Gröbner-Shirshov bases for associative algebras, Lie algebras and metabelian Lie algebras

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Operads and Rewriting

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What is a Gröbner-Shirshov basis

Let $A = \langle X|R \rangle$ (say, associative,Lie...) and "$<$" a monomial ordering (linear ordering compatible with multiplications) on the set of monomials on X with DCC.

For $\forall f \in \langle X \rangle$, denote $\bar{f}$ to be the leading monomial w.r.t. "$<$".
In particular, $\forall r \in R$, $r = \bar{r} - \Sigma_{r_i < \bar{r}} \alpha_i r_i$ in $\langle X \rangle$, which means $\bar{r} = \Sigma_{r_i < \bar{r}} \alpha_i r_i$ in $A = \langle X|R \rangle$.

For $\forall f \in \langle X \rangle$, $f \to^{*}$ a linear combination of so called **irreducible** words.
$Irr(R) =$ the set of all irreducible words w.r.t. R.

Then we have

$$A = span\ Irr(R).$$

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★ What is Gröbner-Shirshov basis?
Answer: Gröbner-Shirshov basis is NOT a basis but a GOOD set of defining relations, say R, such that
$Irr(R)$ is a linear basis of A;
R has GOOD leading terms property: $\bar{I}$ is "generated" by $\bar{R}$.

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Main Goal:
★ Criterion = a list of necessary and sufficient conditions for a set of defining relations to be a Gröbner-Shirshov basis = the Composition-Diamond Lemma.
★ Algorithm: Shirshov's, Buchberger's, rewriting system (Composition=critical monomials= S-polynomials).

Affine Lie algebras over commutative algebras

Let $\mathbf{k}$ be a field, K a commutative associative $\mathbf{k}$-algebra with 1, $\mathcal{L}$ a Lie K-algebra, and $Lie_K(X)$ the free Lie K-algebra generated by X. Then, of course, $\mathcal{L}$ can be presented as K-algebra by generators X and some defining relations S,

$$\mathcal{L} = Lie_K(X|S).$$

On the other hand, K has a presentation

$$K = \mathbf{k}[Y|R]$$

as a quotient algebra of a polynomial algebra $\mathbf{k}[Y]$ over $\mathbf{k}$.
Then the Lie K-algebra $\mathcal{L}$ as a $\mathbf{k}[Y]$-algebra has a presentation as follow

$$\mathcal{L} = Lie_{\mathbf{k}[Y]}(X|S, rx_i, r \in R, x_i \in X).$$

Affine Lie algebras over commutative algebras

$$\mathcal{L} = Lie_K(X|S) = Lie_{\mathbf{k}[Y|R]}(X|S) = Lie_{\mathbf{k}[Y]}(X|S, rx_i, r \in R, x_i \in X)$$

A Gröbner-Shirshov basis T of the last presentation will be called a Gröbner-Shirshov basis of $\mathcal{L} = Lie_K(X|S)$ relative to $K = \mathbf{k}[Y|R]$.

It means that to defined Gröbner-Shirshov bases of Lie algebras over commutative algebras, it is enough to define Gröbner-Shirshov bases (sets) in "double free" Lie algebras $Lie_{\mathbf{k}[Y]}(X)$, i.e., free Lie algebras over polynomial algebras.

A double free Lie algebra $Lie_{\mathbf{k}[Y]}(X)$ is a $\mathbf{k}$-tensor product

$$\mathbf{k}[Y] \otimes Lie_{\mathbf{k}}(X).$$

Affine Lie algebras over commutative algebras

Let $X = \{x_i | i \in I\}$ be a linearly ordered set, consder $Lie(X) \subset \mathbf{k}\langle X\rangle$ the free Lie algebra under the Lie bracket $[xy] = xy - yx$. Let

$Y = \{y_j | j \in J\}$ be a linearly ordered set. Then $[Y]$, the free commutative monoid generated by Y, is a linear basis of $\mathbf{k}[Y]$. Regard $Lie_{\mathbf{k}[Y]}(X) \cong \mathbf{k}[Y] \otimes Lie_{\mathbf{k}}(X)$ as the Lie subalgebra of

$\mathbf{k}[Y]\langle X\rangle \cong \mathbf{k}[Y] \otimes \mathbf{k}\langle X\rangle$ the free associative algebra over polynomial algebra $\mathbf{k}[Y]$, which is generated by X under the Lie bracket $[u, v] = uv - vu$. Then $NLSW(X)$ forms a $\mathbf{k}[Y]$-basis of $Lie_{\mathbf{k}[Y]}(X)$.

