The Lagrangian Coordinates Applied To First Order Traffic Flow Models

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Theory, Numerical resolution, Applications

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Outline

- The LWR Model in Lagrangian Coordinates
  - As a conservation law
  - As a variational problem
- Numerical Resolution in Lagrangian Coordinates
- Conclusion

The LWR Model in Lagrangian coordinates

The LWR model in (x,t) coordinates as a conservation law

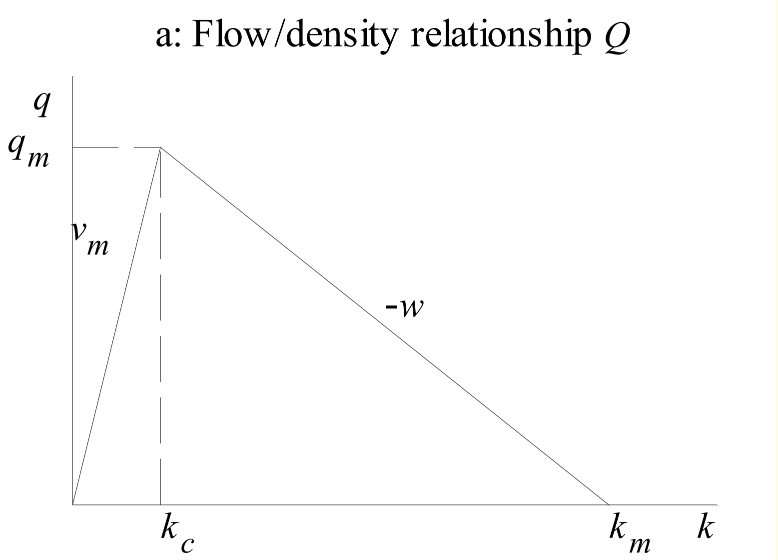
Variables: k, density; v, speed; $q=kv$, flow

Conservation equation:

$$\partial_t k + \partial_x kv = 0$$

Fundamental diagram (FD):

$$v = V(k) \quad \text{or} \quad q = kV(k) = Q(k)$$



The model can be synthesized as a scalar hyperbolic equation:

$$\partial_t k + \partial_x Q(k) = 0 \quad (1)$$

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The cumulative count function $N(x,t)$

- $N(x,t)$ represents the cumulative number of vehicles that cross location x by time t
- $k=-\partial_x N$ and $q=\partial_t N$
- The conservation equation reduces to: $\partial_{x,t}N=\partial_{t,x}N$ (existence of N)
- The LWR model can then be expressed as:

$$\partial_t N = Q(-\partial_x N)$$

The LWR Model in (*N*,*t*) coordinates as a conservation law

Variables: $s=1/k$, spacing; $N(x,t)$, cumulative count function

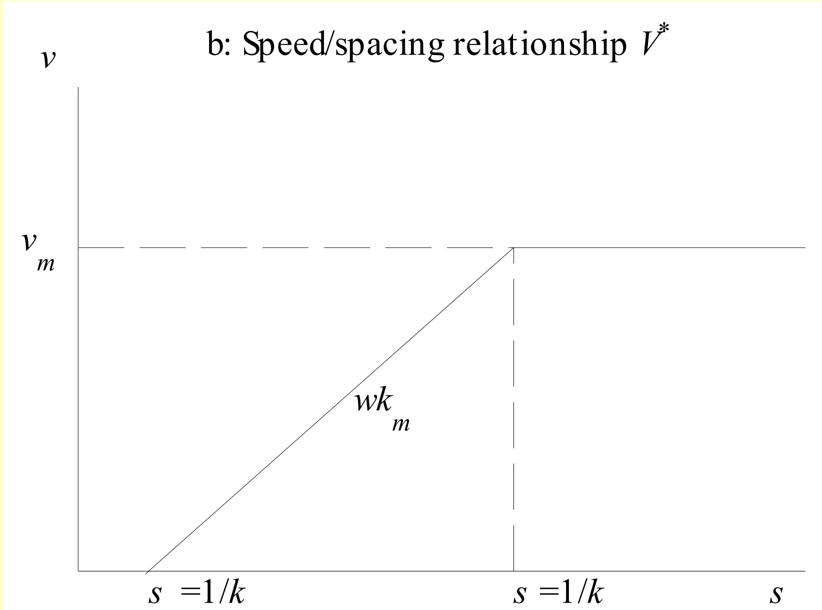
The conservation equation becomes : $\partial_t s + \partial_N v = 0$

FD can be expressed as :

$v = V(1/s) = V^*(s)$

The model reduces to a scalar hyperbolic equation:

