Hyperbolicity singularities in rarefaction waves

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Outline

Structure of rarefaction waves near elliptic boundary

- regular points of elliptic boundary
- exceptional points of elliptic boundary
- classification and analysis of singularities near exceptional points

Exceptional points in Riemann problem for 2 conservation laws. Novel types of Riemann solutions:

- generic 1-wave solutions
- extreme non-uniqueness: infinite number of unstable solutions for the same initial conditions

Example

Rarefaction waves in systems of conservation laws

n equations in one space dimension x

state vector $U \in R^n$, flux function F

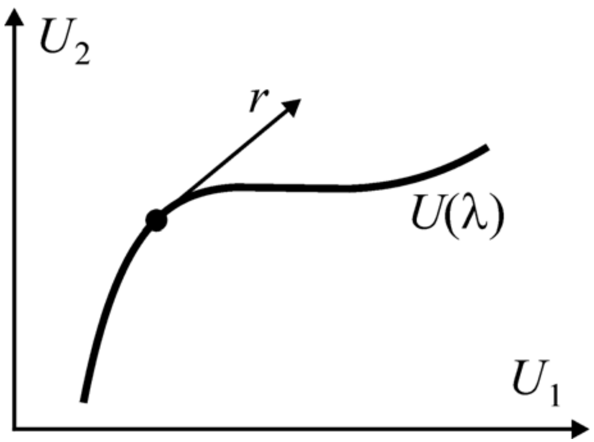
$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0$$

Simple wave solution $$U(x,t) = \widetilde{U}(\lambda), \ \ \lambda = x/t$$

Equation for $U(\lambda)$ $$A(U)\frac{\partial U}{\partial \lambda} = \lambda \frac{\partial U}{\partial \lambda}, \ \ A(U) = \frac{\partial F}{\partial U}$$

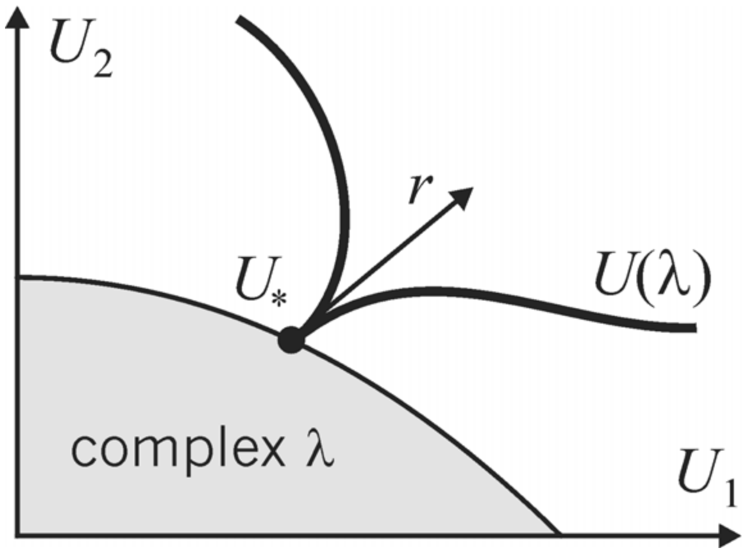
Eigenvalue problem $$A(U)r = \lambda r, \ \ dU \,||\, r$$

Real eigenvalues
(characteristic speeds) $$\lambda_1 < \lambda_2 < \ldots < \lambda_n$$

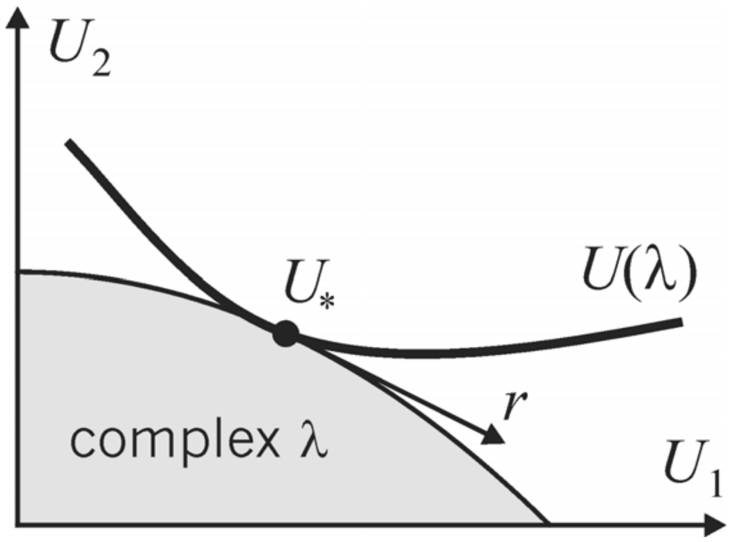


Elliptic boundary

Boundary between elliptic region (complex λ) and hyperbolic region (real λ) is given by double (in general, multiple) real eigenvalues $\lambda_1 = \lambda_2$ with a 2×2 Jordan block