For any $f \in Lie_{\mathbf{k}[Y]}(X)$,

$$f = \sum \alpha_i \beta_i [u_i],$$

where $\alpha_i \in \mathbf{k}$, $\beta_i \in [Y]$ and $[u_i] \in NSLW(X)$.

Affine Lie algebras over commutative algebras

Order $[Y]$ (X^*) by deg-lex ordering "$\succ_Y$" ("$\succ_X$") respectively. Now define an order "$\succ$" on $[Y]X^*$: For $u, v \in [Y]X^*$,

$$u \succ v \text{ if } (u^X \succ_X v^X) \text{ or } (u^X = v^X \text{ and } u^Y \succ_Y v^Y).$$

Then consider $f = \sum \alpha_i \beta_i [u_i]$ as a polynomial in $\mathbf{k}$-algebra $\mathbf{k}[Y]\langle X\rangle$,

the leading word $\bar{f}$ of f in $\mathbf{k}[Y]\langle X\rangle$ is of the form βu, where $\beta \in [Y],\ u \in ALSW(X)$. Denote $\bar{f}^Y = \beta$ and $\bar{f}^X = u$.

Affine Lie algebras over commutative algebras

Note that for any ALSW w, there is a unique bracketing way such that $[w]$ is a NLSW.

Lemma

(Shirshov 1958) Suppose that $w = aub$, $w, u \in ALSW$. Then

$$[w] = [a[uc]d],$$

where $[uc] \in NLSW(X)$ and $b = cd$.
Represent c in the form $c = c_1c_2 \ldots c_k$, where $c_1, \ldots, c_k \in ALSW(X)$ and $c_1 \leq c_2 \leq \ldots \leq c_k$. Then

$$[w] = [a[u[c_1][c_2] \ldots [c_k]]d].$$

Moreover, the leading term of $[w]_u = [a[\cdots[[[u][c_1]][c_2]] \ldots [c_k]]d]$ is exactly w, i.e.,

$$\overline{[w]_u} = w.$$

Definition

Let $S \subset Lie_{\mathbf{k}[Y]}(X)$ be a monic subset. Define the **normal s-word** $\lfloor u \rfloor_s$*, where $u = asb$ inductively.*

(i) *s is normal of s-length 1;*

(ii) *If $\lfloor u \rfloor_s$ is normal with s-length k and $[v] \in NLSW(X)$ such that $|v| = l$, then $[v]\lfloor u \rfloor_s$ when $v > \overline{\lfloor u \rfloor_s}^X$ and $\lfloor u \rfloor_s[v]$ when $v < \overline{\lfloor u \rfloor_s}^X$ are normal of s-length $k + l$.*

Remarks:

1. $\lfloor u \rfloor_s$ is normal if and only if $\overline{\lfloor u \rfloor_s}^X = a\bar{s}^X b \in ALSW(X)$ for some $a, b \in X^*$;
2. If $a\bar{s}^X b \in ALSW(X)$, then by Shirshov's lemma, we have the special bracketing $[a\bar{s}^X b]_{\bar{s}^X}$ of $a\bar{s}^X b$ relative to $\bar{s}^X$. It follows that $[a\bar{s}^X b]_{\bar{s}^X}|_{[\bar{s}^X] \mapsto s}$ is a **special normal s-word**, which is denoted by $[asb]_{\bar{s}}$.

Affine Lie algebras over commutative algebras

Let f, g be monic polynomials of $Lie_{\mathbf{k}[Y]}(X)$ and $L = lcm(\bar{f}^Y, \bar{g}^Y)$.

