Differential Type Operators, Rewriting Systems and Gröbner-Shirshov Bases

Li GUO
(joint work with William Sit and Ronghua Zhang)

Rutgers University at Newark

Motivation: Classification of Linear Operators

- Throughout the history, mathematical objects are often understood through studying operators defined on them.

Motivation: Classification of Linear Operators

- Throughout the history, mathematical objects are often understood through studying operators defined on them.
- Well-known examples include Galois theory where a field is studied by its automorphisms (the Galois group),

Motivation: Classification of Linear Operators

- Throughout the history, mathematical objects are often understood through studying operators defined on them.
- Well-known examples include Galois theory where a field is studied by its automorphisms (the Galois group),
- and analysis and geometry where functions and manifolds are studied through their derivations, integrals and related vector fields.

Rota's Question

- By the 1970s, several other operators had been discovered from studies in analysis, probability and combinatorics.

Average operator $P(x)P(y) = P(xP(y))$,

Inverse average operator $P(x)P(y) = P(P(x)y)$,

(Rota-)Baxter operator $P(x)P(y) = P(xP(y) + P(x)y + \lambda xy)$,

where λ is a fixed constant,

Reynolds operator $P(x)P(y) = P(xP(y) + P(x)y - P(x)P(y))$.

Rota's Question

- By the 1970s, several other operators had been discovered from studies in analysis, probability and combinatorics.

  Average operator $P(x)P(y) = P(xP(y))$,

  Inverse average operator $P(x)P(y) = P(P(x)y)$,

  (Rota-)Baxter operator $P(x)P(y) = P(xP(y) + P(x)y + \lambda xy)$,

  where λ is a fixed constant,

  Reynolds operator $P(x)P(y) = P(xP(y) + P(x)y - P(x)P(y))$.

- Rota posed the question of finding all the identities that could be satisfied by a linear operator defined on *associative algebras*. He also suggested that there should not be many such operators other than these previously known ones.

Quotation from Rota and Known Operators

- ”In a series of papers, I have tried to show that other linear operators satisfying algebraic identities may be of equal importance in studying certain algebraic phenomena, and I have posed the problem of finding all possible algebraic identities that can be satisfied by a linear operator on an algebra. Simple computations show that the possibility are very few, and the problem of classifying all such identities is very probably completely solvable. A partial (but fairly complete) list of such identities is the following. Besides endomorphisms and derivations, one has averaging operators, Reynolds operators and Baxter operators.”

Quotation from Rota and Known Operators

- "In a series of papers, I have tried to show that other linear operators satisfying algebraic identities may be of equal importance in studying certain algebraic phenomena, and I have posed the problem of finding all possible algebraic identities that can be satisfied by a linear operator on an algebra. Simple computations show that the possibility are very few, and the problem of classifying all such identities is very probably completely solvable. A partial (but fairly complete) list of such identities is the following. Besides endomorphisms and derivations, one has averaging operators, Reynolds operators and Baxter operators."
- Little progress was made on finding all such operators while new operators have merged from physics and combinatorial studies, such as

Nijenhuis operator	$P(x)P(y) = P(xP(y) + P(x)y - P(xy)),$
Leroux's TD operator	$P(x)P(y) = P(xP(y) + P(x)y - xP(1)y).$

Other Post-Rota developments

- These previously known operators continued to find remarkable applications in pure and applied mathematics.

Other Post-Rota developments

- These previously known operators continued to find remarkable applications in pure and applied mathematics.
- Vast theories were established for differential algebra and difference algebra, with wide applications, including Wen-Tsun Wu's mechanical proof of geometric theorems and mathematics mechanization (based on work of Ritt).

Other Post-Rota developments

- These previously known operators continued to find remarkable applications in pure and applied mathematics.
- Vast theories were established for differential algebra and difference algebra, with wide applications, including Wen-Tsun Wu's mechanical proof of geometric theorems and mathematics mechanization (based on work of Ritt).
- Rota-Baxter algebra has found applications in classical Yang-Baxter equations, operads, combinatorics, and most prominently, the renormalization of quantum field theory through the Hopf algebra framework of Connes and Kreimer.

PI Algebras

- **What does an algebraic identity satisfied by a linear operator mean?**—**Polynomial identity (PI) algebras** gives **a simplified analogue:**

PI Algebras

- **What does an algebraic identity satisfied by a linear operator mean?—Polynomial identity (PI) algebras** gives **a simplified analogue:**
- A **k**-algebra R is called a PI algebra (Procesi, Rowen, ...) if there is a fixed element $f(x_1, \cdots, x_n)$ in the noncommutative polynomial algebra (that is, the free algebra) $\mathbf{k}\langle x_1, \cdots, x_n\rangle$ such that

$$f(a_1, \cdots, a_n) = 0, \quad \forall a_1, \cdots, a_n \in R.$$

  Thus an algebraic identity satisfied by an algebra is an element in the free algebra.

PI Algebras

- **What does an algebraic identity satisfied by a linear operator mean?—Polynomial identity (PI) algebras** gives **a simplified analogue:**
- A **k**-algebra R is called a PI algebra (Procesi, Rowen, ...) if there is a fixed element $f(x_1, \cdots, x_n)$ in the noncommutative polynomial algebra (that is, the free algebra) $\mathbf{k}\langle x_1, \cdots, x_n\rangle$ such that

$$f(a_1, \cdots, a_n) = 0, \quad \forall a_1, \cdots, a_n \in R.$$

  Thus an algebraic identity satisfied by an algebra is an element in the free algebra.
- Then an algebraic identity satisfied by a linear operator should be an element in a free algebra with an operator, a so called free operated algebra.

Operated algebras

Operated algebras

- An operated $\mathbf{k}$-algebra is a $\mathbf{k}$-algebra R with a linear operator P on R. Examples are given by differential algebras and Rota-Baxter algebras. We can also consider algebras with multiple operators, such as differential Rota-Baxter algebras and integro-differential algebras.

Operated algebras

- An operated **k**-algebra is a **k**-algebra R with a linear operator P on R. Examples are given by differential algebras and Rota-Baxter algebras. We can also consider algebras with multiple operators, such as differential Rota-Baxter algebras and integro-differential algebras.
- An operated ideal of R is an ideal I of R such that $P(I) \subseteq I$.

Operated algebras

- An operated **k**-algebra is a **k**-algebra R with a linear operator P on R. Examples are given by differential algebras and Rota-Baxter algebras. We can also consider algebras with multiple operators, such as differential Rota-Baxter algebras and integro-differential algebras.
- An operated ideal of R is an ideal I of R such that $P(I) \subseteq I$.
- A homomorphism from a operated **k**-algebra (R, α) to a operated **k**-algebra (S, β) is a **k**-linear map $f : U \to V$ such that $f \circ \alpha = \beta \circ f$.

Operated algebras

- An operated **k**-algebra is a **k**-algebra R with a linear operator P on R. Examples are given by differential algebras and Rota-Baxter algebras. We can also consider algebras with multiple operators, such as differential Rota-Baxter algebras and integro-differential algebras.
- An operated ideal of R is an ideal I of R such that $P(I) \subseteq I$.
- A homomorphism from a operated **k**-algebra (R, α) to a operated **k**-algebra (S, β) is a **k**-linear map $f : U \to V$ such that $f \circ \alpha = \beta \circ f$.
- The adjoint functor of the forgetful functor from the category of operated algebras to the category of sets gives the free operated **k**-algebras.