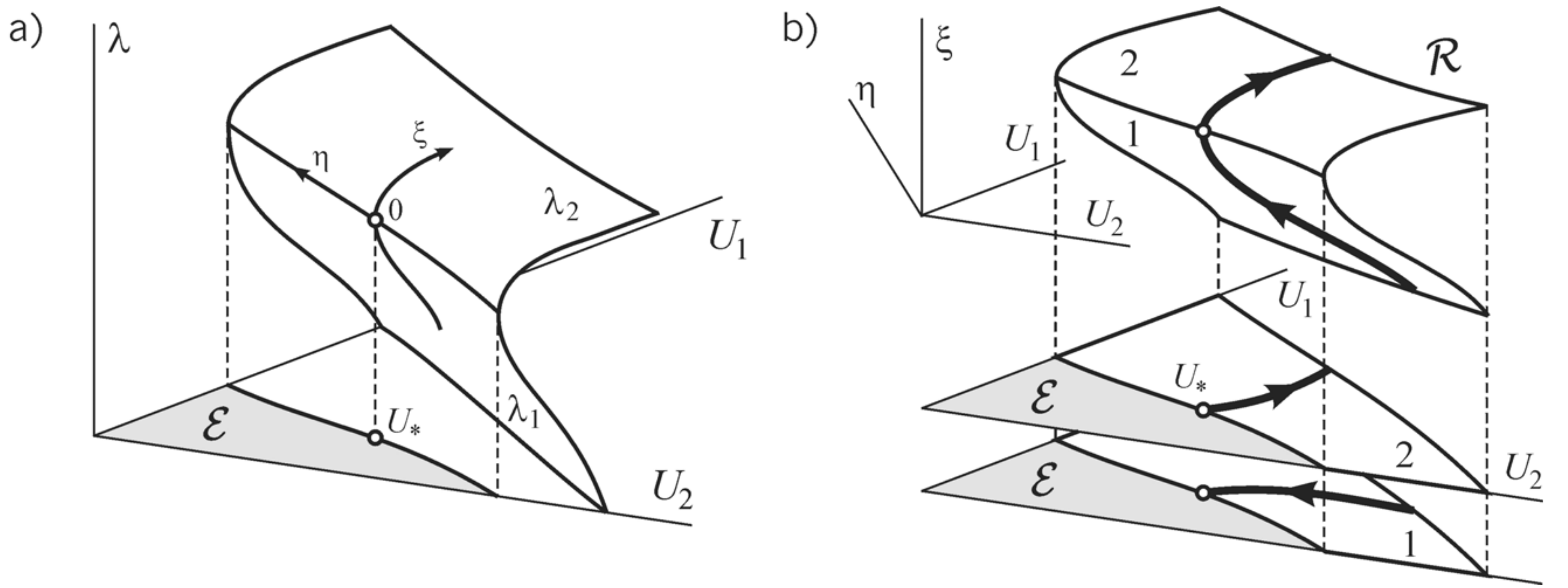


Regular point: eigenvector r is transversal to elliptic boundary



Exceptional point: eigenvector r is tangent to elliptic boundary

Fold structure at regular points



Surface smooth parametrization

$$\lambda = \lambda_0 + \xi + \eta,\ \ r = R_0(U) + \xi R_1(U)$$

$$\xi^2 = p(U),\quad \eta = s(U)$$

$$s(U) = \frac{\lambda_2(U) + \lambda_1(U)}{2} - \lambda_0,\ \ p(U) = \frac{(\lambda_2(U) - \lambda_1(U))^2}{4},\ \ \lambda_0 = \lambda(U_*)$$

Rarefaction wave structure near regular points

Jordan block structure at U_*

$$A_0 r_0 = \lambda_0 r_0, \quad A_0 r_1 = \lambda_0 r_1 + r_0,$$

$$l_0^T A_0 = \lambda_0 l_0^T, \quad l_1^T A_0 = \lambda_0 l_1^T + l_0^T, \quad l_0^T r_1 = 1, \quad l_1^T r_1 = 0$$