Definition

If $\bar{f}^X = a\bar{g}^X b$ *for some* $a, b \in X^*$*, then*

$$C_1\langle f, g\rangle_w = \frac{L}{\bar{f}^Y} f - \frac{L}{\bar{g}^Y}[agb]_{\bar{g}}$$

is called the **inclusion composition** *of* f *and* g *with respect to* w*, where* $w = L\bar{f}^X = La\bar{g}^X b$*.*

Definition

If $\bar{f}^X = aa_0$*,* $\bar{g}^X = a_0 b$*,* $a, b, a_0 \neq 1$*, then*

$$C_2\langle f, g\rangle_w = \frac{L}{\bar{f}^Y}[fb]_{\bar{f}} - \frac{L}{\bar{g}^Y}[ag]_{\bar{g}}$$

is called the **intersection composition** *of* f *and* g *with respect to* w*, where* $w = L\bar{f}^X b = La\bar{g}^X$*.*

Definition

If $gcd(\bar{f}^Y, \bar{g}^Y \neq 1$, then for any $a, b, c \in X^*$ such that $w = La\bar{f}^X b\bar{g}^X c \in T_A$, the polynomial

$$C_3\langle f, g\rangle_w = \frac{L}{\bar{f}^Y}[afb\bar{g}^X c]_{\bar{f}} - \frac{L}{\bar{g}^Y}[a\bar{f}^X bgc]_{\bar{g}}$$

is called the **external composition** *of f and g with respect to w.*

Definition

If $\bar{f}^Y \neq 1$, then for any normal f-word $[afb]_{\bar{f}}$, $a, b \in X^*$, the polynomial

$$C_4\langle f\rangle_w = [a\bar{f}^X b][afb]_{\bar{f}}$$

is called the **multiplication composition** *of f with respect to w, where $w = a\bar{f}^X ba\bar{f}b$.*

Immediately, we have that $\overline{C_i(..)_w} \prec w, \ i \in \{1, 2, 3, 4\}$.

Definition

Given a monic subset $S \subset Lie_{\mathbf{k}[Y]}(X)$ and $w \in [Y]X^*$, an element $h \in Lie_{\mathbf{k}[Y]}(X)$ is called **trivial modulo** (S, w)*, denoted by $h \equiv 0 \ mod(S, w)$, if h can be presented as a $\mathbf{k}[Y]$-linear combination of normal S-words with leading words less than w, i.e., $h = \sum_i \alpha_i\beta_i[a_i s_i b_i]_{\bar{s}_i}$, where $\alpha_i \in \mathbf{k}$, $\beta_i \in [Y]$, $a_i, b_i \in X^*$, $s_i \in S$, and $\beta_i a_i \bar{s}_i b_i \prec w$.*
In general, for $p, q \in Lie_{\mathbf{k}[Y]}(X)$, we write $p \equiv q \ mod(S, w)$ if $p - q \equiv 0 \ mod(S, w)$.
S is a **Gröbner-Shirshov basis** *in $Lie_{\mathbf{k}[Y]}(X)$ if all the possible compositions of elements in S are trivial modulo S and corresponding w.*

Lemma

Let S be a monic subset of $Lie_{\mathbf{k}[Y]}(X)$ in which each multiplication composition is trivial. Then for any normal s-word $\lfloor u \rfloor_s = (asb)$ and $w = a\bar{s}b = \overline{\lfloor u \rfloor_s}$, where $a, b \in X^*$, we have

$$\lfloor u \rfloor_s \equiv [asb]_{\bar{s}} \quad mod(S, w).$$

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From above lemma,we have for such $S \subset Lie_{\mathbf{k}[Y]}(X)$, the elements of the $\mathbf{k}[Y]$-ideal generated by S can be written as a $\mathbf{k}[Y]$-linear combination of special normal S-words.

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Lemma

Let S be a Gröbner-Shirshov basis in $Lie_{\mathbf{k}[Y]}(X)$. For any $s_1, s_2 \in S$, $\beta_1, \beta_2 \in [Y], a_1, a_2, b_1, b_2 \in X^*$ such that $w = \beta_1 a_1 \bar{s}_1 b_1 = \beta_2 a_2 \bar{s}_2 b_2 \in T_A$, we have

$$\beta_1[a_1 s_1 b_1]_{\bar{s}_1} \equiv \beta_2[a_2 s_2 b_2]_{\bar{s}_2} \ mod(S, w).$$

Main Theorem (BCC 2011)

Theorem

Let $S \subset Lie_{\mathbf{k}[Y]}(X)$ be nonempty set of monic polynomials and $Id(S)$ be the ideal of $Lie_{\mathbf{k}[Y]}(X)$ generated by S. Then the following statements are equivalent.

