}, \ldots, t_{2n-1}, \omega(t_1, \ldots, t_{n-1}, t_{n+i-1})).$$

An algebra with identity $nlie = 0$ is called n-Lie-Filippov.

Proposition. *A is n-Lie-Filippov, iff the adjoint map*

$$ad\,\{a_1, \dots, a_{n-1}\} : b \mapsto [a_1, \dots, a_{n-1}, b]$$

is a derivation for any $a_1, \dots, a_{n-1} \in A$.

Theorem. A n-ary Vector products algebra

$$[e_1, \dots, \hat{e}_i, \dots, e_n] = (-1)^i e_i, \quad i = 1, \dots, n+1$$

is n-Lie-Filippov.

n-Lie-Filippov algebras under Jacobian

Theorem. *Let* $A = (K[x_1, \ldots, x_n]$ *and*

$$jac^S(a_1, \ldots, a_n) = \begin{vmatrix} \partial_1(a_1) & \partial_1(a_2) & \cdots & \partial_1(a_n) \\ \partial_2(a_1) & \partial_2(a_2) & \cdots & \partial_2(a_n) \\ \vdots & \vdots & \cdots & \vdots \\ \partial_n(a_1) & \partial_n(a_2) & \cdots & \partial_n(a_n) \end{vmatrix}$$

$$jac^W(a_1, \ldots, a_n) = \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ \partial_1(a_1) & \partial_1(a_2) & \cdots & \partial_1(a_n) \\ \vdots & \vdots & \cdots & \vdots \\ \partial_{n-1}(a_1) & \partial_{n-1}(a_2) & \cdots & \partial_{n-1}(a_n) \end{vmatrix}$$

Then jac^S *and* jac^W *are satify the identity* $nlie = 0$.

n-Associative algebra

Define n-ary polynomials $ass_i, 0 < i < n$, of ω-degree 2 by

$$ass_i = \omega(t_1, \ldots, t_{i-1}, t_i, \omega(t_{i+1}, \ldots, t_{i+n}), t_{i+n+1}, \ldots, t_{2n-1}) -$$

$$\omega(t_1, \ldots, t_{i-1}, \omega(t_i, t_{i+1}, \ldots, t_{i+n-1}), t_{i+n}, \ldots, t_{2n-1}).$$

Definition. A n-ary algebra is called *totally associative* if $ass_i = 0$ is a n-polynomial identity for any $0 < i < n$.

Example. If A is binary associative, and $\alpha(a_1, \ldots, a_n) = a_1 \cdots a_n$, then (A, α) is totally associative. If a n-ary totally associative algebra has unit, then it can be imbedded to a such algebra (Kurosh, Polin).

Remark. There are other kinds of n-associativity (Gnedbaye).

Vector fields and Lie commutator

A vector field is a differential operator of first order. A composition of operators is a natural operation on a space of differential operators. Space of vector fields is not close under composition.

(S. Lie) $Vect(n)$ is close under Lie commutator,

$$[D_1, D_2] = D_1 D_2 - D_2 D_1.$$

$$X_1, X_2 \in Vect(n) \Rightarrow X_1 = u_i\partial_i, X_2 = v_j\partial_j \Rightarrow$$

$$X_1X_2 = u_i\partial_i(v_j)\partial_j + u_iv_j\partial_i\partial_j, \quad X_2X_1 = v_j\partial_j(u_i)\partial_i + v_ju_i\partial_j\partial_i \Rightarrow$$

$$[X_1, X_2] = u_i\partial_i(v_j)\partial_j - v_j\partial_j(u_i)\partial_i \in Vect(n).$$

Generalisation of Lie commutator

Let $\mathcal{D}_{n,p}$ be a space of differential operators with n variables of differential degree p.

Example. $\mathcal{D}_{n,1}$ coincides with a space of vector fields on n-dimensional manifold $Vect(n)$.

Example. Laplacian $\Delta = \sum_{i=1}^n u_i \partial_i^2$ belongs to $\mathcal{D}_{n,2}$.