Operated algebras

- An operated **k**-algebra is a **k**-algebra R with a linear operator P on R. Examples are given by differential algebras and Rota-Baxter algebras. We can also consider algebras with multiple operators, such as differential Rota-Baxter algebras and integro-differential algebras.
- An operated ideal of R is an ideal I of R such that $P(I) \subseteq I$.
- A homomorphism from a operated **k**-algebra (R, α) to a operated **k**-algebra (S, β) is a **k**-linear map $f : U \to V$ such that $f \circ \alpha = \beta \circ f$.
- The adjoint functor of the forgetful functor from the category of operated algebras to the category of sets gives the free operated **k**-algebras.
- More precisely, a free operated **k**-algebra on a set X is an operated **k**-algebra $(\mathbf{k}\|X\|, \alpha_X)$ together with a map $j_X : X \to \mathbf{k}\|X\|$ with the property that, for any operated algebra (R, β) together with a map $f : X \to R$, there is a unique morphism $\bar{f} : (\mathbf{k}\|X\|, \alpha_X) \to (R, \beta)$ of operated algebras such that $f = \bar{f} \circ j_X$.

Bracketed words

- For any set Y, let
$$[Y] := \{\lfloor y \rfloor \,|\, y \in Y\}$$
denote a set indexed by Y and disjoint from Y.

Bracketed words

- For any set Y, let
$$[Y] := \{\lfloor y \rfloor \mid y \in Y\}$$
denote a set indexed by Y and disjoint from Y.
- For a fixed set X, let $\mathfrak{M}_0 = \mathfrak{M}(X)_0 = M(X)$. For $n \geq 0$, let
$$\mathfrak{M}_{n+1} := M(X \cup [\mathfrak{M}_n]).$$

Bracketed words

- For any set Y, let
$$[Y] := \{\lfloor y \rfloor \, | y \in Y\}$$
denote a set indexed by Y and disjoint from Y.
- For a fixed set X, let $\mathfrak{M}_0 = \mathfrak{M}(X)_0 = M(X)$. For $n \geq 0$, let
$$\mathfrak{M}_{n+1} := M(X \cup [\mathfrak{M}_n]).$$

- With the embedding $X \cup [\mathfrak{M}_{n-1}] \to X \cup [\mathfrak{M}_n]$, we obtain an embedding of monoids $i_n : \mathfrak{M}_n \to \mathfrak{M}_{n+1}$, giving the direct limit
$$\mathfrak{M}(X) := \varinjlim \mathfrak{M}_n.$$

Bracketed words

- For any set Y, let
$$[Y] := \{\lfloor y \rfloor \,|y \in Y\}$$
denote a set indexed by Y and disjoint from Y.
- For a fixed set X, let $\mathfrak{M}_0 = \mathfrak{M}(X)_0 = M(X)$. For $n \geq 0$, let
$$\mathfrak{M}_{n+1} := M(X \cup [\mathfrak{M}_n]).$$
- With the embedding $X \cup [\mathfrak{M}_{n-1}] \to X \cup [\mathfrak{M}_n]$, we obtain an embedding of monoids $i_n : \mathfrak{M}_n \to \mathfrak{M}_{n+1}$, giving the direct limit
$$\mathfrak{M}(X) := \varinjlim \mathfrak{M}_n.$$
- Elements of $\mathfrak{M}(X)$ are called bracketed words.

Bracketed words

- For any set Y, let
$$[Y] := \{\lfloor y \rfloor \, | y \in Y\}$$
denote a set indexed by Y and disjoint from Y.
- For a fixed set X, let $\mathfrak{M}_0 = \mathfrak{M}(X)_0 = M(X)$. For $n \geq 0$, let
$$\mathfrak{M}_{n+1} := M(X \cup [\mathfrak{M}_n]).$$
- With the embedding $X \cup [\mathfrak{M}_{n-1}] \to X \cup [\mathfrak{M}_n]$, we obtain an embedding of monoids $i_n : \mathfrak{M}_n \to \mathfrak{M}_{n+1}$, giving the direct limit
$$\mathfrak{M}(X) := \varinjlim \mathfrak{M}_n.$$
- Elements of $\mathfrak{M}(X)$ are called bracketed words.
- $\mathfrak{M}(X)$ can also be identified with elements of $M(X \cup \{\lfloor, \rfloor\})$ such that
  - the total number of $\lfloor$ in the word equals to the total number of $\rfloor$ in the word;
  - counting from the left, the number of $\lfloor$ is always greater or equal to the number of $\rfloor$.

Bracketed words

- For any set Y, let
$$[Y] := \{\lfloor y \rfloor \mid y \in Y\}$$
denote a set indexed by Y and disjoint from Y.
- For a fixed set X, let $\mathfrak{M}_0 = \mathfrak{M}(X)_0 = M(X)$. For $n \geq 0$, let
$$\mathfrak{M}_{n+1} := M(X \cup [\mathfrak{M}_n]).$$
- With the embedding $X \cup [\mathfrak{M}_{n-1}] \to X \cup [\mathfrak{M}_n]$, we obtain an embedding of monoids $i_n : \mathfrak{M}_n \to \mathfrak{M}_{n+1}$, giving the direct limit
$$\mathfrak{M}(X) := \varinjlim \mathfrak{M}_n.$$
- Elements of $\mathfrak{M}(X)$ are called bracketed words.
- $\mathfrak{M}(X)$ can also be identified with elements of $M(X \cup \{\lfloor, \rfloor\})$ such that
  - the total number of $\lfloor$ in the word equals to the total number of $\rfloor$ in the word;
  - counting from the left, the number of $\lfloor$ is always greater or equal to the number of $\rfloor$.
- $\mathfrak{M}(X)$ can also be constructed by rooted trees and Motzkin paths.

- Theorem. 1. The set $\mathfrak{M}(X)$, equipped with the concatenation product, the operator $w \mapsto \lfloor w \rfloor, w \in \mathfrak{M}(X)$ and the natural embedding $j_X : X \to \mathfrak{M}(X)$, is the free operated monoid on X.
  2. $\mathbf{k}\|X\| := \mathbf{k}\mathfrak{M}(X)$ is the free unitary **k**-algebra on X.

- Theorem. 1. The set $\mathfrak{M}(X)$, equipped with the concatenation product, the operator $w \mapsto \lfloor w \rfloor, w \in \mathfrak{M}(X)$ and the natural embedding $j_X : X \to \mathfrak{M}(X)$, is the free operated monoid on X.
  2. $\mathbf{k}\|X\| := \mathbf{k}\mathfrak{M}(X)$ is the free unitary $\mathbf{k}$-algebra on X.
- Let $\mathfrak{D}(Z)$ be the submonoid of $\mathfrak{M}(Z)$ generated by the set

$$\Delta(Z) := Z \cup \lfloor Z \rfloor \cup \lfloor \lfloor Z \rfloor \rfloor \cup \cdots .$$

- Theorem. 1. The set $\mathfrak{M}(X)$, equipped with the concatenation product, the operator $w \mapsto \lfloor w \rfloor, w \in \mathfrak{M}(X)$ and the natural embedding $j_X : X \to \mathfrak{M}(X)$, is the free operated monoid on X.
  2. $\mathbf{k}\|X\| := \mathbf{k}\mathfrak{M}(X)$ is the free unitary $\mathbf{k}$-algebra on X.
- Let $\mathfrak{D}(Z)$ be the submonoid of $\mathfrak{M}(Z)$ generated by the set

$$\Delta(Z) := Z \cup \lfloor Z \rfloor \cup \lfloor\lfloor Z \rfloor\rfloor \cup \cdots .$$

- Elements in $\mathfrak{D}(Z)$ are called differential words since they form a basis of the free differential algebra on Z.