(i) *S is a Gröbner-Shirshov basis in $Lie_{\mathbf{k}[Y]}(X)$.*

(ii) *$f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b \in T_A$ for some $s \in S$ and $a, b \in [Y]X^*$.*

(iii) *$Irr(S) = \{[u] \mid [u] \in T_N,\ u \neq a\bar{s}b,\ \text{for any } s \in S,\ a, b \in [Y]X^*\}$ is a $\mathbf{k}$-basis for $(Lie_{\mathbf{k}[Y]}(X))/Id(S)$.*

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We give applications of Gröbner-Shirshov bases theory for Lie algebras over a commutative algebra K (over a field $\mathbf{k}$) to the Poincaré-Birkhoff-Witt theorem.

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★ P.M. Cohn proved that any Lie algebra over ${}_{\mathbf{k}}K$, where $char(\mathbf{k}) = 0$, is special. Also he gave an example of non-special Lie algebra over a truncated polynomial algebra over a filed of characteristic $p > 0$.

Affine Lie algebras over commutative algebras

Let $K = \mathbf{k}[y_1, y_2, y_3 | y_i^p = 0, i = 1, 2, 3]$ be an algebra of truncated polynomials over a field k of characteristic $p > 0$. Let

$$\mathcal{L}_p = Lie_K(x_1, x_2, x_3 \mid y_3 x_3 = y_2 x_2 + y_1 x_1).$$

Then is $\mathcal{L}_p$ non-special?

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Note that in $U(\mathcal{L}_p)$ we have

$$0 = (y_3x_3)^p = (y_2x_2)^p + \Lambda_p(y_2x_2, y_1x_1) + (y_1x_1)^p = \Lambda_p(y_2x_2, y_1x_1),$$

where Λ_p is a Jacobson-Zassenhaus Lie polynomial.
Cohn conjectured that $\Lambda_p(y_2x_2, y_1x_1) \neq 0$ in $\mathcal{L}_p$.

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Theorem

(BCC 2011) $\Lambda_p(y_2x_2, y_1x_1) \neq 0$ in $\mathcal{L}_p$ and $\mathcal{L}_p$ is non-special when $p = 2, 3, 5$. Moreover, Λ_p for $\mathcal{L}_p$ can be computed by an algorithm for any p.

Consider $p = 3$.

$$\begin{aligned}
&S_{X^3} = \{s \in S^C | \deg(\bar{s}^X) \leq 3\} \\
&= \{y_3x_3 = y_2x_2 + y_1x_1,\ y_i^3x_j = 0,\ y_3^2y_2x_2 = y_3^2y_1x_1,\ y_3^2y_2^2y_1x_1 = 0, \\
&y_2[x_3x_2] = -y_1[x_3x_1],\ y_3^2y_1[x_2x_1] = 0,\ y_2^2y_1[x_3x_1] = 0, \\
&y_3y_2^2[x_2x_2x_1] = y_3y_2y_1[x_2x_1x_1],\ y_3y_2^2y_1[x_2x_1x_1] = 0, \\
&y_3y_2y_1[x_2x_2x_1] = y_3y_1^2[x_2x_1x_1].\}
\end{aligned}$$

Thus, $y_2^2y_1[x_2x_2x_1], y_2y_1^2[x_2x_1x_1] \in Irr(S^C)$, which implies $\Lambda_3 = y_2^2y_1[x_2x_2x_1] + y_2y_1^2[x_2x_1x_1] \neq 0$ in $\mathcal{L}_3$.