Let us define N-commutator of diferential operators as a skew-symmetric sum of $N!$ compositions

$$s_N(D_1, \ldots, D_N) = \sum_{\sigma \in S_N} sign\, \sigma\, D_{\sigma(1)} \cdots D_{\sigma(N)}.$$

N-commutator on differential operators

Theorem. *For any positive integers n and p there exists $N = N(n,p)$ such that N-commutator is well-defined on a space of differential operators $\mathcal{D}_{n,p}$. The N-algebra $(\mathcal{D}_{n,p}, s_N)$ is left-commutative.*

Warning. Might be that $s_N = 0$ will be identity.
For $p = 1$ this result can be improved.

Theorem. *If, $N = n^2 + 2n - 2$, then N-commutator is well-defined on $Vect(n)$ and $s_{N+1} = 0$ is identity on $Vect(n)$.*

Are there another N such that s_N is well-defined on $Vect(n)$? For $n = 3$ there is another one: $N = 10$.
List of N-commutators on $Vect(n)$ for small n

$$(n, N) = (1,2), (2,2), (2,6), (3,2), (3,10), (3,13).$$

N-commutator on quadratic differential operators

If $p = 2$, N-commutator can be presented in a nice form.

Theorem. *If $p = 2$ and $n = 1$ then $2n$-commutator is well defined on $\mathcal{D}_{1,2}$*

Example. 4-commutator of four differential operators of second order on the line can be calculated by wronskians

$$s_4(u_1\partial^2, u_2\partial^2, u_3\partial^2, u_4\partial^2) = -2 \begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ \partial(u_1) & \partial(u_2) & \partial(u_3) & \partial(u_4) \\ \partial^2(u_1) & \partial^2(u_2) & \partial^2(u_3) & \partial^2(u_4) \\ \partial^3(u_1) & \partial^3(u_2) & \partial^3(u_3) & \partial^3(u_4) \end{vmatrix} \partial^2.$$

sh-Lie structure on $Vect(n)$

Theorem. $(Vect(n), s_2, s_{n^2+2n-2})$ *is Sh-Lie.*

In other words,

$$[s_2, s_2] = 0,$$

$$[s_2, s_{n^2+2n-2}] = 0,$$

$$[s_{n^2+2n-2}, s_{n^2+2n-2}] = 0.$$

Theorem. *If s_N is N-commutator on $Vect_0(2)$, then $N = 2, 5$ and $(Vect(2), s_2, s_5)$ is Sh-Lie.*

Any divergenceless vector field in two variables can be presented in a form $X = D_{1,2}(u) = \partial_1(u)\partial_2 - \partial_2(u)\partial_1$.

$$s_5(D_{12}(u_1), D_{12}(u_2), D_{12}(u_3), D_{12}(u_4), D_{12}(u_5))$$
$$= -3D_{12}([u_1, u_2, u_3, u_4, u_5]),$$

where $[u_1, u_2, u_3, u_4, u_5]$ is the following determinant

$$[u_1, u_2, u_3, u_4, u_5] = \begin{vmatrix} \partial_1 u_1 & \partial_1 u_2 & \partial_1 u_3 & \partial_1 u_4 & \partial_1 u_5 \\ \partial_2 u_1 & \partial_2 u_2 & \partial_2 u_3 & \partial_2 u_4 & \partial_2 u_5 \\ \partial_1^2 u_1 & \partial_1^2 u_2 & \partial_1^2 u_3 & \partial_1^2 u_4 & \partial_1^2 u_5 \\ \partial_1\partial_2 u_1 & \partial_1\partial_2 u_2 & \partial_1\partial_2 u_3 & \partial_1\partial_2 u_4 & \partial_1\partial_2 u_5 \\ \partial_2^2 u_1 & \partial_2^2 u_2 & \partial_2^2 u_3 & \partial_2^2 u_4 & \partial_2^2 u_5 \end{vmatrix}$$

6-commutator on $Vect(2)$

Theorem. *If s_N is N-commutator on $Vect(2)$, then $N = 2, 6$ and $(Vect(2), s_2, s_6)$ is Sh-Lie.*

Let $X_i = u_{i,1}\partial_1 + u_{i,2}\partial_2$, $i = 1, \ldots, 6$. Then $s_6(X_1, \ldots, X_6)$ can be presented as a sum of fourteen 6×6 determinants. For example,

$$s_6(\partial_1, \partial_2, x_1\partial_1, x_2\partial_1, x_2\partial_2, x_1^2\partial_2) = -6\partial_2.$$

N-commutators of simple Lie algebras

Theorem.