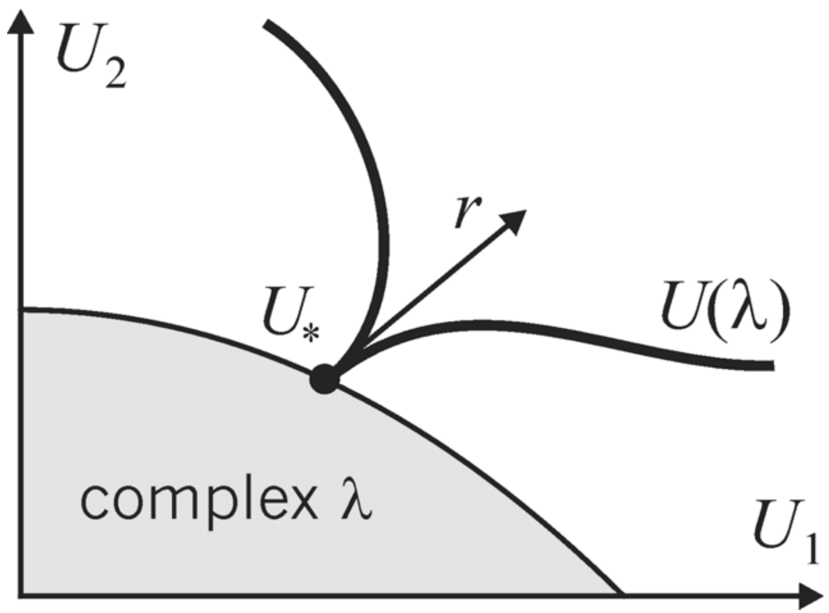
Regularity condition

$$\partial_0 p = l_0^T \partial_0^2 F \neq 0,$$

$$\partial_{0,1} = r_{0,1} \cdot \partial / \partial U$$

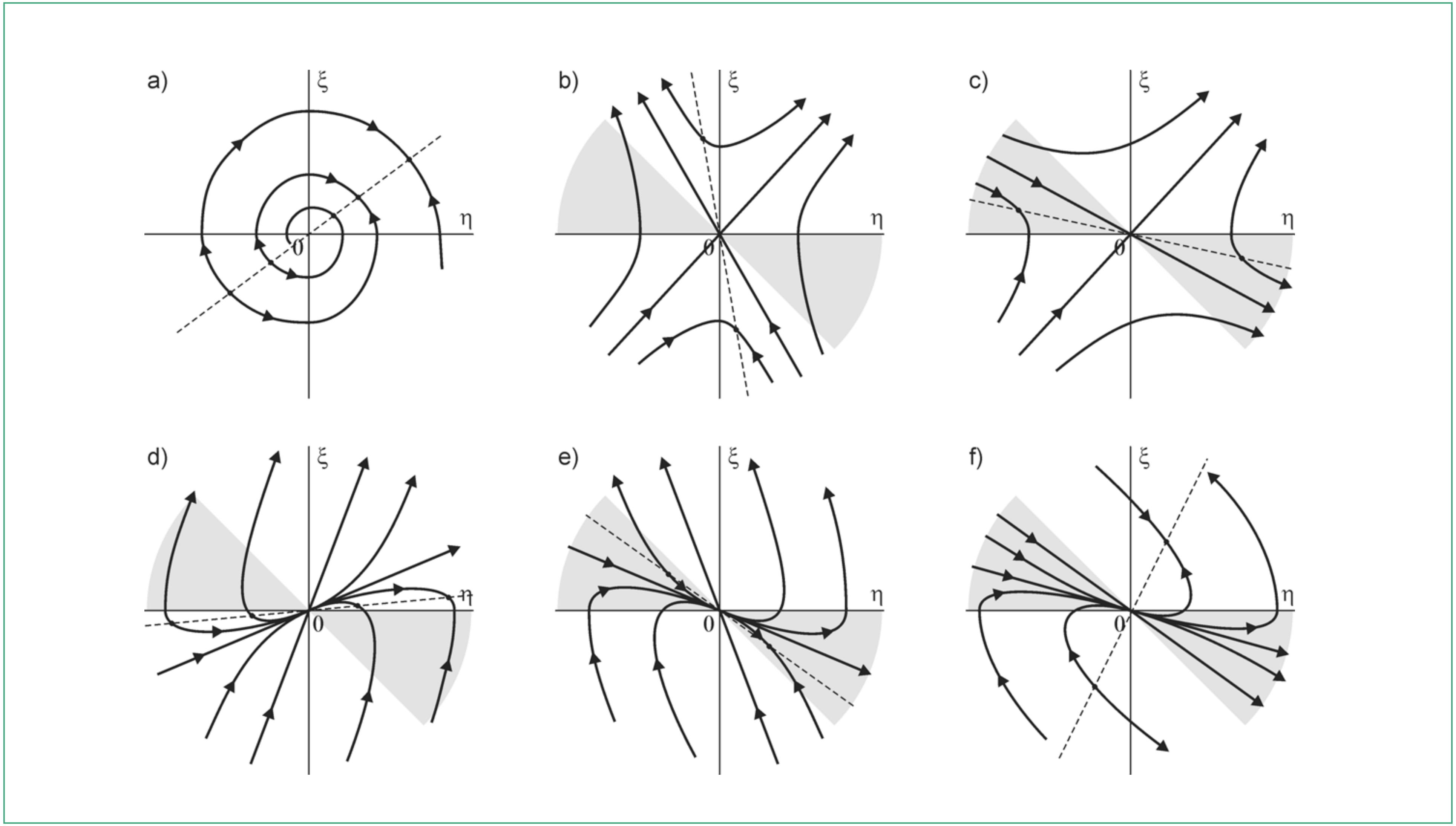
Rarefaction wave

$$U(\lambda) = U_* + \frac{(\lambda - \lambda_0)^2}{\partial_0 p} r_0 + o(|\lambda - \lambda_0|^2)$$