Affine Lie algebras over commutative algebras

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It gives an extension of the list of known "special" Lie algebras (ones with valid PBW Theorems) (see P.-P. Grivel (2004)):

1. $\mathcal{L}$ is a free K-module (G. Birkhoff (1937), E. Witt(1937)),
2. K is a principal ideal domain (M. Lazard (1952)),
3. K is a Dedekind domain (P. Cartier (1958)),
4. K is over a field $\mathbf{k}$ of characteristic 0 (P.M. Cohn (1963)),
5. $\mathcal{L}$ is K-module without torsion (P.M. Cohn (1963)),
6. 2 is invertible in K and for any $x, y, z \in \mathcal{L}$, $[x[yz]] = 0$ (Y. Nouaze, P. Revoy (1971)).

Affine Lie algebras over commutative algebras

Theorem

For an arbitrary commutative $\mathbf{k}$*-algebra* $K = \mathbf{k}[Y|R]$*, if* S *is a Gröbner-Shirshov basis in* $Lie_{\mathbf{k}[Y]}(X)$ *such that for any* $s \in S$*,* s *is* $\mathbf{k}[Y]$*-monic, then* $\mathcal{L} = Lie_K(X|S)$ *is special.*

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Corollary

Any Lie K*-algebra* $L_K = Lie_K(X|f)$ *with one monic defining relation* $f = 0$ *is special.*

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Corollary

If L_K *is a free* K*-module, then* L_K *is special.*

Non-commutative Gröbner-Shirshov bases for commutative algebras

★ Mikhalev and Zolotykh (1998) established the GSB theory and proved the CD-lemma for associative algebra over K where K is a commutative associative k-algebra. The free object they considered is $k[X] \otimes k\langle Y \rangle$.

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Question: What is the relation between these two Gröbner-Shirshov bases.

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First, let us consider the relation between Gröbner-Shirshov bases in $k\langle X\rangle$ and $k[X]$.

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(Eisenbud, Peeva, Sturmfels, 1998) Let k be an infinite field and $I \subset k[X]$ be an ideal. Then after a general linear change of variables, the ideal $\gamma^{-1}(I)$ in $k\langle X\rangle$ has a finite Gröbner-Shirshov basis.

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Idea of the proof: Let R be a minimal GSB for I in $k[X]$. Then R together with all the commutators on X generates $\gamma^{-1}(I)$ in $k\langle X\rangle$, but it is not a GSB in general.
Construct a set $EPS(R)$ such that $EPS(R)$ together with all the commutators is a GSB for $\gamma^{-1}(I)$ in $k\langle X\rangle$. But the set is not finite in general.
Lemma: If k is infinite, then there exists a generic change of variables such that $|EPS(R)| < \infty$.

Non-commutative Gröbner-Shirshov bases for commutative algebras

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Non-commutative Gröbner-Shirshov bases for commutative algebras

Let $\gamma \otimes \mathbf{1}: \ k\langle X\rangle \otimes k\langle Y\rangle \rightarrow k[X] \otimes k\langle Y\rangle, \ u^X u^Y \mapsto \gamma(u^X)u^Y$.
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Theorem

(BCC, 2010) Let R be a minimal GSB for $I \lhd k[X] \otimes k\langle Y\rangle$. Then $EPS(R)$ together with all the commutators on X is a GSB for $\gamma^{-1}(I)$ in $k\langle X\rangle \otimes k\langle Y\rangle$.

Gröbner-Shirshov bases for metabelian Lie algebras

From now on, all algebras will be considered over a field $\mathbf{k}$ of arbitrary characteristic. Suppose that $\mathcal{L}$ is a Lie algebra. Then $\mathcal{L}$ is called a metabelian Lie algebra if $\mathcal{L}^{(2)} = 0$, where $\mathcal{L}^{(0)} = \mathcal{L}$, $\mathcal{L}^{(n+1)} = [\mathcal{L}^{(n)}, \mathcal{L}^{(n)}]$. More precisely, the variety of metabelian Lie algebras is given by the identity

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We begin with the construction of a free metabelian Lie algebra. Let $Lie(X)$ the free Lie algebra generated by X. Then $\mathcal{L}_{(2)}(X) = Lie(X)/Lie(X)^{(2)}$ is the free metabelian Lie algebra generated by X. Then any metabelian Lie algebra $\mathcal{ML}$ can be presented by

$$\mathcal{ML} = \mathcal{L}_{(2)}(X \mid S).$$

★ **Linear basis of free metabelian Lie algebras:**

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A non-associative monomial on X is *left-normed* if it is of the form $(\cdots((ab)c)\cdots)d$.

