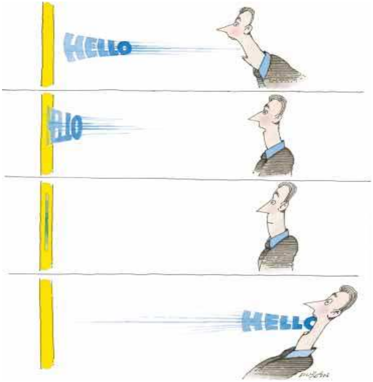
Shock structure due to the stochastic forcing of waves

André Nachbin IMPA/BRAZIL

Motivation: WAVEFORM INVERSION/Refocusing

A FANTASTIC APPLICATION!

2D linear HYPERBOLIC waves $\Rightarrow$ 1D nonlinear waves



DUSAN PETRICIC

TIME-REVERSED ACOUSTICS

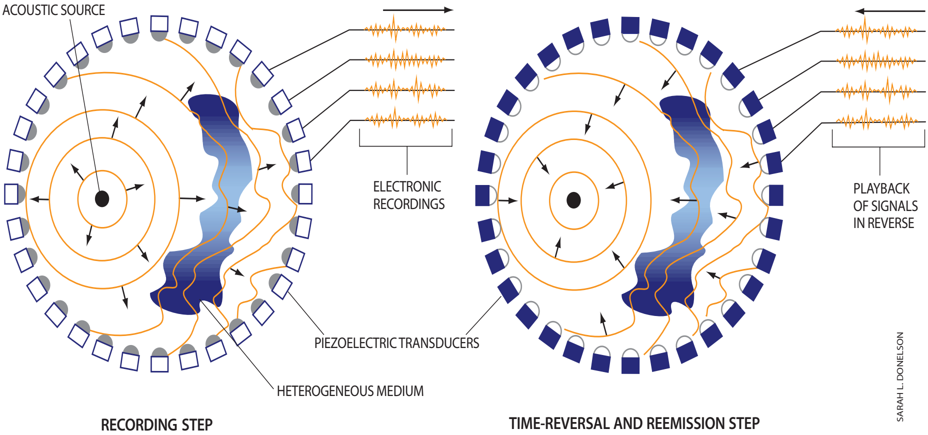
Arrays of transducers can re-create a sound and send it back to its source as if time had been reversed. The process can be used to destroy kidney stones, detect defects in materials and communicate with submarines

by Mathias Fink

In a room inside the Waves and Acoustics Laboratory in Paris is an array of microphones and loudspeakers. If you stand in front of this array and speak into it, anything you say comes back at you, but played in reverse. Your "hello" echoes—almost instantaneously—as "olleh." At first this may seem as ordinary as playing a tape backward, but the loudspeakers, the sound of the "olleh" converges onto your mouth, almost as if time itself had been reversed. Indeed, the process is known as time-reversed acoustics, and the array in front of you is acting as a "time-reversal mirror."

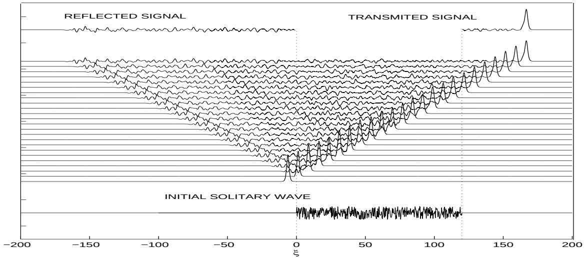
Such mirrors are more than just a novelty item. They have a range of applications, including destruction of tumors and

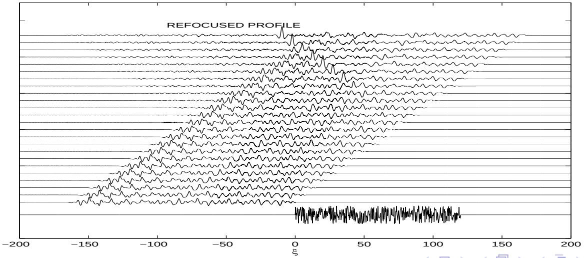
Acoustic chamber



ACOUSTIC TIME-REVERSAL MIRROR operates in two steps. In the first step (*left*) a source emits sound waves (*orange*) that propagate out, perhaps being distorted by inhomogeneities in the medium. Each transducer in the mirror array detects the sound arriving at its location and feeds the signal to a computer. In the second step (*right*), each transducer plays back its sound signal in reverse in synchrony with the other transducers. The original wave is re-created, but traveling backward, retracing its passage back through the medium, untangling its distortions and refocusing on the original source point.