- Theorem. 1. The set $\mathfrak{M}(X)$, equipped with the concatenation product, the operator $w \mapsto \lfloor w \rfloor, w \in \mathfrak{M}(X)$ and the natural embedding $j_X : X \to \mathfrak{M}(X)$, is the free operated monoid on X.
  2. $\mathbf{k}\|X\| := \mathbf{k}\mathfrak{M}(X)$ is the free unitary $\mathbf{k}$-algebra on X.
- Let $\mathfrak{D}(Z)$ be the submonoid of $\mathfrak{M}(Z)$ generated by the set

$$\Delta(Z) := Z \cup \lfloor Z \rfloor \cup \lfloor \lfloor Z \rfloor \rfloor \cup \cdots .$$

- Elements in $\mathfrak{D}(Z)$ are called differential words since they form a basis of the free differential algebra on Z.
- Elements in $\mathbf{k}\mathfrak{D}(Z)$ are called in differentially reduced form (DRF).

- Theorem. 1. The set $\mathfrak{M}(X)$, equipped with the concatenation product, the operator $w \mapsto \lfloor w \rfloor, w \in \mathfrak{M}(X)$ and the natural embedding $j_X : X \to \mathfrak{M}(X)$, is the free operated monoid on X.
  2. $\mathbf{k}\|X\| := \mathbf{k}\mathfrak{M}(X)$ is the free unitary $\mathbf{k}$-algebra on X.
- Let $\mathfrak{D}(Z)$ be the submonoid of $\mathfrak{M}(Z)$ generated by the set

$$\Delta(Z) := Z \cup \lfloor Z \rfloor \cup \lfloor \lfloor Z \rfloor \rfloor \cup \cdots .$$

- Elements in $\mathfrak{D}(Z)$ are called differential words since they form a basis of the free differential algebra on Z.
- Elements in $\mathbf{k}\mathfrak{D}(Z)$ are called in differentially reduced form (DRF).
- Note that $\mathfrak{D}(Z)$ is closed under multiplication by definition, but not under the operator $\lfloor\ \rfloor$.

Operated Polynomial Identities

- An operated **k**-algebra (R, P) is called an operated PI (OPI) **k**-algebra if there is a fixed element $\phi(x_1, \cdots, x_n) \in \mathbf{k}\lfloor x_1, \cdots, x_n \rfloor$ such that

$$\phi(a_1, \cdots, a_n) = 0, \quad \forall a_1, \cdots, a_n \in R.$$

where a pair of brackets $\lfloor\ \rfloor$ is replaced by P everywhere.

Operated Polynomial Identities

- An operated **k**-algebra (R, P) is called an operated PI (OPI) **k**-algebra if there is a fixed element $\phi(x_1, \cdots, x_n) \in \mathbf{k}\lfloor\lfloor x_1, \cdots, x_n \rfloor\rfloor$ such that
$$\phi(a_1, \cdots, a_n) = 0, \quad \forall a_1, \cdots, a_n \in R.$$
where a pair of brackets $\lfloor\ \rfloor$ is replaced by P everywhere.
- In this case, we also call (R, P) a ϕ-**k**-algebra and call P a ϕ-operator.

Operated Polynomial Identities

- An operated $\mathbf{k}$-algebra (R, P) is called an operated PI (OPI) $\mathbf{k}$-algebra if there is a fixed element $\phi(x_1, \cdots, x_n) \in \mathbf{k}\|x_1, \cdots, x_n\|$ such that

$$\phi(a_1, \cdots, a_n) = 0, \quad \forall a_1, \cdots, a_n \in R.$$

  where a pair of brackets $\lfloor\ \rfloor$ is replaced by P everywhere.
- In this case, we also call (R, P) a ϕ-$\mathbf{k}$-algebra and call P a ϕ-operator.
- Examples
  1. When $\phi = [xy] - x[y] - [x]y$, a ϕ-operator (resp. algebra) is a differential operator (resp. algebra).
  2. When $\phi = [x][y] - [x[y]] - [[x]y] - \lambda[xy]$, a ϕ-operator (resp. ϕ-algebra) is a Rota-Baxter operator (resp. algebra) of weight λ.
  3. When $\phi = [x] - x$, then a ϕ-algebra is just an associative algebra. Together with a second identity from the noncommutative polynomial algebra $\mathbf{k}\langle X\rangle$, we get a PI-algebra.

Free ϕ-algebras

- **Proposition** Let $\phi = \phi(x_1, \cdots, x_k) \in \mathbf{k}\|X\|$ be given. For any set Z, the free ϕ-algebra on Z is given by the quotient operated algebra $\mathbf{k}\|Z\|/I_{\phi,Z}$ where $I_{\phi,Z}$ is the operated ideal of $\mathbf{k}\|Z\|$ generated by the set
$$\{\phi(u_1, \cdots, u_k) \mid u_1, \cdots, u_k \in \mathbf{k}\|Z\|\}.$$

Free ϕ-algebras

- **Proposition** Let $\phi = \phi(x_1, \cdots, x_k) \in \mathbf{k}\lfloor X \rfloor$ be given. For any set Z, the free ϕ-algebra on Z is given by the quotient operated algebra $\mathbf{k}\lfloor Z \rfloor / I_{\phi,Z}$ where $I_{\phi,Z}$ is the operated ideal of $\mathbf{k}\lfloor Z \rfloor$ generated by the set
$$\{\phi(u_1, \cdots, u_k) \mid u_1, \cdots, u_k \in \mathbf{k}\lfloor Z \rfloor\}.$$

- Examples
  - When $\phi = [x] - x$, then the quotient $\mathbf{k}\lfloor Z \rfloor / I_{\phi,Z}$ gives the free algebra $\mathbf{k}\langle Z \rangle$ on Z.
  - When $\phi = [xy] - x[y] - [x]y$, then the quotient gives the free noncommutative polynomial differential algebra $\mathbf{k}\langle \mathfrak{D}(Z) \rangle$ on Z.

Remarks:

- A classification of linear operators can be regarded as a classification of elements in $\mathbf{k}[\![X]\!]$.

Remarks:

- A classification of linear operators can be regarded as a classification of elements in $\mathbf{k}[\![X]\!]$.
- This problem is precise, but is too broad.

Remarks:

- A classification of linear operators can be regarded as a classification of elements in $\mathbf{k}\|X\|$.
- This problem is precise, but is too broad.
- We remind ourselves that Rota also wanted the operator to be defined on *associative algebras.*

Remarks:

- A classification of linear operators can be regarded as a classification of elements in $\mathbf{k}\|X\|$.
- This problem is precise, but is too broad.
- We remind ourselves that Rota also wanted the operator to be defined on *associative algebras.*
- This means that the operated identity $\phi \in \mathbf{k}\|x_1, \cdots, x_n\|$ should be compatible with the associativity condition.

Remarks:

- A classification of linear operators can be regarded as a classification of elements in $\mathbf{k}\|X\|$.
- This problem is precise, but is too broad.
- We remind ourselves that Rota also wanted the operator to be defined on *associative algebras.*
- This means that the operated identity $\phi \in \mathbf{k}\|x_1, \cdots, x_n\|$ should be compatible with the associativity condition.
- What does this mean?