Let X be well-ordered. For an arbitrary set of indices $j_1, j_2, \cdots, j_m$, define an associative word $\langle a_{j_1}\cdots a_{j_m}\rangle = a_{i_1}\cdots a_{i_m}$, where $a_{i_1} \leq \cdots \leq a_{i_m}$ and $i_1, i_2, \cdots, i_m$ is a permutation of the indices $j_1, j_2, \cdots, j_m$.

Let

$$R = \{u = a_0a_1a_2\cdots a_n \mid u \text{ is left-normed}, a_i \in X,\ a_0 > a_1 \leq \cdots \leq a_n, n \geq 1\}$$

and $N = X \cup R$. Then N forms a linear basis of the free metabelian Lie algebra $\mathcal{L}_{(2)}(X)$ (Bokut, 1963).

For any $f \in \mathcal{L}_{(2)}(X)$, f has a unique presentation $f = f^{(1)} + f^{(0)}$, where $f^{(1)} \in \mathbf{k}R$ and $f^{(0)} \in \mathbf{k}X$.

Moreover, the multiplication table of N is the following, $u \cdot v = 0$ if both $u, v \in R$, and

$$a_0 a_1 a_2 \cdots a_n \cdot b = \begin{cases} a_0 \langle a_1 a_2 \cdots a_n b \rangle & \text{if } a_1 \leq b, \\ a_0 b a_1 a_2 \cdots a_n - a_1 b \langle a_0 a_2 \cdots a_n \rangle & \text{if } a_1 > b. \end{cases}$$

Now we order the set N degree-lexicographically.

★ **Elements of an ideal:**

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Let $S \subset \mathcal{L}_{(2)}(X)$. Then the following two kinds of polynomials are called **normal** S**-words**:

(i) $sa_1a_2\cdots a_n$, where $a_i \in X$ $(1 \leq i \leq n)$, $a_1 \leq a_2 \leq \cdots \leq a_n$, $s \in S$, $\bar{s} \neq a_1$ and $n \geq 0$;

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Simple observation:

$$\overline{sa_1a_2\cdots a_n} = \begin{cases} c_0\langle c_1\cdots c_k a_1a_2\cdots a_n\rangle & \text{if } \bar{s} = c_0c_1\cdots c_k, \\ c_0a_1a_2\cdots a_n & \text{if } \bar{s} = c_0 > a_1, \\ a_1c_0a_2\cdots a_n & \text{if } \bar{s} = c_0 < a_1, \end{cases}$$

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That is to say, if u_s is normal, then $\overline{u_s}$ is a S-**irreducible**(which contains neither $\bar{s}$ as a subword nor $\overline{s^{(0)}}$ as a strict subword).

★ **Compositions:**

Gröbner-Shirshov bases for metabelian Lie algebras

★ **Compositions:**

Let f and g be momic polynomials of $\mathcal{L}_{(2)}(X)$ and α and β are the coefficients of $\overline{f^{(0)}}$ and $\overline{g^{(0)}}$ respectively. We define seven different types of compositions as follow:

1. If $\bar{f} = a_0 a_1 \cdots a_n$, $\bar{g} = a_0 b_1 \cdots b_m$, $(n, m \geq 0)$ and $lcm(AB) \neq \langle a_1 \cdots a_n b_1 \cdots b_m \rangle$, where $lcm(AB)$ denotes the least common multiple in $[X]$ of associative words $a_1 \cdots a_n$ and $b_1 \cdots b_m$, then let $w = a_0 \langle lcm(AB) \rangle$. The composition of type I of f and g relative to w is defined by
$$C_I(f, g)_w = f \langle \frac{lcm(AB)}{a_1 \cdots a_n} \rangle - g \langle \frac{lcm(AB)}{b_1 \cdots b_m} \rangle.$$
2. If $\bar{f} = \overline{f^{(1)}} = a_0 a_1 \cdots a_n$, $\overline{g^{(0)}} = a_i$ for some $i \geq 2$ or $\overline{g^{(0)}} = a_1$ and $a_0 > a_2$, then let $w = \bar{f}$ and the composition of type II of f and g relative to w is defined by