$\partial_t s + \partial_x V^*(s) = 0$ (2)



The weak solutions of (1) and (2) are equivalent even in vacuum cases (Wagner, 1987)

The LWR model in (*x*,*t*) coordinates as a variational principle [Daganzo, 2005]

$\partial_t N = Q(-\partial_x N)$ is a Hamilton-Jacobi equation

The model solution in N satisfies a least-cost path problem:

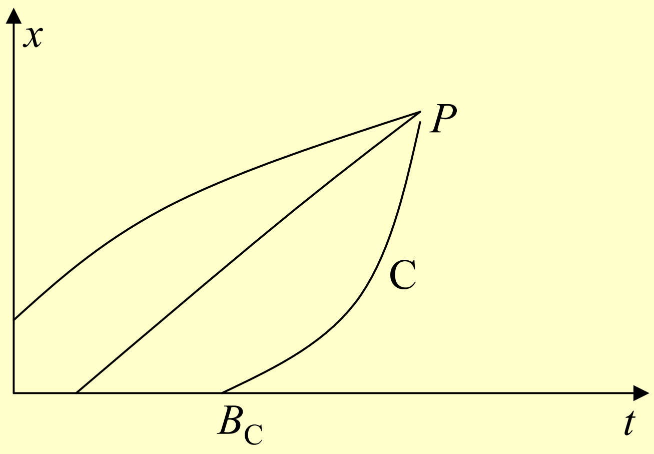
$N_P = \min\left(B_C + \Delta(C) : \forall C \in V \cap S_P\right)$, where

V: set of all valid paths

S_P: set of all path from the boundary condition to P

B_C: N value at the beginning of the path C

$\Delta(C)$: cost of path C



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The LWR model in (*x*,*t*) coordinates as a variational principal - Definitions

- Valid paths are continuous and piecewise differentiable path $x(t)$ whose slopes are bounded by the extremum wave speeds
- Wave paths are valid paths whose slopes are possible wave speeds. A wave path is composed of a succession of waves
- The cost rate on a wave path is *r*

$$r = \mathrm{d}_t N = \partial_t N + \partial_x N \partial_t x = q - kv$$

- The cost rate on a valid path is defined as (maximum passing rate):

$$r = R(u) = \max_k \left(Q(k) - ku \right)$$

- The cost of a valid path from *B* to *P* is $\Delta_{B \to P} = \int_{t_B}^{t_P} R\left(x'(t)\right) dt$

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The LWR model in (N,t) coordinates as a variational principle

- N is bijective except in vaccum case (k=0,s=∞)
- Let $X(n,t)$ be the inverse of $N(x,t)$ (X is obtained by solving for x in $n=N(x,t)$)
- X verifies: $\partial_t X = V^*(-\partial_N X)$ (Hamilton-Jacobi equation)
- The model solution in X also satisfies a least-cost path problem

$$X_P = \min\left(B_C + \Delta(C) : \forall C \in \mathsf{V} \cap \mathsf{S}_P\right), \text{ where}$$

V: set of all valid paths

S_P: set of all path from the boundary condition to P

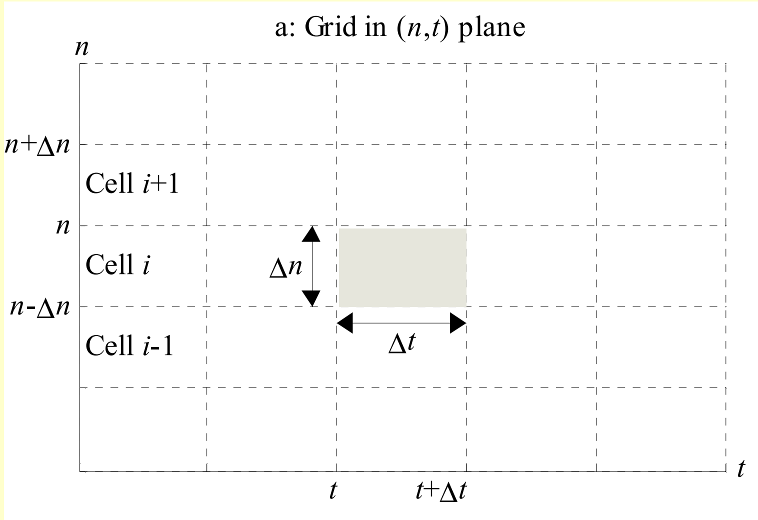
B_C: X value at the beginning of the path C