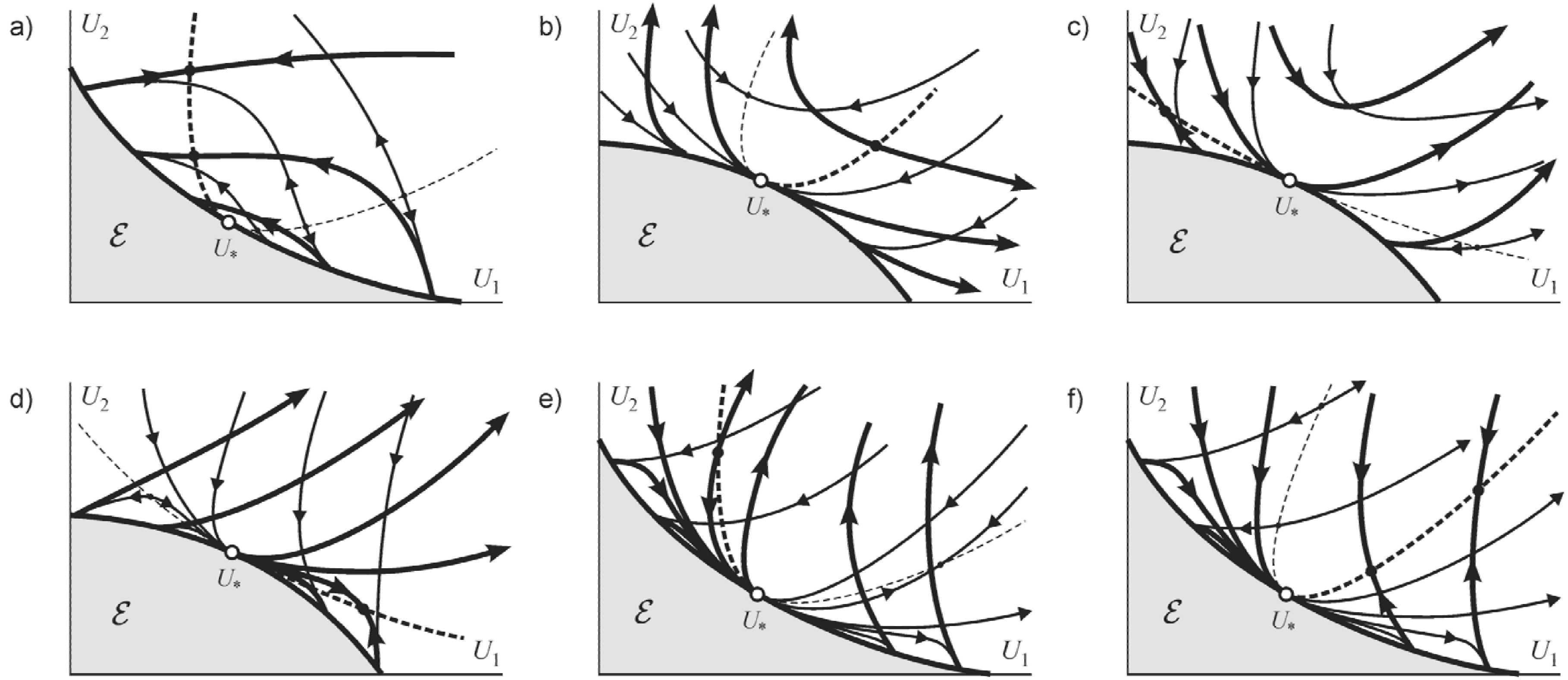


Method: analysis is carried out in the extended (ξ, η, U)-space

Rarefaction wave structure near exceptional points: singularities in fold coordinates (ξ,η)



Rarefaction wave structure near exceptional points: singularities in state space



Quantitative description of singularities

Eigenvalues characterizing singularity type

$$\rho_{\pm} = \frac{1}{2}\left(l_0^T \partial_1 \partial_0 F \pm \sqrt{D}\right), \quad D = \left(3 l_0^T \partial_1 \partial_0 F - 2 l_1^T \partial_0^2 F\right)^2 + 8 l_0^T \partial_0^3 F$$

Rarefaction wave curves passing through the exceptional point

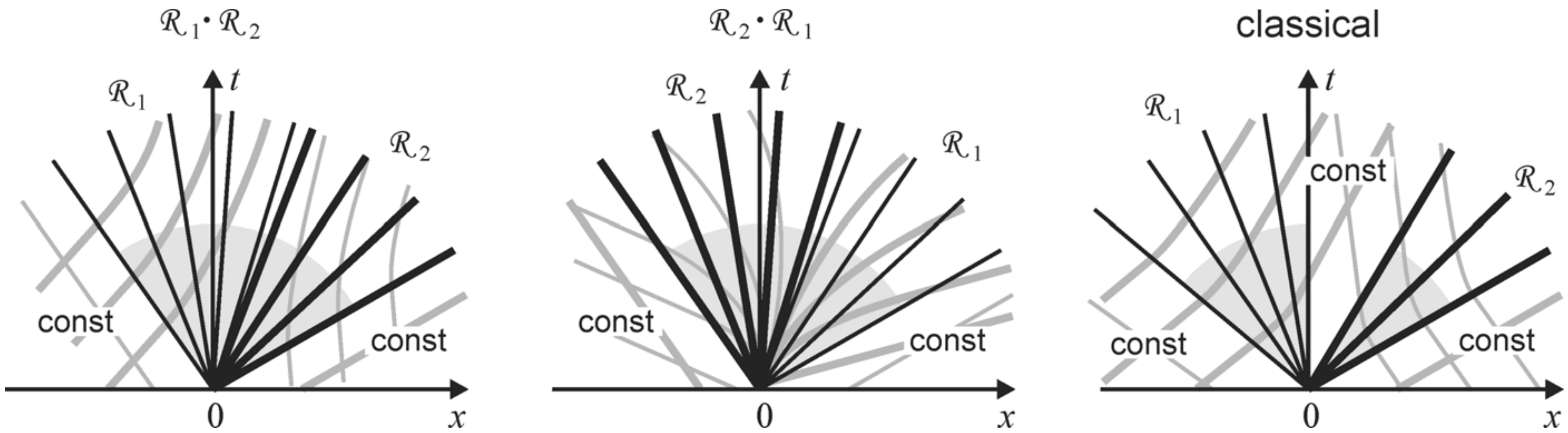
$$U(\lambda) = U_* + \frac{2(\lambda - \lambda_0)}{\rho_{\pm} + 2\partial_0 s} r_0 + o(\lambda - \lambda_0),$$

$$2\partial_0 s = l_0^T \partial_1 \partial_0 F + l_1^T \partial_0^2 F$$

Equation for inflection locus (points of rarefaction curves, where characteristic speed λ attains maximum or minimum)