Solitary wave: Fouque, Garnier, Muñoz & N., PRL ’04



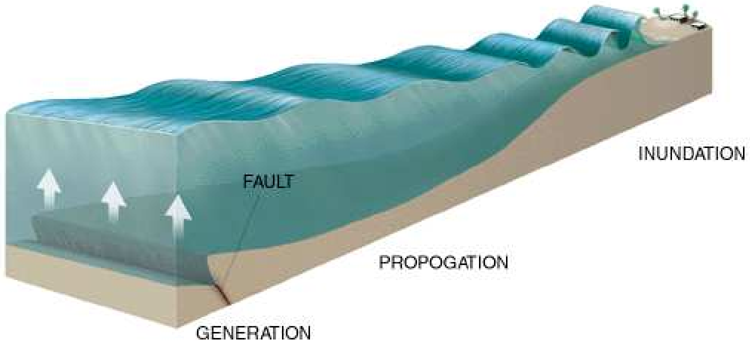


In order to understand NONLINEAR PDEs with TIME-REVERSED data

First we address the DIRECT NONLINEAR SCATTERING PROBLEM

$\Rightarrow$ NonLin Hyperbolic PDEs with HIGHLY VARIABLE coefficients

PHYSICAL MODEL: Long propagation distances + detailed TOPOGRAPHY



Scientific American '99

α = **nonlinearity** = amplitude/depth

β = **dispersion** = depth/wavelength

γ = **disorder**/wavelength

$h(x) \equiv$ DISORDERED TOPOGRAPHY PROFILE

Reflection-Transmission of waves
&
Time-reversal of waves

...in the diffusion approximation regime:

(a) Linear Hyperbolic: $(\alpha = \beta = 0)$ $\sim$ Acoustics

(b) Linear Dispersive: $(\alpha = 0; \beta = \varepsilon)$

(c) Nonlinear Hyperbolic: $(\alpha = \varepsilon; \beta = 0)$

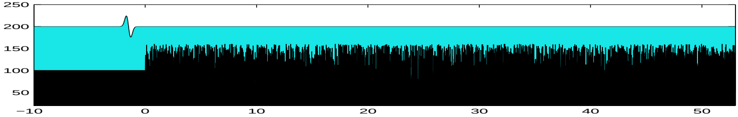
(d) Convection-diffusion: $(\alpha = \varepsilon; \mu = \varepsilon)$

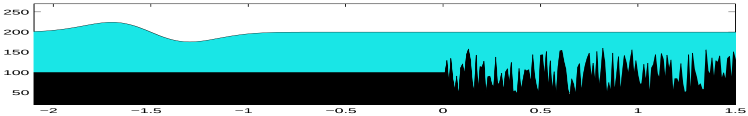
(e) Solitary waves: $(\alpha = \beta = \varepsilon)$

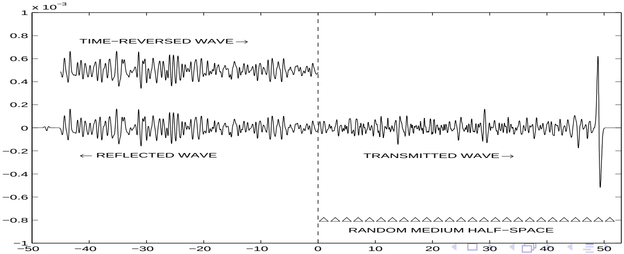
OVERVIEW of RESULTS and THEORY

SETUP for THEORY and SIMULATIONS:

Typical wave profiles: Gaussian, dGaussian/dx and Solitary wave.