$\Delta(C)$: cost of path C

Valid path slopes represent passing rates and cost rates represent wave speeds

Numerical resolution

The Godunov scheme in *s*



Godunov scheme in *s* reduces to the upwind method as the flux function is increasing:

$$s_i^{t+\Delta t} = s_i^t + \frac{\Delta t}{\Delta n}\left(V^*\left(s_i^t\right) - V^*\left(s_{i-1}^t\right)\right)$$

CFL condition: $\Delta n \geq \max_s \left|\partial_s\left(V^*(s)\right)\right| \Delta t$

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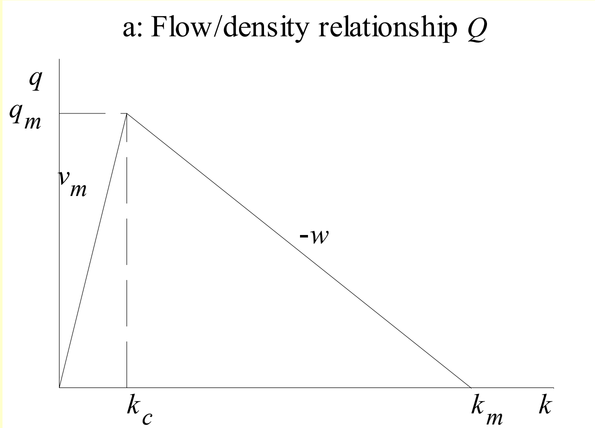
Godunov scheme in X

$$\frac{X(n,t+\Delta t)-X(n,t)}{\Delta t}=V^*\left(-\frac{X(n,t)-X(n-\Delta n,t)}{\Delta n}\right)$$



We suppose that Q is triangular

$$\Rightarrow\quad X(n,t+\Delta t)=\min\begin{pmatrix}X(n,t)+v_m\Delta t,\\(1-\alpha)X(n,t)+\alpha X(n-\Delta n,t)-w\Delta t\end{pmatrix}$$

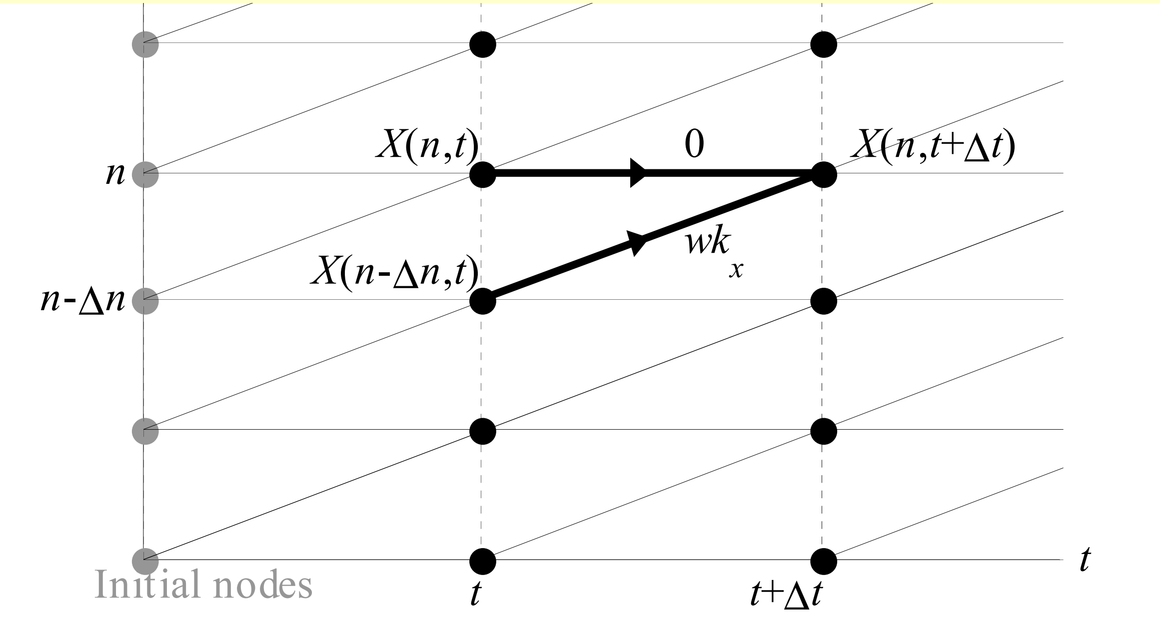
with $\alpha = wk_m\Delta t/\Delta n$

if α=1 (CFL condition satisfied as an equality)

$$\Rightarrow\quad X(n,t+\Delta t)=\min\left(X(n,t)+v_m\Delta t, X(n-\Delta n,t)-w\Delta t\right)$$