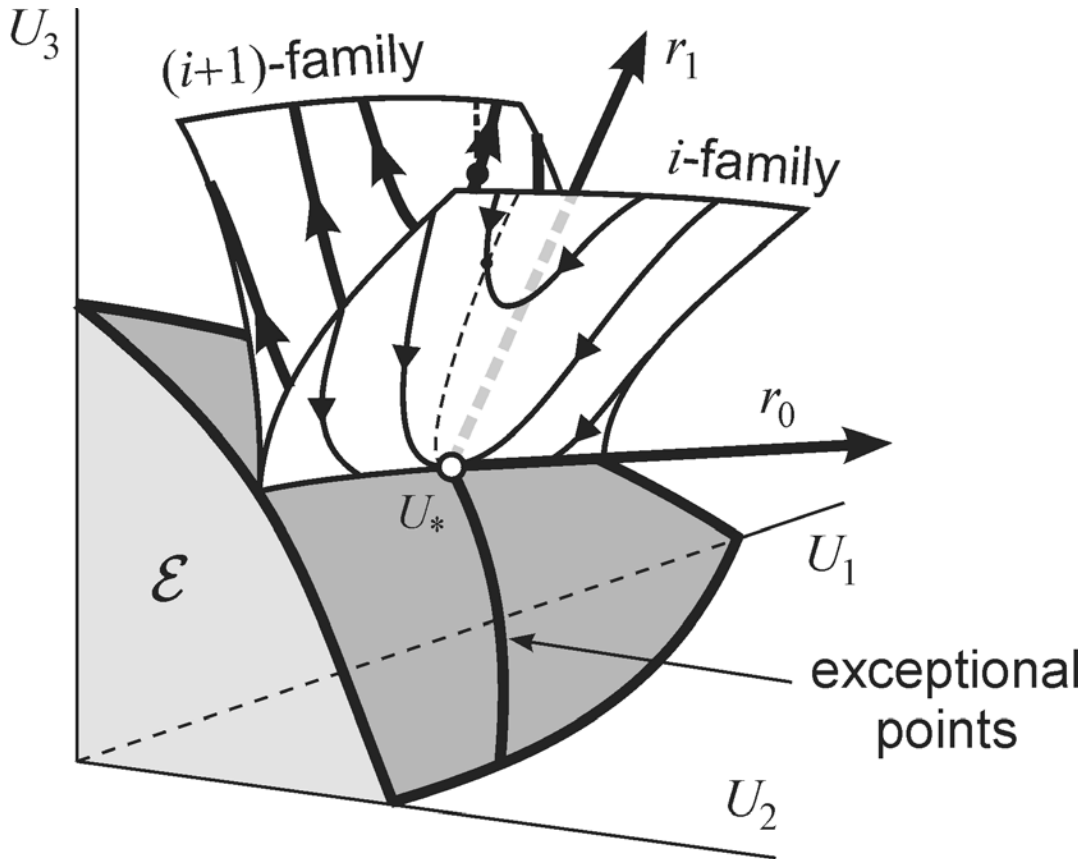
$$p = \left(\frac{2p\nabla s \cdot R_1 + \nabla p \cdot R_0}{2\nabla s \cdot R_0 + \nabla p \cdot R_1}\right)^2$$

Rarefaction waves in (x,t) space



Both cases are possible: family of a simple wave can **increase** or **decrease** when passing the exceptional point

Exceptional points for more than two conservation laws



Geometry of singularity in state space is determined by the eigenvector r_0 and the associated vector (generalized eigenvector) r_1

Riemann problem

Riemann problem
initial conditions

$$U(x,t=0)=\begin{cases} U_L, & x<0, \\ U_R, & x>0, \end{cases}$$

Solution is a sequence of shock waves (discontinuities) and rarefaction waves separated by constant states.