3. If $\bar{f} = \overline{f^{(1)}} = a_0 a_1 \cdots a_n$, $\bar{g} = \overline{g^{(0)}} = a_1$ and $a_0 \leq a_2$ or $n = 1$, then let $w = \bar{f}$ and the composition of type III of f and g relative to w is defined by

$$C_{III}(f, g)_{\bar{f}} = f + g a_0 a_2 \cdots a_n.$$

4. If $\bar{f} = \overline{f^{(1)}} = a_0 a_1 \cdots a_n$, $g^{(1)} \neq 0$, $\overline{g^{(0)}} = a_1$ and $a_0 \leq a_2$ or $n = 1$, then for any $a < a_0$ and $w = a_0 \langle a_1 \cdots a_n a \rangle$, the composition of type IV of f and g relative to w is defined by

$$C_{IV}(f, g)_w = fa - \beta^{-1} a_0 a a_2 \cdots a_n \cdot g.$$

5. If $\bar{f} = \overline{f^{(1)}} = a_0 a_1 \cdots a_n$, $f^{(0)} \neq 0$, $g^{(1)} \neq 0$ and $\overline{g^{(0)}} = b \notin \{a_i\}_{i=1}^{n}$, then let $w = a_0 \langle a_1 \cdots a_n b \rangle$ and the composition of type V of f and g relative to w is defined by

$$C_V(f, g)_w = fb - \beta^{-1} a_0 a_1 \cdots a_n \cdot g.$$

6. If $\overline{f^{(0)}} = \overline{g^{(0)}} = a$ and $f^{(1)} \neq 0$, then for any $a_0 a_1 \in R$ and $w = a_0\langle a_1 a\rangle$, the composition of type VI of f and g relative to w is defined by

$$C_{VI}(f,g)_w = (a_0 a_1)(\alpha^{-1} f - \beta^{-1} g).$$

7. If $f^{(1)} \neq 0$, $g^{(1)} \neq 0$ and $\overline{f^{(0)}} = a > \overline{g^{(0)}} = b$, then for any $a_0 > a$ and $w = a_0 ba$, the composition of type VII of f and g relative to w is defined by

$$C_{VII}(f,g)_w = \alpha^{-1}(a_0 b) f - \beta^{-1}(a_0 a) g.$$

Theorem

(Composition-Diamond lemma for metabelian Lie algebras) *Let $S \subset \mathcal{L}_{(2)}(X)$ be a nonempty set of monic polynomials and $Id(S)$ be the ideal of $\mathcal{L}_{(2)}(X)$ generated by S. Then the following statements are equivalent.*

(i) *S is a Gröbner-Shirshov basis.*

(ii) *$f \in Id(S) \Rightarrow \bar{f} = \overline{u_s}$ for some normal S-word u_s.*

(iii) *$Irr(S) = \{u \mid u \in N, u \neq \overline{v_s}$ for any normal S-word $v_s\}$ is a* **k***-basis for $\mathcal{L}_{(2)}(X|S) = \mathcal{L}_{(2)}(X)/Id(S)$.*

Gröbner-Shirshov bases for metabelian Lie algebras

Let $\mathcal{A}$ be a metabelian Lie algebra with $Y = \{a_i, i \in I\} \cup \{b_j, j \in J\}$ as a **k**-basis, where $\{a_i\}$ is a basis of $\mathcal{A}^{(1)}$ and b_j's are linear independent modulo $\mathcal{A}^{(1)}$. And

$$m_{1ij}: \ a_i b_j - \sum \gamma_{ij}^k a_k,$$
$$m_{2ij}: \ b_i b_j - \sum \delta_{ij}^k a_k, \ (i > j),$$
$$m_{3ij}: \ a_i a_j, \ (i > j).$$

Let $\mathcal{S} = \mathcal{A} * \mathcal{L}_{(2)}(X) = \mathcal{L}_{(2)}(X \cup Y | M)$.