Examples of compatibility with associativity

- Example 1: For $\phi(x,y) = [xy] - [x]y - x[y]$, we have

$$[xy] \mapsto [x]y + x[y].$$

Thus

$$[(xy)z] \mapsto [xy]z + (xy)[z] \mapsto [x]yz + x[y]z + xy[z].$$

$$[x(yz)] \mapsto [x](yz) + x[yz] \mapsto [x]yz + x[y]z + xy[z].$$

So $[(xy)z]$ and $[x(yz)]$ have the same reduction, indicating that the differential operator is consistent with the associativity condition.

Examples of compatibility with associativity

- Example 1: For $\phi(x,y) = [xy] - [x]y - x[y]$, we have

$$[xy] \mapsto [x]y + x[y].$$

Thus

$$[(xy)z] \mapsto [xy]z + (xy)[z] \mapsto [x]yz + x[y]z + xy[z].$$

$$[x(yz)] \mapsto [x](yz) + x[yz] \mapsto [x]yz + x[y]z + xy[z].$$

So $[(xy)z]$ and $[x(yz)]$ have the same reduction, indicating that the differential operator is consistent with the associativity condition.

- Example 2: The same is true for $\phi(x,y) = [xy] - [x]y$:

$$\lfloor x \rfloor yz \leftarrow\!\shortmid \lfloor xy \rfloor z \leftarrow\!\shortmid [(xy)z] = \lfloor x(yz) \rfloor \mapsto [x]yz.$$

Examples of compatibility with associativity

- Example 1: For $\phi(x,y) = [xy] - [x]y - x[y]$, we have

$$[xy] \mapsto [x]y + x[y].$$

Thus

$$[(xy)z] \mapsto [xy]z + (xy)[z] \mapsto [x]yz + x[y]z + xy[z].$$

$$[x(yz)] \mapsto [x](yz) + x[yz] \mapsto [x]yz + x[y]z + xy[z].$$

So $[(xy)z]$ and $[x(yz)]$ have the same reduction, indicating that the differential operator is consistent with the associativity condition.

- Example 2: The same is true for $\phi(x,y) = [xy] - [x]y$:

$$\lfloor x \rfloor yz \leftarrow\!\shortmid \lfloor xy \rfloor z \leftarrow\!\shortmid [(xy)z] = \lfloor x(yz) \rfloor \mapsto [x]yz.$$

- Example 3: Suppose $\phi(x,y) = [xy] - [y]x$. Then $[xy] \mapsto [y]x$. So

$$[w]uv \leftarrow\!\shortmid [(uv)w] = [u(vw)] \mapsto [vw]u \mapsto [w]vu.$$

Thus a ϕ-algebra (R,δ) satisfies the weak commutativity:

$$\delta(w)(uv - vu) = 0, \forall u, v, w \in Z.$$

Differential type operators

- differential operator $[xy] = [x]y + x[y]$,
  differential operator of weight λ $[xy] = [x]y + x[y] + \lambda[x][y]$,
  homomorphism $[xy] = [x][y]$,
  semihomomorphism $[xy] = x[y]$.

Differential type operators

- differential operator $[xy] = [x]y + x[y]$,
  differential operator of weight λ $[xy] = [x]y + x[y] + \lambda[x][y]$,
  homomorphism $[xy] = [x][y]$,
  semihomomorphism $[xy] = x[y]$.
- They are of the form $[xy] = N(x, y)$ where
  1. $N(x, y) \in \mathbf{k}\|x, y\|$ is in DRF, namely, it does not contain $[uv], u, v \neq 1$, that is, $N(x, y)$ is in $\mathbf{k}\mathfrak{D}(x, y)$;
  2. $N(uv, w) = N(u, vw)$ is reduced to zero under the reduction $[xy] \mapsto N(x, y)$.

  An operator identity $\phi(x, y) = 0$ is said of differential type if $\phi(x, y) = [xy] - N(x, y)$ where $N(x, y)$ satisfies these properties. We call $N(x, y)$ and an operator satisfying $\phi(x, y) = 0$ of differential type.

Differential type operators

- differential operator $[xy] = [x]y + x[y]$,
  differential operator of weight λ $[xy] = [x]y + x[y] + \lambda[x][y]$,
  homomorphism $[xy] = [x][y]$,
  semihomomorphism $[xy] = x[y]$.
- They are of the form $[xy] = N(x, y)$ where
  1. $N(x, y) \in \mathbf{k}\|x, y\|$ is in DRF, namely, it does not contain $[uv], u, v \neq 1$, that is, $N(x, y)$ is in $\mathbf{k}\mathfrak{D}(x, y)$;
  2. $N(uv, w) = N(u, vw)$ is reduced to zero under the reduction $[xy] \mapsto N(x, y)$.

  An operator identity $\phi(x, y) = 0$ is said of differential type if $\phi(x, y) = [xy] - N(x, y)$ where $N(x, y)$ satisfies these properties. We call $N(x, y)$ and an operator satisfying $\phi(x, y) = 0$ of differential type.
- The above examples also satisfy
  1. The free ϕ-algebra on Z can be defined by the noncommutative polynomial algebra $\mathbf{k}\langle \Delta(Z) \rangle$ with a suitable operator. So $\mathfrak{D}(Z)$ is a canonical basis of the free object.
  2. The restriction $\mathbf{k}\langle \Delta(Z) \rangle \hookrightarrow \mathbf{k}\|Z\| \to \mathbf{k}\|Z\|/I_\phi(Z)$ is bijective.

Classification of differential type operators

- **(Rota's Problem: the Differential Case)** Find all operated polynomial identities of differential type by finding all expressions $N(x, y) \in \mathbf{k}\lfloor x, y \rfloor$ of differential type.

Classification of differential type operators

- **(Rota's Problem: the Differential Case)** Find all operated polynomial identities of differential type by finding all expressions $N(x,y) \in \mathbf{k}\|x,y\|$ of differential type.
- **Conjecture (OPIs of Differential Type)** Let $\mathbf{k}$ be a field of characteristic zero. Every expression $N(x,y) \in \mathbf{k}\|x,y\|$ of differential type takes one of the forms below for some $a, b, c, e \in \mathbf{k}$:
  1. $b(x\lfloor y\rfloor + \lfloor x\rfloor y) + c\lfloor x\rfloor\lfloor y\rfloor + exy$ where $b^2 = b + ce$,
  2. $ce^2yx + exy + c\lfloor y\rfloor\lfloor x\rfloor - ce(y\lfloor x\rfloor + \lfloor y\rfloor x)$,
  3. $axy\lfloor 1\rfloor + b\lfloor 1\rfloor xy + cxy$,
  4. $x\lfloor y\rfloor + \lfloor x\rfloor y + ax\lfloor 1\rfloor y + bxy$,
  5. $\lfloor x\rfloor y + a(x\lfloor 1\rfloor y - xy\lfloor 1\rfloor)$,
  6. $x\lfloor y\rfloor + a(x\lfloor 1\rfloor y - \lfloor 1\rfloor xy)$.

Rewriting systems

- $\phi(x,y) := \lfloor xy \rfloor - N(x,y) \in \mathbf{k}\llbracket x,y \rrbracket$ defines a rewriting system:

$$\Sigma_\phi := \{\lfloor ab \rfloor \mapsto N(a,b) \mid a,b \in \mathfrak{M}(Z)\backslash\{1\}\}, \tag{1}$$

where Z is a set.