Classical solution contains n waves, one for each characteristic family

Shock waves: left state U_-, right state U_+, speed s

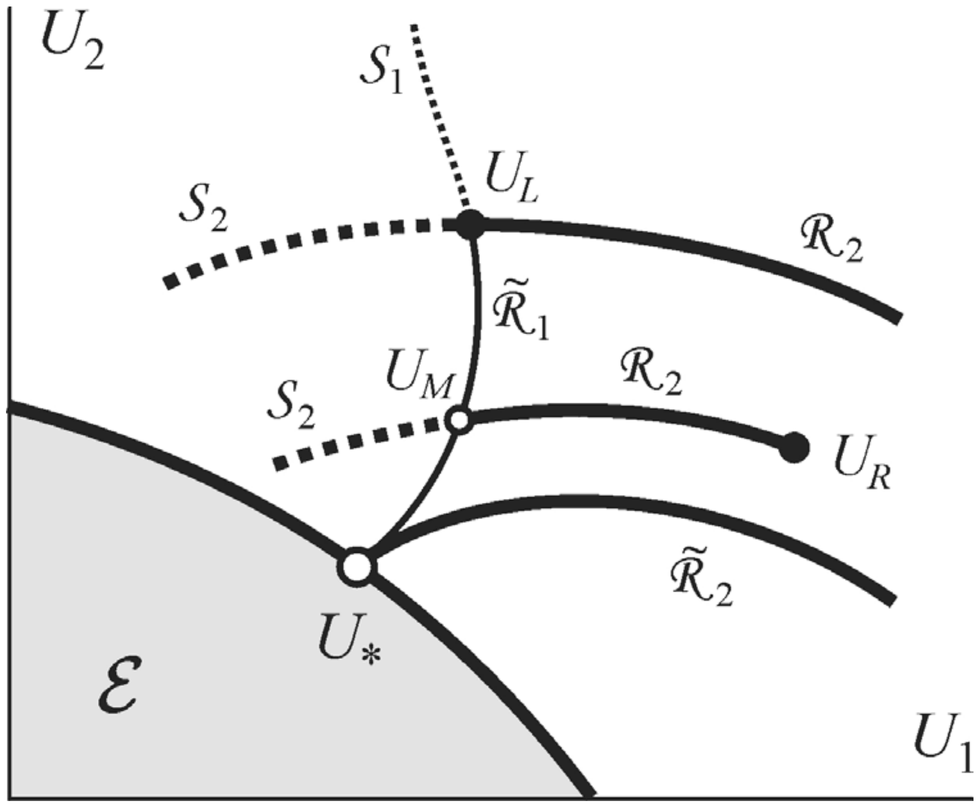
Rankine-Hugoniot conditions $$F(U_+)-F(U_-)=s(U_+-U_-)$$

Lax conditions

$$S_1: \quad \lambda_1(U_+)<s<\lambda_1(U_-),\ s<\lambda_2(U_+)$$

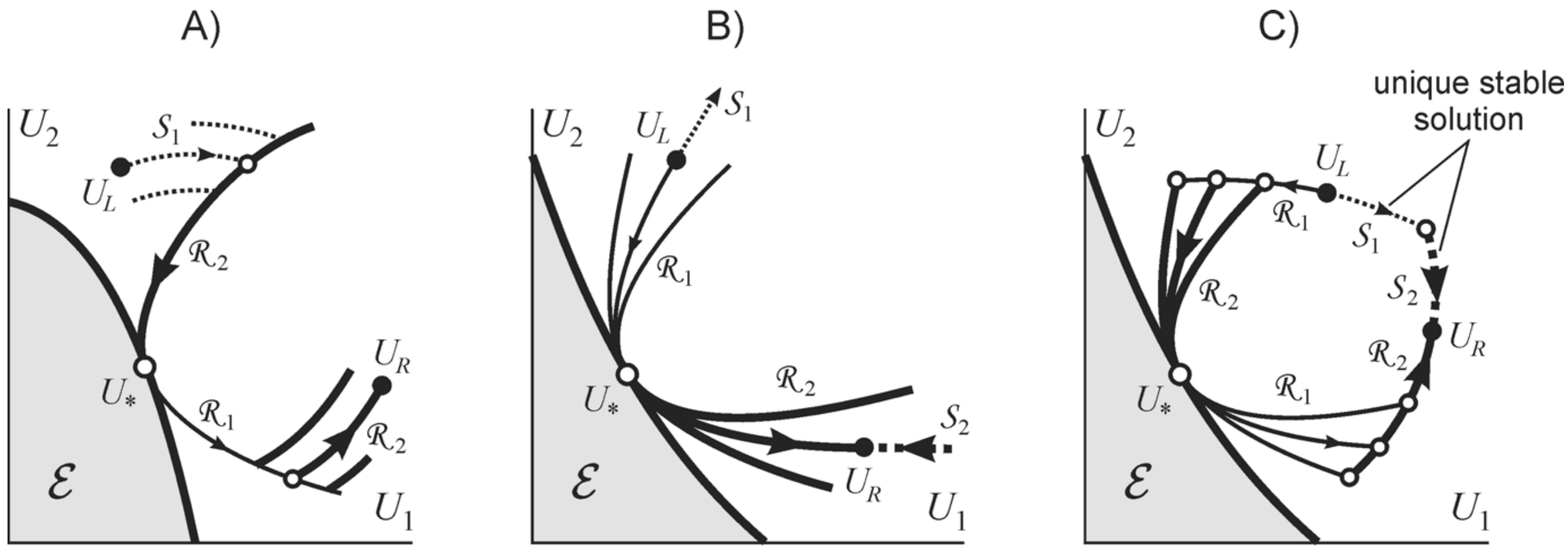
$$S_2: \quad \lambda_2(U_+)<s<\lambda_2(U_-),\ s>\lambda_1(U_-)$$