- *G_2 has 2- and 10-commutator.*
- *sl_n has N-commutators for $N = 2, 4, \dots, 2n-2$.*
- *so_n has N-commutators for $N = 2, 5, 6, 9, \dots, 2n-2$.*
- *sp_{2n} has N-commutators for $N = 2, 5, 6, 9, 10, \dots, 4n-3, 4n-2$.*

Lists of N-commutators are final.
Identities:

$$s_k \star s_p = 0 \text{ if } k, p \text{ are even},$$

$$s_k \star s_p = k\, s_{k+p-1} \text{ if } p \text{ is odd},$$

$$s_k \star s_p = s_{k+p-1} \text{ if } k \text{ is odd and } p \text{ is even}$$

Super-derivations

D is odd super-derivation if

$$D(a \circ b) = D(a) \circ b + (-1)^{q(a)} a \circ D(b).$$

$$D = \textit{odd deriation} \Rightarrow D^2 = \textit{derivation}$$

Is it possible similar situation for $N > 2$

$$D = \textit{odd deriation} \Rightarrow D^N = \textit{derivation ?}$$

Answer. Yes. Take in algebra of super-lagrangians $N = n^2 + 2n - 2$.

Power of super-derivations

Theorem. *Let* η_i *are odd,* ∂_i *are even and*

$$D = \sum_{i=1}^{n} \eta_i \partial_i$$

is odd derivation. Then

$$D^{n^2+2n-2} \text{ is a derivation.}$$

Moreover,

$$D^{n^2+2n-1} = 0,$$

$$D^{n^2+2n-2} = -Div\, D\, D^{n^2+2n-3}.$$

Left-symmetric power of super-derivations

$$a\partial_i \bullet b\partial_j = ab\partial_i\partial_j \text{ bullet multiplication}$$

$$a\partial_i \circ b\partial_j = a\partial_i(b)\partial_j \text{ left-symmetric multiplication}$$

$$a\partial_i \cdot b\partial_j = a\partial_i \circ b\partial_j + a\partial_i \bullet b\partial_j$$

Theorem. *For $N = n^2 + 2n - 2$,*

$$D^N = D^{\circ N}.$$

where

$$D^{\circ k} = D \circ (D \circ \cdots (D \circ D) \cdots),$$

$\circ$ is left-symmetric multiplication.

N-commutator under left-symmetric multiplication

Another formulation

Theorem $N = n^2 + 2n - 2$. *For any* $X_1, \ldots, X_N \in Vect(n)$,

$$s_N(X_1, \ldots, X_N) = s_N^{\circ}(X_1, \ldots, X_N),$$

where $s_N^{\circ}(X_1, \ldots, X_N)$ *is calculated in terms of left-symmetric multiplication* $\circ$,

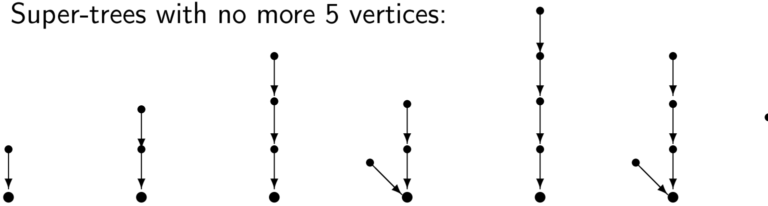
$$s_k^{\circ}(t_1, \ldots, t_k) = \sum_{\sigma \in Sym_k} sign\,\sigma\; t_{\sigma(1)} \circ (\cdots (t_{\sigma(k-1)} \circ t_{\sigma(k)}) \cdots)$$

How to calculate powers of super-derivations ?

Use super-tree calculus.
Tree is a graph without cycle
Rooted tree is a tree with a distinguished vertex called root
Orientation towards to root.
All vertices except root are labeled by super variables.

A tree is called super-tree if no vertex has two equal sub-branches with odd degree.

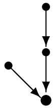
Example. Super-trees with no more 5 vertices:



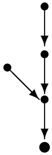
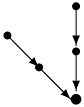
Let Q^k be its k-th associative power of super-tree Q. Then

$Q =$  ,

$Q^2 =$  ,

$Q^3 =$  $+$  ,

Power of super-trees

$Q^4 =$  $+$  $+$  .