Rewriting systems

- $\phi(x,y) := \lfloor xy \rfloor - N(x,y) \in \mathbf{k}\llbracket x,y \rrbracket$ defines a rewriting system:

$$\Sigma_\phi := \{\lfloor ab \rfloor \mapsto N(a,b) \mid a,b \in \mathfrak{M}(Z) \backslash \{1\}\}\,, \tag{1}$$

where Z is a set.

- More precisely, for $g, g' \in \mathbf{k}\llbracket Z \rrbracket$, denote $g \to_{\Sigma_\phi} g'$ if g' is obtained from g by replacing a subword $\lfloor ab \rfloor$ in a monomial of g by $N(a,b)$.

Rewriting systems

- $\phi(x,y) := \lfloor xy \rfloor - N(x,y) \in \mathbf{k}\|x,y\|$ defines a rewriting system:

$$\Sigma_\phi := \{\lfloor ab \rfloor \mapsto N(a,b) \mid a,b \in \mathfrak{M}(Z)\backslash\{1\}\}\,, \tag{1}$$

  where Z is a set.
- More precisely, for $g, g' \in \mathbf{k}\|Z\|$, denote $g \rightarrow_{\Sigma_\phi} g'$ if g' is obtained from g by replacing a subword $\lfloor ab \rfloor$ in a monomial of g by $N(a,b)$.
- A rewriting system Σ is call
  - terminating if every reduction $g_0 \mapsto_\Sigma g_1 \mapsto \cdots$ stops after finite steps,
  - confluent if any two reductions of g can be reduced to the same element.
  - convergent if it is both terminating and confluent.

Rewriting systems

- $\phi(x,y) := \lfloor xy \rfloor - N(x,y) \in \mathbf{k}\|x,y\|$ defines a rewriting system:

$$\Sigma_\phi := \{\lfloor ab \rfloor \mapsto N(a,b) \mid a,b \in \mathfrak{M}(Z)\backslash\{1\}\}, \tag{1}$$

  where Z is a set.
- More precisely, for $g, g' \in \mathbf{k}\|Z\|$, denote $g \to_{\Sigma_\phi} g'$ if g' is obtained from g by replacing a subword $\lfloor ab \rfloor$ in a monomial of g by $N(a,b)$.
- A rewriting system Σ is call
  - terminating if every reduction $g_0 \mapsto_\Sigma g_1 \mapsto \cdots$ stops after finite steps,
  - confluent if any two reductions of g can be reduced to the same element.
  - convergent if it is both terminating and confluent.
- Theorem $\phi = [xy] - N(x,y)$ defines a differential type operator if and only if the rewriting system Σ_ϕ is convergent.

Monomial well orderings

- Let Z be a set. Let $\mathfrak{M}^\star(Z)$ denote the bracketed words in $Z \cup \{\star\}$ where $\star$ appears exactly once.

Monomial well orderings

- Let Z be a set. Let $\mathfrak{M}^\star(Z)$ denote the bracketed words in $Z \cup \{\star\}$ where $\star$ appears exactly once.
- For $q \in \mathfrak{M}^\star(Z)$ and $u \in \mathfrak{M}(Z)$, let $q|_u$ denote the bracketed word in $\mathfrak{M}(Z)$ when $\star$ in q is replaced by u.

Monomial well orderings

- Let Z be a set. Let $\mathfrak{M}^\star(Z)$ denote the bracketed words in $Z \cup \{\star\}$ where $\star$ appears exactly once.
- For $q \in \mathfrak{M}^\star(Z)$ and $u \in \mathfrak{M}(Z)$, let $q|_u$ denote the bracketed word in $\mathfrak{M}(Z)$ when $\star$ in q is replaced by u.
- Then $g \mapsto_{\Sigma_\phi} g'$ if there are $q \in \mathfrak{M}^\star(Z)$ and $a, b \in \mathfrak{M}(Z)$ such that
  1. $q|_{\lfloor ab \rfloor}$ is a monomial of g with coefficient $c \neq 0$,
  2. $g' = g - cq|_{N(a,b)}$.

Monomial well orderings

- Let Z be a set. Let $\mathfrak{M}^\star(Z)$ denote the bracketed words in $Z \cup \{\star\}$ where $\star$ appears exactly once.
- For $q \in \mathfrak{M}^\star(Z)$ and $u \in \mathfrak{M}(Z)$, let $q|_u$ denote the bracketed word in $\mathfrak{M}(Z)$ when $\star$ in q is replaced by u.
- Then $g \mapsto_{\Sigma_\phi} g'$ if there are $q \in \mathfrak{M}^\star(Z)$ and $a, b \in \mathfrak{M}(Z)$ such that
  1. $q|_{\lfloor ab \rfloor}$ is a monomial of g with coefficient $c \neq 0$,
  2. $g' = g - cq|_{N(a,b)}$.
- A **monomial ordering** on $\mathfrak{M}(Z)$ is a well-ordering $<$ on $\mathfrak{M}(X)$ such that

$$1 \leq u \text{ and } u < v \Rightarrow q|_u < q|_v, \ \forall u, v \in \mathfrak{M}(X), q \in \mathfrak{M}^\star(X).$$

Monomial well orderings

- Let Z be a set. Let $\mathfrak{M}^\star(Z)$ denote the bracketed words in $Z \cup \{\star\}$ where $\star$ appears exactly once.
- For $q \in \mathfrak{M}^\star(Z)$ and $u \in \mathfrak{M}(Z)$, let $q|_u$ denote the bracketed word in $\mathfrak{M}(Z)$ when $\star$ in q is replaced by u.
- Then $g \mapsto_{\Sigma_\phi} g'$ if there are $q \in \mathfrak{M}^\star(Z)$ and $a, b \in \mathfrak{M}(Z)$ such that
  1. $q|_{\lfloor ab \rfloor}$ is a monomial of g with coefficient $c \neq 0$,
  2. $g' = g - cq|_{N(a,b)}$.
- A **monomial ordering** on $\mathfrak{M}(Z)$ is a well-ordering $<$ on $\mathfrak{M}(X)$ such that

$$1 \leq u \text{ and } u < v \Rightarrow q|_u < q|_v, \ \forall u, v \in \mathfrak{M}(X), q \in \mathfrak{M}^\star(X).$$

- Given a monomial ordering $<$ and a bracketed polynomial $s \in \mathbf{k}\lfloor X \rfloor$, we let $\bar{s}$ denote the leading bracketed word (monomial) of s.

Monomial well orderings

- Let Z be a set. Let $\mathfrak{M}^\star(Z)$ denote the bracketed words in $Z \cup \{\star\}$ where $\star$ appears exactly once.
- For $q \in \mathfrak{M}^\star(Z)$ and $u \in \mathfrak{M}(Z)$, let $q|_u$ denote the bracketed word in $\mathfrak{M}(Z)$ when $\star$ in q is replaced by u.
- Then $g \mapsto_{\Sigma_\phi} g'$ if there are $q \in \mathfrak{M}^\star(Z)$ and $a, b \in \mathfrak{M}(Z)$ such that
  1. $q|_{\lfloor ab \rfloor}$ is a monomial of g with coefficient $c \neq 0$,
  2. $g' = g - cq|_{N(a,b)}$.
- A **monomial ordering** on $\mathfrak{M}(Z)$ is a well-ordering $<$ on $\mathfrak{M}(X)$ such that

$$1 \leq u \text{ and } u < v \Rightarrow q|_u < q|_v, \ \forall u, v \in \mathfrak{M}(X), q \in \mathfrak{M}^\star(X).$$

- Given a monomial ordering $<$ and a bracketed polynomial $s \in \mathbf{k}\|X\|$, we let $\bar{s}$ denote the leading bracketed word (monomial) of s.
- If the coefficient of $\bar{s}$ in s is 1, we call s monic with respect to the monomial order $<$

Gröbner-Shirshov bases

- Bokut, Chen and Qiu determined Gröbner-Shirshov bases for free nonunitary operated algebras. This can be similarly given for free unitary operated algebras $\mathbf{k}\llbracket Z\rrbracket$.