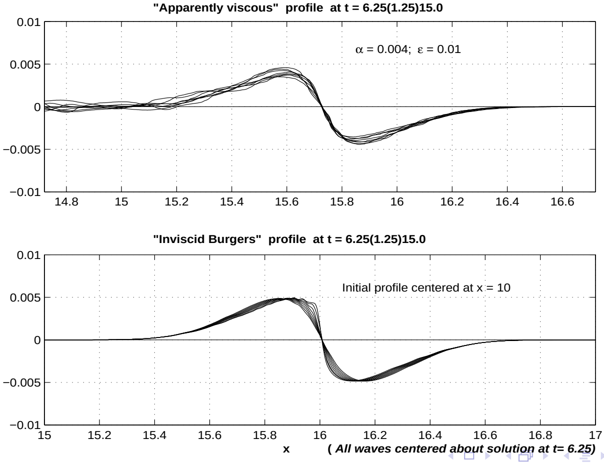


INVISCID NonLinear Shallow Water system w/ a dGaussian/dx pulse

2x2 CONSERVATION LAW with DISORDERED variable COEFFICIENTS

RANDOM Forcing $\Rightarrow$ shock structure: Fouque, Garnier & N., Physica D '04.

ASYMPTOTICS $\Rightarrow$ wave elevation $\equiv \eta(x,t)$ governed by **VISCOUS Burgers'**



MATH TOOL: a LIMIT THEOREM for Stochastic ODEs

Khasminskii's Theorem (*): consider the IVP $\omega \in (\Omega, \mathcal{A}, \mathcal{P})$

$$\frac{dx_\varepsilon}{dt} = \varepsilon F(t, x_\varepsilon; \omega), \qquad x_\varepsilon(0) = x_0$$

and

$$\frac{dy}{d\tau} = \overline{F}(y), \qquad y(0) = x_0,$$

where $F(t, \cdot; \omega)$ is a stationary process, ergodic... with

$$\overline{F}(x) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}\{F(t, x; \omega)\}dt.$$

Then

$$\sup_{0\le t} \mathbb{E}\{|x_\varepsilon(t) - y(t)|\} \sim \sqrt{\varepsilon} \quad \textbf{on the time scale } 1/\varepsilon.$$

(*) R.Z. Khasminskii, On stochastic processes defined by differential equations with a small parameter, Theory Prob. Applications, Volume XI (1966), pp.211-228.
R.Z. Khasminskii, A limit-theorem for the solutions of differential equations with random right-hand sides, Theory Prob. Applications, Volume XI (1966), pp.390-406.

Setting up Shallow Water Eqn. for Khasminskii's theorem...

...include viscosity $\mu = \varepsilon^2 \mu_0$...

$$\frac{\partial \eta}{\partial t} + \frac{\partial(1 + \varepsilon h + \alpha \eta)u}{\partial x} = 0,$$
$$\frac{\partial u}{\partial t} + \frac{\partial \eta}{\partial x} + \alpha u \frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2}.$$

With the underlying Riemann Invariants, to leading order...

$$\begin{aligned}\frac{\partial}{\partial x}\begin{pmatrix} A \\ B \end{pmatrix} &= Q(x)\frac{\partial}{\partial t}\begin{pmatrix} A \\ B \end{pmatrix} - \varepsilon \frac{h'}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \\ &+ \varepsilon^2 \frac{\alpha_0}{4}\begin{pmatrix} 3A + B & 0 \\ 0 & A + 3B \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} A \\ B \end{pmatrix} \\ &+ \varepsilon^2 \frac{\mu_0}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\frac{\partial^2}{\partial t^2}\begin{pmatrix} A \\ B \end{pmatrix} + O(\varepsilon^3),\end{aligned}$$

...and using a Lagrangian frame ⇒ random ODE-like setting.

Khasminskii's theorem $\Rightarrow$ The front pulse/right **Riemann Inv.** $B^\varepsilon(x,\tau) := B(x/\varepsilon^2, \tau + x/\varepsilon^2)$ **converges to** $\tilde{B}$

$$\tilde{B}(x,\tau) = \tilde{B}_0\left(x, \tau - \frac{\sqrt{b_0(0)}}{\sqrt{2}} W_x - \frac{\phi_0(0)}{2} x\right).$$

where $\tilde{B}_0$ **satisfies the deterministic Burgers equation**

$$\frac{\partial \tilde{B}_0}{\partial x} = \mathcal{L}\tilde{B}_0 + \frac{3\alpha_0}{4}\tilde{B}_0 \frac{\partial \tilde{B}_0}{\partial \tau},$$

$$\tilde{B}_0(0,\tau) = f(\tau), \qquad \tau \equiv t - z, \quad z \equiv \int_0^x c^{-1}(s)ds.$$