Left-symmetric multiplication and bullet-multiplication

In addition to composition we introduce new operations

$$u\partial_i \circ v\partial^\alpha = u\partial_i(v)\partial^\alpha,$$

(left-symmetric multiplication)

$$u\partial_i \bullet v\partial^\alpha = uv\partial_i\partial^\alpha,$$

(bullet-multiplication)

$$D_1 \cdot D_2 = D_1 \circ D_2 + D_1 \bullet D_2.$$

Key observation: Composition of (super)-differential operators can be expressed in terms of left-symmetric and bullet multiplications. In particular, power of (super)-differential operators can be presented in terms of left-symmetric and bullet-powers.

Set partitions and product of derivations

Let $D_i = a_i \partial_i$ be a derivation. For a set of integers $A = \{i_s, \ldots, i_1\}$, $i_1 \le i_2 \le \cdots i_s$, set

$$D_A = D_{i_s} \cdots D_{i_2} \circ D_{i_1} = D_{i_s} \cdots D_{i_2}(a_{i_1}) \partial_{i_1}$$

Theorem.

$$D_n \cdots D_1 = \sum_{[n]=A_1 \cup \cdots \cup A_k} D_{A_1} \bullet \cdots \bullet D_{A_k}$$

Corollary

$$D^n = \sum_{\lambda \vdash n} k_\lambda (D^{\lambda_1 - 1} \circ D) \bullet \cdots \bullet (D^{\lambda_l - 1} \circ D)$$

Here λ is a partition of n with length l and k_λ is a number of partitions of the set $[n]$ with type λ.

Power of derivation and Faa di Bruno Formula

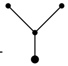
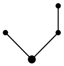
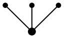
Example.

$D^0 =$ 

$D^1 = \emptyset \star D^0 =$ 

$D^2 = \emptyset \star D^1 + D^0 \star D^0 =$  $+$ 

$D^3 = \emptyset \star D^2 + 3D^0 \star D^1 + D^0 \star D^0 \star D^0 =$

 $+$  $+ 3$  $+$ 

$\lambda \vdash n$ is odd super-partition of n if no more one block has odd number of elements.

Example.5 has 4 odd super-partitions: $5, 14, 23, 122$

Theorem. *If $D_1, \ldots, D_n$ are odd super-derivations, then*

$$D_n \cdots D_1 = \sum_{[n]=\bar{A}_1 \cup \cdots \cup \bar{A}_l} D_{\bar{A}_1} \bullet \cdots \bullet D_{\bar{A}_l}$$

Corollary *For odd super-derivation D,*

$$D^n = \sum_{\bar{\lambda} \vdash n} \bar{k}_\lambda (D^{\bar{\lambda}_1 - 1} \circ D) \bullet \cdots \bullet (D^{\bar{\lambda}_l - 1} \circ D).$$

Here $[n] = \bar{A}_1 \cup \cdots \cup \bar{A}_l$ is super-partition of $[n]$, i.e., numbers of elements of blocks $|A_i|$, except might be one are even, and $\bar{\lambda} \vdash n$ is a super-partition, $\bar{k}_{\bar{\lambda}}$ is a number of super-partitions of $[n]$ with type $\bar{\lambda}$.

Super Bell numbers

Note that $\bar{\lambda}$ is super-partition of length l, iff $\bar{\lambda} = 1^{\alpha_1} \cdots n^{\alpha_n}$, i.e., $\sum_i \alpha_i i = n$, $\sum_i \alpha_i = l$ and $\sum_{i \equiv 1 (mod\, 2)} \alpha_i \leq 1$.

Number of super-partitions of $[n]$ with super-type $\bar{\lambda}$

$$\bar{k}_{\bar{\lambda}} = \frac{1}{\alpha_1! \ldots \alpha_n!} \binom{\lfloor n/2 \rfloor}{\underbrace{\lfloor 1/2 \rfloor \ldots \lfloor 1/2 \rfloor}_{\alpha_1} \ldots \underbrace{\lfloor n/2 \rfloor \ldots \lfloor n/2 \rfloor}_{\alpha_n}}$$

where $\bar{\lambda} = 1^{\alpha_1} \cdots n^{\alpha_n}$.