Gröbner-Shirshov bases

- Bokut, Chen and Qiu determined Gröbner-Shirshov bases for free nonunitary operated algebras. This can be similarly given for free unitary operated algebras $\mathbf{k}\| Z \|$.
- Let $>$ be a monomial ordering on $\mathfrak{M}(Z)$. Let f, g be two monic bracketed polynomials.

Gröbner-Shirshov bases

- Bokut, Chen and Qiu determined Gröbner-Shirshov bases for free nonunitary operated algebras. This can be similarly given for free unitary operated algebras $\mathbf{k}\lfloor Z\rfloor$.
- Let $>$ be a monomial ordering on $\mathfrak{M}(Z)$. Let f, g be two monic bracketed polynomials.
- If there are $p, q \in \mathfrak{M}^{\star}Z$ and $s, t \in \mathbf{k}\lfloor Z\rfloor$ such that $w := p|_{\overline{s}} = q|_{\overline{t}}$, then call

$$(f, g)_w^{p,q} := p_s - q|_t$$

an composition of f and g.

Gröbner-Shirshov bases

- Bokut, Chen and Qiu determined Gröbner-Shirshov bases for free nonunitary operated algebras. This can be similarly given for free unitary operated algebras $\mathbf{k}\|Z\|$.
- Let $>$ be a monomial ordering on $\mathfrak{M}(Z)$. Let f, g be two monic bracketed polynomials.
- If there are $p, q \in \mathfrak{M}^{\star}Z$ and $s, t \in \mathbf{k}\|Z\|$ such that $w := p|_{\overline{s}} = q|_{\overline{t}}$, then call

  $$(f, g)_w^{p,q} := p_s - q|_t$$

  an composition of f and g.
- For $S \subseteq \mathbf{k}\|Z\|$ and $u \in \mathbf{k}\|Z\|$, we call u trivial modulo (S, w) if $u = \sum_i c_i q_i|_{s_i}$, with $c_i \in \mathbf{k}$, $q_i \in \mathfrak{M}^{\star}(Z)$, $s_i \in S$ and $q_i|_{\overline{s_i}} < w$.

Gröbner-Shirshov bases

- Bokut, Chen and Qiu determined Gröbner-Shirshov bases for free nonunitary operated algebras. This can be similarly given for free unitary operated algebras $\mathbf{k}\|Z\|$.
- Let $>$ be a monomial ordering on $\mathfrak{M}(Z)$. Let f, g be two monic bracketed polynomials.
- If there are $p, q \in \mathfrak{M}^\star Z$ and $s, t \in \mathbf{k}\|Z\|$ such that $w := p|_{\overline{s}} = q|_{\overline{t}}$, then call

$$(f, g)_w^{p,q} := p_s - q|_t$$

  an composition of f and g.
- For $S \subseteq \mathbf{k}\|Z\|$ and $u \in \mathbf{k}\|Z\|$, we call u trivial modulo (S, w) if $u = \sum_i c_i q_i|_{s_i}$, with $c_i \in \mathbf{k}$, $q_i \in \mathfrak{M}^\star(Z)$, $s_i \in S$ and $q_i|_{\overline{s_i}} < w$.
- A set $S \subseteq \mathbf{k}\|X\|$ is called a Gröbner-Shirshov basis if, for all $f, g \in S$, all compositions $(f, g)_w^{p,q}$ of f and g are trivial modulo (S, w).

Gröbner-Shirshov bases

- Bokut, Chen and Qiu determined Gröbner-Shirshov bases for free nonunitary operated algebras. This can be similarly given for free unitary operated algebras $\mathbf{k}\|Z\|$.
- Let $>$ be a monomial ordering on $\mathfrak{M}(Z)$. Let f, g be two monic bracketed polynomials.
- If there are $p, q \in \mathfrak{M}^\star Z$ and $s, t \in \mathbf{k}\|Z\|$ such that $w := p|_{\overline{s}} = q|_{\overline{t}}$, then call

$$(f, g)_w^{p,q} := p_s - q|_t$$

  an composition of f and g.
- For $S \subseteq \mathbf{k}\|Z\|$ and $u \in \mathbf{k}\|Z\|$, we call u trivial modulo (S, w) if $u = \sum_i c_i q_i|_{s_i}$, with $c_i \in \mathbf{k}$, $q_i \in \mathfrak{M}^\star(Z)$, $s_i \in S$ and $q_i|_{\overline{s_i}} < w$.
- A set $S \subseteq \mathbf{k}\|X\|$ is called a Gröbner-Shirshov basis if, for all $f, g \in S$, all compositions $(f, g)_w^{p,q}$ of f and g are trivial modulo (S, w).
- The Gröbner-Shirshov condition can be weakened to requiring for only intersection and including compositions.

Differential well ordering

- Let $>$ be a well order on a set Z. We extend $>$ to a well order on $\mathfrak{M}(Z) = \varinjlim \mathfrak{M}(Z)$ by inductively defining a well ordering $>$ on $\mathfrak{M}_n := \mathfrak{M}_n(Z)$, $n \geq 0$.

Differential well ordering

- Let $>$ be a well order on a set Z. We extend $>$ to a well order on $\mathfrak{M}(Z) = \varinjlim \mathfrak{M}(Z)$ by inductively defining a well ordering $>$ on $\mathfrak{M}_n := \mathfrak{M}_n(Z)$, $n \geq 0$.
- For $n = 0$, have $\mathfrak{M}_0 = M(Z)$. We take the lexicographic order on $M(Z)$ with 1 to be the least element.

Differential well ordering

- Let $>$ be a well order on a set Z. We extend $>$ to a well order on $\mathfrak{M}(Z) = \varinjlim \mathfrak{M}(Z)$ by inductively defining a well ordering $>$ on $\mathfrak{M}_n := \mathfrak{M}_n(Z)$, $n \geq 0$.
- For $n = 0$, have $\mathfrak{M}_0 = M(Z)$. We take the lexicographic order on $M(Z)$ with 1 to be the least element.
- Suppose a well order $>$ has been defined on $\mathfrak{M}_n$ for $n \geq 0$. Then for $u, v \in Z \cup \lfloor \mathfrak{M}_n \rfloor$, define

$$u > v \Leftrightarrow \begin{cases} u, v \in X, \text{ such that } u > v, \text{ or} \\ u \in \lfloor \mathfrak{M}_n \rfloor, v \in x, \text{ or} \\ u = \lfloor u' \rfloor, v = \lfloor v' \rfloor \in \lfloor \mathfrak{M}_n \rfloor \text{ such that } u' > v'. \end{cases}$$

  Then extend this $>$ to $\mathfrak{M}_{n+1} := M(X \cup \lfloor \mathfrak{M}_n \rfloor)$ lexicographically.