Riemann solutions near regular points of elliptic boundary



Given left (right) state, rarefaction curves passing through regular points of elliptic boundary serve as **bifurcation boundaries** for Riemann solutions with different right (left) initial states.

Riemann solutions near exceptional points



A) Unique 3-waves solution (stable)

B) Unique 1-wave solution (stable)

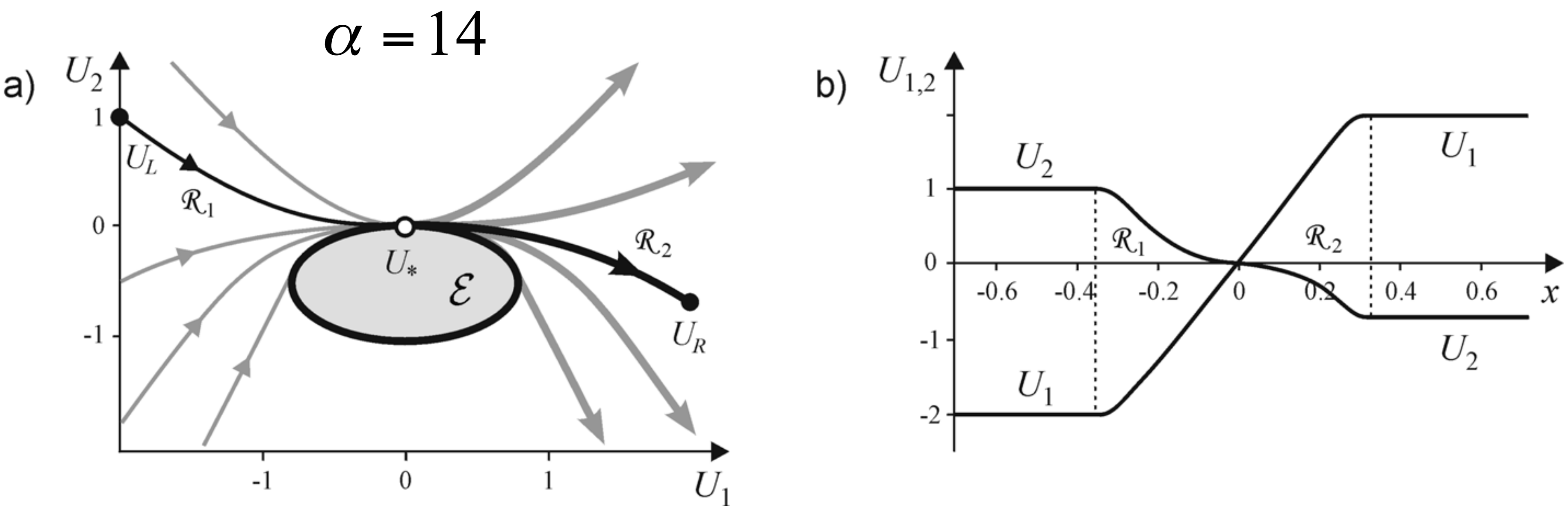
C) Infinite number of 3-waves solutions (**unstable**)
+ one separate 2-waves solution (stable)

Example

Flux function $F(U) = \begin{pmatrix} U_2 + \alpha U_1^2/2 + U_2^2/2 \\ 10U_1U_2 \end{pmatrix}$, α is a parameter

Exceptional point: $U_* = 0$

Singularity types: (b) $\alpha < 0$, (c) $0 < \alpha < 10$ or $20 < \alpha$, (d) $10 < \alpha < 20$



Conclusion

Structure of rarefaction curves near regular and exceptional points of elliptic boundary is analyzed both qualitatively and quantitatively

Novel types of Riemann solutions for 2 conservation laws containing exceptional points are found:

- generic stable 1-wave solutions
- extreme non-uniqueness: infinite number of unstable solutions for the same initial conditions