Number of super-partitions

$$\bar{p}(n) = \begin{cases} p(\lfloor n/2 \rfloor), & \text{if } n \text{ is even} \\ \sum_{i=0}^{\lfloor n/2 \rfloor} p(i), & \text{if } n \text{ is odd} \end{cases}$$

Super-Binomial coefficients

Binomial coefficients

$$C_n^i = C_{n-1}^i + C_{n-1}^{i-1}, \text{ if } n > 0,$$

$$C_0^0 = 1.$$

Generating function for binomial coefficients

$$\sum_{i=0}^{n} C_n^i x^i = (x+1)^n.$$

Newton formula

$$(x+y)^n = \sum_i C_n^i x^i y^{n-i},$$

where x and y are commuting variables.

Super-Binomial coefficients

$$\bar{C}_n^i = \bar{C}_{n-1}^i + (-1)^{(n-1)i}\bar{C}_{n-1}^{i-1}, \text{ if } n > 0,$$

$$\bar{C}_0^0 = 1.$$

$n \diagdown i$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	0	1					
3	1	1	1	1				
4	1	0	2	0	1			
5	1	1	2	2	1	1		
6	1	0	3	0	3	0	1	
7	1	1	3	3	3	3	1	1

Super-Newton formula

$$(x+y)^n = \sum_{i=0}^{n} \bar{C}_n^i x^i y^{n-i},$$

where x, y are anti-commuting variables $xy = -yx$.

$$(x+y)^n = \sum_{i=0}^{n} \binom{\lfloor n/2 \rfloor}{\lfloor i/2 \rfloor} x^i y^{n-i}.$$

$$\bar{C}_n^i = \binom{\lfloor n/2 \rfloor}{\lfloor i/2 \rfloor}$$

For super variables $x_1, x_2, \ldots,$ such that $x_i x_j = (-1)^{ij} x_j x_i$,

$$\bar{Y}_{n+1}(x_1, \ldots, x_{n+1}) =$$

$$\begin{vmatrix} \bar{C}_n^0 x_1 & \bar{C}_n^1 x_2 & \bar{C}_n^2 x_3 & \cdots & \bar{C}_n^{n-1} x_n & \bar{C}_n^n x_{n+1} \\ -1 & \bar{C}_{n-1}^0 x_1 & \bar{C}_{n-1}^1 x_2 & \cdots & \bar{C}_{n-1}^{n-2} x_{n-1} & \bar{C}_{n-1}^{n-1} x_n \\ 0 & -1 & \bar{C}_{n-2}^0 x_1 & \cdots & \bar{C}_{n-2}^{n-3} x_{n-2} & \bar{C}_{n-2}^{n-2} x_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & \bar{C}_0^0 x_1 \end{vmatrix}$$

Example. $n = 3$

$$\bar{Y}_3(x_1, x_2, x_3) = \begin{vmatrix} x_1 & 0 & x_3 \\ -1 & x_1 & x_2 \\ 0 & -1 & x_1 \end{vmatrix} = x_1 x_2 + x_3.$$

Determinants are calculated by column expansion.

Coefficients of $\bar{Y}_n(x_1,\ldots,x_n)$ are super-Bell set partition numbers,

$$\bar{Y}_n(x_1,\ldots,x_n)=\sum_{\bar{\lambda}\vdash n}\bar{k}_{\bar{\lambda}}x_{\lambda}.$$

Change in $\bar{Y}_n(x_1,\ldots,x_n)$

- the variable x_i to $D^{i-1}\circ D$
- super-multiplication to $\bullet$

Then we obtain formula for D^n.