Variational scheme

We suppose that Q is triangular



Sufficient networks

$$X\left(n,t+\Delta t\right)=\min\left(X\left(n,t\right)+v_m\Delta t, X\left(n-\Delta n,t\right)-w\Delta t\right)$$

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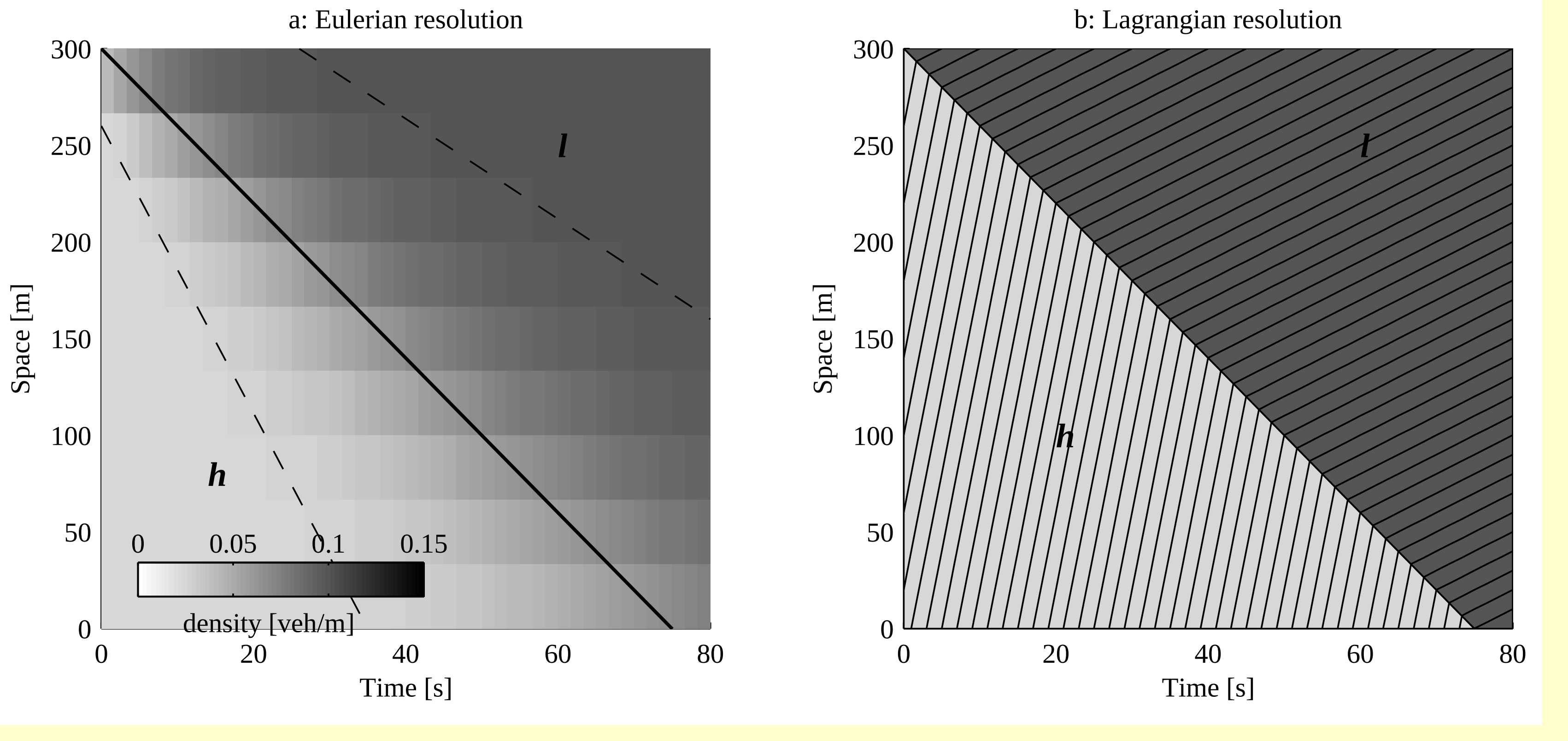
Remarks

- The variational scheme is exact provided that the initial data is linear between two consecutive initial nodes
- When Q is triangular, the Godunov and the variational schemes are equivalent

Numerical examples



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Conclusion

	Lagrangian approach		Eulerian approach	
	Godunov scheme	Variational method	Godunov scheme	Variational method
Q triangular + rectangular lattice	V	V	X	V only if v_m/w is an integer and memory is used
Q triangular	V	V	X	V only for geometric network
Q Piecewise linear	X	V	X	X

Exactitude by approach (V:exact; ×: non exact)