Differential well ordering

- Let $>$ be a well order on a set Z. We extend $>$ to a well order on $\mathfrak{M}(Z) = \varinjlim \mathfrak{M}(Z)$ by inductively defining a well ordering $>$ on $\mathfrak{M}_n := \mathfrak{M}_n(Z)$, $n \geq 0$.
- For $n = 0$, have $\mathfrak{M}_0 = M(Z)$. We take the lexicographic order on $M(Z)$ with 1 to be the least element.
- Suppose a well order $>$ has been defined on $\mathfrak{M}_n$ for $n \geq 0$. Then for $u, v \in Z \cup \lfloor \mathfrak{M}_n \rfloor$, define

$$u > v \Leftrightarrow \begin{cases} u, v \in X, \text{ such that } u > v, \text{ or} \\ u \in \lfloor \mathfrak{M}_n \rfloor, v \in x, \text{ or} \\ u = \lfloor u' \rfloor, v = \lfloor v' \rfloor \in \lfloor \mathfrak{M}_n \rfloor \text{ such that } u' > v'. \end{cases}$$

  Then extend this $>$ to $\mathfrak{M}_{n+1} := M(X \cup \lfloor \mathfrak{M}_n \rfloor)$ lexicographically.
- We obtain a well order, still denoted by $>$, on the direct limit $\mathfrak{M}(Z) = \varinjlim \mathfrak{M}_n$.

Differential well ordering (cont'd)

- Let $\deg_z(u)$ denote the number of $z \in Z$ in u. Denote the **weight** of u by

$$wt(u) = (\deg_z(u), u).$$

Define

$$u > v \iff wt(u) > wt(v) \ \text{lexicographically.} \tag{2}$$

Differential well ordering (cont'd)

- Let $\deg_z(u)$ denote the number of $z \in Z$ in u. Denote the **weight** of u by
$$wt(u) = (\deg_z(u), u).$$
Define
$$u > v \Longleftrightarrow wt(u) > wt(v) \text{ lexicographically.} \tag{2}$$

- This order is a monomial well ordering on $\mathfrak{M}(Z)$.

Differential well ordering (cont'd)

- Let $\deg_z(u)$ denote the number of $z \in Z$ in u. Denote the **weight** of u by
$$wt(u) = (\deg_z(u), u).$$
Define
$$u > v \Longleftrightarrow wt(u) > wt(v) \text{ lexicographically.} \quad (2)$$
- This order is a monomial well ordering on $\mathfrak{M}(Z)$.
- Under this order, $\lfloor xy \rfloor$ is greater than elements in $\Delta(x, y)$. Thus $\lfloor xy \rfloor$ is the leading term for $\phi(x, y) = \lfloor xy \rfloor - N(x, y)$ when $N(x, y)$ is in DRF.

Differential type, rewriting systems and Gröbner-Shirshov bases

- For $\phi(x,y) := \delta(xy) - N(x,y) \in \mathbf{k}\llbracket x,y \rrbracket$, the following statements are equivalent.

Differential type, rewriting systems and Gröbner-Shirshov bases

- For $\phi(x,y) := \delta(xy) - N(x,y) \in \mathbf{k}\llbracket x,y \rrbracket$, the following statements are equivalent.
- $\phi(x,y)$ is of differential type;

Differential type, rewriting systems and Gröbner-Shirshov bases

- For $\phi(x,y) := \delta(xy) - N(x,y) \in \mathbf{k}\llbracket x,y \rrbracket$, the following statements are equivalent.
- $\phi(x,y)$ is of differential type;
- The rewriting system Σ_ϕ is convergent;

Differential type, rewriting systems and Gröbner-Shirshov bases

- For $\phi(x,y) := \delta(xy) - N(x,y) \in \mathbf{k}\|x,y\|$, the following statements are equivalent.
- $\phi(x,y)$ is of differential type;
- The rewriting system Σ_ϕ is convergent;
- Let Z be a set with a well ordering. With the differential order $>$, the set

$$S := S_\phi := \{\phi(u,v) = \delta(uv) - N(u,v) |\, u,v \in \mathfrak{M}(Z)\backslash\{1\}\}$$

  is a Gröbner-Shirshov basis in $\mathbf{k}\|Z\|$.
- The free ϕ-algebra on a set Z is the noncommutative polynomial $\mathbf{k}$-algebra $\mathbf{k}\langle\Delta(Z)\rangle$, together with the operator $d := d_Z$ on $\mathbf{k}\langle\Delta(Z)\rangle$ defined by the following recursion:
  Let $u = u_1u_2\cdots u_k \in M(\Delta(Z))$, where $u_i \in \Delta(Z), 1 \leq i \leq k$.
  1. If $k = 1$, i.e., $u = \delta^i(x)$ for some $i \geq 0, x \in Z$, then define $d(u) = \delta^{(i+1)}(x)$.
  2. If $k \geq 1$, then define $d(u) = N(u_1, u_2\cdots u_k)$.

Rota-Baxter type words

- Let $\mathfrak{M}'(Z)$ be the set of disjoint bracketed words consisting of bracketed words with no pairs of brackets right next to each other, such as $\lfloor * \rfloor \lfloor * \rfloor$.

Rota-Baxter type words

- Let $\mathfrak{M}'(Z)$ be the set of disjoint bracketed words consisting of bracketed words with no pairs of brackets right next to each other, such as $\lfloor * \rfloor \lfloor * \rfloor$.
- Elements of $\mathfrak{M}'(Z)$ are called Rota-Baxter words since they form a **k**-basis of the free Rota-Baxter **k**-algebra on Z.

Rota-Baxter type words

- Let $\mathfrak{M}'(Z)$ be the set of disjoint bracketed words consisting of bracketed words with no pairs of brackets right next to each other, such as $\lfloor * \rfloor \lfloor * \rfloor$.
- Elements of $\mathfrak{M}'(Z)$ are called Rota-Baxter words since they form a **k**-basis of the free Rota-Baxter **k**-algebra on Z.
- Note that $\mathfrak{M}'(Z)$ is not closed under multiplication, but is closed under the operator $\lfloor\ \rfloor$.

Rota-Baxter type words

- Let $\mathfrak{M}'(Z)$ be the set of disjoint bracketed words consisting of bracketed words with no pairs of brackets right next to each other, such as $\lfloor * \rfloor \lfloor * \rfloor$.
- Elements of $\mathfrak{M}'(Z)$ are called Rota-Baxter words since they form a **k**-basis of the free Rota-Baxter **k**-algebra on Z.
- Note that $\mathfrak{M}'(Z)$ is not closed under multiplication, but is closed under the operator $\lfloor\ \rfloor$.
- Words in $Z \cup \lfloor \mathfrak{M}(Z) \rfloor$ are called **indecomposable**. Any $\mathfrak{z} \in \mathfrak{M}(Z) - \{1\}$ has a unique factorization $\mathfrak{z} = \mathfrak{z}_1 \cdots \mathfrak{z}_b$ of indecomposable words, called the **standard decomposition.**

Rota-Baxter type operators

- What Rota-Baxter operator, average operator, Nijenhuis operator, etc. have in common is that

Rota-Baxter type operators

- What Rota-Baxter operator, average operator, Nijenhuis operator, etc. have in common is that
- 1). they are of the form

$$[u][v] = [M(u,v)]$$

where $M(u,v)$ is an expression involving u, v and P, i.e. $M(u,v) \in \mathbf{k}\llbracket u, v \rrbracket$.