Shuffle permutations and n-polynomials

Let

$$S_{k,l} = \{\sigma \in Sym_{k+l} | \sigma(1) < \ldots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+l)\}$$

$$S_{n-1,n-1,n} = \{\sigma \in Sym_{3n-2} | \sigma(1) < \ldots < \sigma(n-1),$$
$$\sigma(n) < \cdots < \sigma(2n-2), \quad \sigma(2n-1) < \cdots < \sigma(3n-2)\},$$

$$S_{n-2,n,n} = \{\sigma \in Sym_{3n-2} | \sigma(1) < \ldots < \sigma(n-2),$$
$$\sigma(n-1) < \cdots < \sigma(2n-2), \quad \sigma(2n-1) < \cdots < \sigma(3n-2),$$
$$\sigma(n-1) < \sigma(2n-1)\}$$

$$homot = \sum_{\sigma \in S_{n-1,n}} sign\, \sigma \omega(t_{\sigma(1)}, \ldots, t_{\sigma(n-1)}, \omega(t_{\sigma(n)}, \ldots, t_{\sigma(2n-1)})).$$

n-Polynomials $F_1^{[1]}$ and $F_1^{[2]}$

$$F_1^{[1]}(t_1,\dots,t_{3n-2}) = \sum_{\sigma\in Sym_{n-1,n-1,n},\sigma(1)=1}$$

$$sign\,\sigma\,(t_1, t_{\sigma(2)}\dots, t_{\sigma(n-1)}, (t_{\sigma(n)},\dots, t_{\sigma(2n-2)}, (t_{\sigma(2n-1)},\dots, t_{\sigma(3n-2)}))),$$

$$F_1^{[2]}(t_1,\dots,t_{3n-2}) = \sum_{\sigma\in Sym_{n-1,n-1,n},\sigma(n)=1}$$

$$sign\,\sigma\,(t_{\sigma(1)},\dots, t_{\sigma(n-1)}, (t_1, t_{\sigma(n+1)},\dots, t_{\sigma(2n-2)}, (t_{\sigma(2n-1)},\dots, t_{\sigma(3n-2)}))$$

n-Polynomials $F_2^{[1]}$ and $F_2^{[2]}$

$$F_2^{[1]}(t_1,\dots,t_{3n-2}) = \sum_{\sigma\in Sym_{n-2,n,n},\sigma(1)=1}$$

$$sign\,\sigma\,(t_1, t_{\sigma(2)}\dots, t_{\sigma(n-2)}, (t_{\sigma(n-1)},\dots, t_{\sigma(2n-2)}), (t_{\sigma(2n-1)},\dots, t_{\sigma(3n-2)}))$$

$$F_2^{[2]}(t_1,\dots,t_{3n-2}) = \sum_{\sigma\in Sym_{n-2,n,n},\sigma(n-1)=1}$$

$$sign\,\sigma\,(t_{\sigma(1)},\dots, t_{\sigma(n-2)}, (t_1, t_{\sigma(n)},\dots, t_{\sigma(2n-2)}), (t_{\sigma(2n-1)},\dots, t_{\sigma(3n-2)})))$$

Problem. (Kurosh) Find n-identites for total associative n-algebras under n-commutator.

Let

$$f_\lambda^{[1]} = -F_1^{[1]} + \lambda F_2^{[1]},$$

$$f_\lambda^{[2]} = F_1^{[2]} + \lambda F_2^{[2]}.$$

Theorem. *Let A be total associative n-algebra. If n is even, then its n-commutators algebra $[A]$ is homotopical n-Lie. Moreover, homot $= 0$ is a minimal identity.*

Theorem. *Let A be total associative algebra. Then its n-commutators algebra $[A]$ satisfies the identity $f_{-1}^{[2]} = 0$.*

Remark. This result for $n = 3$ was proved eariler by M.R. Bremner.

Matrices under 3-commutator

Theorem. *Let $[Mat_n]$ be ternary matrix algebra under 3-commutator $[a_1, a_2, a_3] = a_1[a_2, a_3] + a_2[a_3, a_1] + a_3[a_1, a_2]$. Then*

- *If $n = 2$, then $[Mat_2]$ is 3-Lie.*
- *Let $n = 3$. Then any multilinear identity of $[Mat_3]$ of ω-degree 2 follows from skew-symmetry identity of 3-commutator. The 3-algebra $[Mat_3]$ satisfies the identity $f_{-3}^{[1]} = 0$ and $f_{-1}^{[2]} = 0$. The identities $f_{-3}^{[1]} = 0$ and $f_{-1}^{[2]} = 0$ are independent and any multilinear identity of $[Mat_3]$ of ω-degree 3 follows these identities.*
- *If $n > 3$, then any multilinear identity of $[Mat_3]$ of ω-degree 2 follows from skew-symmetry identity of 3-commutator. The 3-algebra $[Mat_n]$ satisfies the identity $f_{-1}^{[2]} = 0$ and any its identity of ω-degree 3 follows from the identity $f_{-1}^{[2]} = 0$.*