Theorem

(CC 2011) Let the notion be as above. Then with respect to $x_h > a_i > b_j$, a Gröbner-Shirshov complement M^C of M in $\mathcal{L}_{(2)}(X \cup Y)$ consists of M and some X-homogenous polynomials without (0)-part, whose leading words are of the form $xy\cdots$ with an a_i as a strict subword, $x \in X$, $y \in Y$. We say such polynomials satisfy property P_X.

Gröbner-Shirshov bases for metabelian Lie algebras

A metabelian Lie algebra is **partial commutative** related to a graph $\Gamma = (V, E)$, if $\mathcal{ML}_\Gamma = \mathcal{L}_{(2)}(V| \, [o(e), t(e)] = 0, e \in E)$.
The following algorithm gives a Gröbner-Shirshov basis for partial commutative metabelian Lie algebras with a finite relation set.

Algorithm

Input: *relations* $f_1, \cdots, f_s$ *of* $\mathcal{L}_{(2)}(X)$, $f_i = xx'$, $F = \{f_1, \cdots, f_s\}$.

Output: *a Gröbner-Shirshov basis* $H = \{h_1, \cdots, h_t\}$ *for* $\mathcal{L}_{(2)}(X|F)$.

Initialization: $H := F$

While: $f_i = x_{i_0}x_{i_1}\cdots x_{i_n}, \; f_i = x_{j_0}x_{j_1}\cdots x_{j_m}$, *and* $x_{i_0} = x_{j_0}, \, x_{i_1} \neq x_{j_1}$

Then Do: $h := max\{x_{i_1}, x_{j_1}\}min\{x_{i_1}, x_{j_1}\}\langle x_{t_1}x_{t_2}\cdots x_{t_l}\rangle$

where $\{x_{t_1}, x_{t_2}, \cdots, x_{t_l}\} = \{x_{i_0}, x_{i_2}, \cdots, x_{i_n}\} \cup \{x_{j_2}, \cdots, x_{j_m}\}$

If: *there is no* $f_j \in H$ *such that* f_j *is a subword of* h

Do: $H := H \cup \{h\}$

End

Gröbner-Shirshov bases for metabelian Lie algebras

By using the above algorithm, we find Gröbner-Shirshov bases for partial commutative metabelian Lie algebras related to any circuits, trees and cubes. For example, we have a Gröbner-Shirshov basis S for the partial commutative metabelian Lie algebra related to 3-cube

$$\mathcal{ML}_{Cu_3} = \mathcal{L}_{(2)}(V_3|\varepsilon\delta,\ d(\varepsilon,\delta)=1, \varepsilon>\delta)$$

is the union of the following:

$$R_2 = \{\lfloor\varepsilon\delta\rfloor \mid d(\varepsilon,\delta)=1\},$$
$$R_3 = \{\lfloor\varepsilon\delta\rfloor\mu \mid d(\varepsilon,\delta)=2,\ \mu\varepsilon,\mu\delta \in R_1\},$$
$$R_4 = \{\lfloor\varepsilon\delta\rfloor\mu\gamma \mid d(\varepsilon,\delta)=3,\ \mu\varepsilon \in R_2, \mu\delta\gamma \in R_3\},$$
$$R_5 = \{\lfloor\delta_1\delta_2\rfloor\gamma\langle\mu_1\mu_2\rangle \mid d(\delta_1,\delta_2)=2,\ \gamma\delta_i\mu_i \in R_3, i=1,2\},$$
$$R_5' = \{\lfloor\delta_1\delta_2\rfloor\gamma\mu\mu' \mid d(\delta_1,\delta_2)=2,\ \gamma\delta_1 \in R_2, \gamma_2\mu\mu' \in R_4, d(\mu,\delta_1)\neq 1\},$$

where $\lfloor\varepsilon\delta\rfloor = max\{\varepsilon,\delta\}min\{\varepsilon,\delta\}$.
Also, we have that a reduced Gröbner-Shirshov basis (it means there is no composition of type I, II, III) for the partial commutative metabelian Lie algebra related to 4-cube consists of 268 relations.

Thank You!