Rota-Baxter type operators

- What Rota-Baxter operator, average operator, Nijenhuis operator, etc. have in common is that
- 1). they are of the form

$$[u][v] = [M(u,v)]$$

  where $M(u,v)$ is an expression involving u, v and P, i.e. $M(u,v) \in \mathbf{k}\|u,v\|$.
- 2). $M(u,v)$ is formally associative:

$$M(M(u,v),w) - M(u,M(v,w))$$

  is reduced to zero under the rewriting system $[u][v] \mapsto [M(u,v)]$.

Rota-Baxter type operators

- What Rota-Baxter operator, average operator, Nijenhuis operator, etc. have in common is that
- 1). they are of the form

$$[u][v] = [M(u, v)]$$

  where $M(u, v)$ is an expression involving u, v and P, i.e. $M(u, v) \in \mathbf{k}\llbracket u, v \rrbracket$.
- 2). $M(u, v)$ is formally associative:

$$M(M(u, v), w) - M(u, M(v, w))$$

  is reduced to zero under the rewriting system $[u][v] \mapsto [M(u, v)]$.
- We call $\phi(x, y) := [x][y] - [M(x, y)]$ of Rota-Baxter type if the above two conditions are satisfied.

Rota-Baxter type operators

- What Rota-Baxter operator, average operator, Nijenhuis operator, etc. have in common is that
- 1). they are of the form

$$[u][v] = [M(u, v)]$$

where $M(u, v)$ is an expression involving u, v and P, i.e. $M(u, v) \in \mathbf{k}\llbracket u, v \rrbracket$.
- 2). $M(u, v)$ is formally associative:

$$M(M(u, v), w) - M(u, M(v, w))$$

is reduced to zero under the rewriting system $[u][v] \mapsto [M(u, v)]$.
- We call $\phi(x, y) := [x][y] - [M(x, y)]$ of Rota-Baxter type if the above two conditions are satisfied.
- Rota-Baxter type operators can be similarly characterized in terms of convergent rewriting systems and Gröbner-Shirshov bases.

Rota-Baxter type words

- Let $\mathfrak{M}'(Z)$ be the set of disjoint bracketed words consisting of bracketed words with no pairs of brackets right next to each other, such as $\lfloor * \rfloor \lfloor * \rfloor$.

Rota-Baxter type words

- Let $\mathfrak{M}'(Z)$ be the set of disjoint bracketed words consisting of bracketed words with no pairs of brackets right next to each other, such as $\lfloor * \rfloor \lfloor * \rfloor$.
- Elements of $\mathfrak{M}'(Z)$ are called Rota-Baxter words since they form a **k**-basis of the free Rota-Baxter **k**-algebra on Z.

Rota-Baxter type words

- Let $\mathfrak{M}'(Z)$ be the set of disjoint bracketed words consisting of bracketed words with no pairs of brackets right next to each other, such as $\lfloor * \rfloor \lfloor * \rfloor$.
- Elements of $\mathfrak{M}'(Z)$ are called Rota-Baxter words since they form a **k**-basis of the free Rota-Baxter **k**-algebra on Z.
- Note that $\mathfrak{M}'(Z)$ is not closed under multiplication, but is closed under the operator $\lfloor\ \rfloor$.

Rota-Baxter type words

- Let $\mathfrak{M}'(Z)$ be the set of disjoint bracketed words consisting of bracketed words with no pairs of brackets right next to each other, such as $\lfloor * \rfloor \lfloor * \rfloor$.
- Elements of $\mathfrak{M}'(Z)$ are called Rota-Baxter words since they form a **k**-basis of the free Rota-Baxter **k**-algebra on Z.
- Note that $\mathfrak{M}'(Z)$ is not closed under multiplication, but is closed under the operator $\lfloor\ \rfloor$.
- Words in $Z \cup \lfloor \mathfrak{M}(Z) \rfloor$ are called **indecomposable**. Any $\mathfrak{z} \in \mathfrak{M}(Z) - \{1\}$ has a unique factorization $\mathfrak{z} = \mathfrak{z}_1 \cdots \mathfrak{z}_b$ of indecomposable words, called the **standard decomposition.**

Rota-Baxter type operators

- What Rota-Baxter operator, average operator, Nijenhuis operator, etc. have in common is that they are of the form
$$[u][v] = [M(u,v)]$$
where $M(u,v)$ is an expression involving u,v and P, i.e. $M(u,v) \in \mathbf{k}\|u,v\|$.
- Also, $M(u,v)$ is formally associative:
$$M(M(u,v),w) = M(u,M(v,w))$$
modulo the relation $\phi_M := [u][v] - [M(u,v)]$.
- Further, free algebras in the corresponding categories (of Rota-Baxter algebras, of average algebras, ...) have a special basis. More precisely, The map
$$\mathbf{k}\{Z\}' := \mathbf{k}\mathfrak{M}'(Z) \to \mathbf{k}\|Z\| \to \mathbf{k}\|Z\|/I_{\phi,Z}$$
is bijective. Thus a suitable multiplication on $\mathbf{k}\{Z\}'$ makes it the free ϕ_M-algebra on Z.
- As we will see, these properties are related.

Classification of Rota-Baxter type operators

- **Conjecture.** Any Rota-Baxter type operator is necessarily of the form
$$P(x)P(y) = P(M(x,y)),$$
for an $M(x,y)$ from the following list (new types in red).
  1. $xP(y)$ (average operator)
  2. $P(x)y$ (inverse average operator)
  3. $xP(y) + yP(x)$
  4. $P(x)y + P(y)x$
  5. $-P(xy) + xP(y) + P(x)y$ (Nijenhuis operator)
  6. $xP(y) + P(x)y + e_1xy$ (RBA with weight e_1)
  7. $xP(y) - xP(1)y + e_1xy$
  8. $P(x)y - xP(1)y + e_1xy$
  9. $xP(y) + P(x)y - xP(1)y + e_1xy$
     (generalized Leroux TD operator with weight e_1)
  10. $xP(y) + P(x)y - xyP(1) - xP(1)y + e_1xy$
  11. $-P(xy) + xP(y) + P(x)y - xP(1)y + e_1xy$
  12. $xP(y) + P(x)y - xP(1)y - P(1)xy + e_1xy$
  13. $d_0xP(1)y + e_1xy$ (generalized endomorphisms)
  14. $d_2yP(1)x + e_0yx$

Summary and outlook

- In the framework of bracketed polynomials, operators of differential type are defined by the convergence of special cases of the rewriting system from the operator identity. The fact that these special cases are enough for the general convergence is proved by Gröbner-Shirshov bases.

Summary and outlook

- In the framework of bracketed polynomials, operators of differential type are defined by the convergence of special cases of the rewriting system from the operator identity. The fact that these special cases are enough for the general convergence is proved by Gröbner-Shirshov bases.
- For operators of Rota-Baxter type (including Rota-Baxter, average, Nijenhuis, Leroux's TD), a similar conjecture and equivalence can be established.

Summary and outlook

- In the framework of bracketed polynomials, operators of differential type are defined by the convergence of special cases of the rewriting system from the operator identity. The fact that these special cases are enough for the general convergence is proved by Gröbner-Shirshov bases.
- For operators of Rota-Baxter type (including Rota-Baxter, average, Nijenhuis, Leroux's TD), a similar conjecture and equivalence can be established.
- In general, the linear operators that interested Rota (or maybe other mathematicians) should be the ones whose defining identities define convergent rewriting systems, or give Gröbner-Shirshov bases.

Thank You!