Theorem. *Let U be associative commutative algebra with derivations $D_1, \ldots, D_n$. Then the skew-symmetric n-algebra $(U, D_1 \wedge \cdots \wedge D_n)$ is homotopical n-Lie if $n > 2$. If $n = 2$ it is s_4-Lie.*

Theorem. *Let U be associative commutative algebra with m commuting derivations $\partial_1, \ldots, \partial_m$. For $n \leq m$ let us endow U by n-multiplication ψ given as a linear combination of $n \times n$ jacobians*

$$\psi(a_1, \ldots, a_n) = \sum_{ß_{1,\ldots i_n \in [m]}} \lambda_{i_1,\ldots,i_n} \begin{vmatrix} \partial_{i_1}(a_1) & \cdots & \partial_{i_1}(a_n) \\ \partial_{i_2}(a_1) & \cdots & \partial_{i_2}(a_n) \\ \cdots & \cdots & \cdots \\ \partial_{i_n}(a_1) & \cdots & \partial_{i_n}(a_n) \end{vmatrix}$$

If n is even, then (U, ψ) is homotopical n-Lie.

Let

$$r(t_1,\dots,t_7)=$$

$$\sum_{\sigma\in S_{2,3}} sign\,\sigma\,\psi(t_{\sigma(1)},t_{\sigma(2)},\psi(t_6,t_7,\psi(t_{\sigma(3)},t_{\sigma(4)},t_{\sigma(5)})))+$$

$$\sum_{\sigma\in\tilde{S}_{1,2,2}} sign\,\sigma\,\{\psi(t_{\sigma(1)},t_6,\psi(t_{\sigma(2)},t_{\sigma(3)},\psi(t_{\sigma(4)},t_{\sigma(5)},t_7)))-$$

$$\psi(t_{\sigma(1)},t_7,\psi(t_{\sigma(2)},t_{\sigma(3)},\psi(t_{\sigma(4)},t_{\sigma(5)},t_6)))\}-$$

$$\sum_{\sigma\in S_{1,2,2}} sign\,\sigma\,\{\psi(t_{\sigma(1)},\psi(t_{\sigma(2)},t_{\sigma(3)},t_6),\psi(t_{\sigma(4)},t_{\sigma(5)},t_7)))$$

$$-\psi(t_{\sigma(1)},\psi(t_{\sigma(2)},t_{\sigma(3)},t_7),\psi(t_{\sigma(4)},t_{\sigma(5)},t_6))\}.$$

Then r is skew-symmetric under variables $\{t_1,t_2,t_3,t_4,t_5\}$ and $\{t_6,t_7\}$.

Sum of Jacobians for odd n

Theorem. *Let $A = K[x_1, \ldots, x_6]$ be algebra of polynomials with 6 variables and 3-multiplication given as a sum of two jacobians*

$$[a_1, a_2, a_3] = \begin{vmatrix} \partial_1 a_1 & \partial_1 a_2 & \partial_1 a_3 \\ \partial_2 a_1 & \partial_2 a_2 & \partial_2 a_3 \\ \partial_3 a_1 & \partial_3 a_2 & \partial_3 a_3 \end{vmatrix} + \begin{vmatrix} \partial_4 a_1 & \partial_4 a_2 & \partial_4 a_3 \\ \partial_5 a_1 & \partial_5 a_2 & \partial_5 a_3 \\ \partial_6 a_1 & \partial_6 a_2 & \partial_6 a_3 \end{vmatrix}$$

Then any multilinear identity of A of ω-degree 2 follows from the skew-symmetric identity of 3-multiplication. The algebra A satisfies the identities $f_{-2}^{[1]} = 0$ and $f_1^{[2]} = 0$ and $r = 0$. Any multilinear identity of A of ω-degree 3 follows from the identities $f_{-2}^{[1]} = 0$, $f_1^{[2]} = 0$ and $r = 0$.

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Thank You !