$\mathcal{L}$ **can be written explicitly in the Fourier domain as**

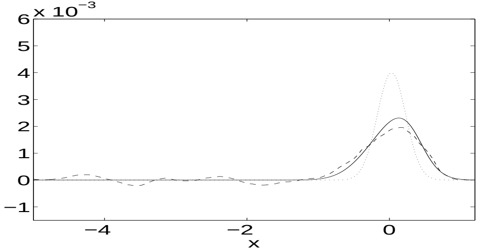
$$\int_{-\infty}^{\infty} \mathcal{L}B(\tau)e^{i\omega\tau}d\tau = -\left(\frac{\mu_0\omega^2}{2} + \frac{b_0(2\omega)\omega^2}{4}\right)\int_{-\infty}^{\infty} B(\tau)e^{i\omega\tau}d\tau.$$

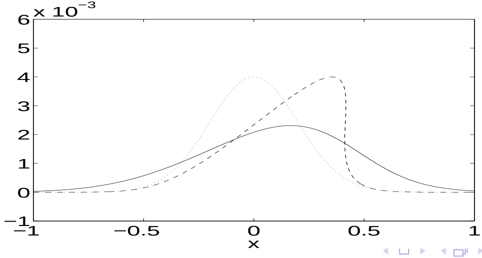
Garnier & N., PRL 2004, PhysFlu, May 2006 $\Rightarrow$ **EDDY VISCOSITY**

$$b_0(\omega) = \int_0^\infty \mathbb{E}[h(0)h(x)]\exp(i\omega x)dx$$

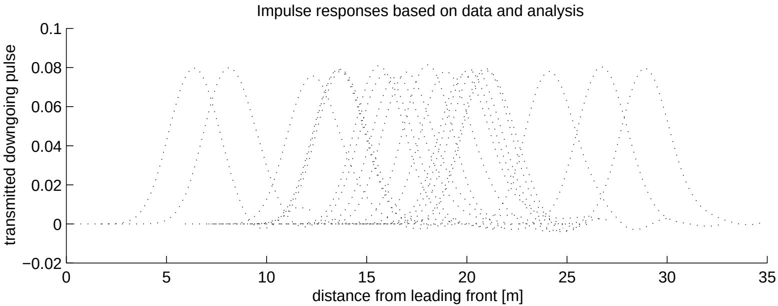
Direct SWE numerics versus effective Burgers equation

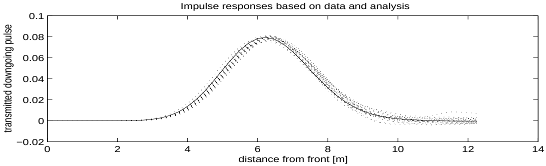
GAUSSIAN WAVE PROFILE





DETERMINISTIC PROFILES with RANDOM ARRIVAL TIMES

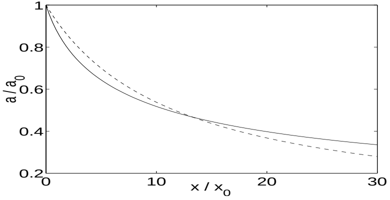


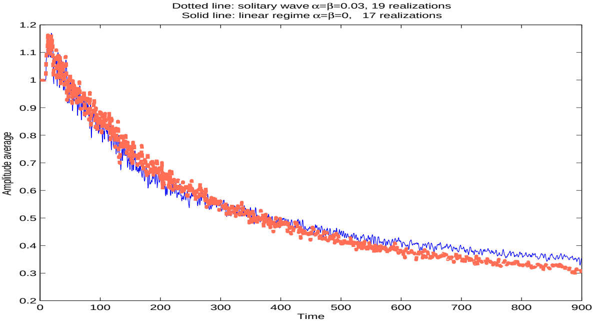


Very Recent! Solitary wave DECAY: Garnier, Muñoz & N., submitted '06

Using underlying Riemann Invariants for the zero-dispersion system

Get coupled variable-coefficient KdV system





How do we address this non-hyperbolic problem?

with IMPROVED

Boussinesq systems

Muñoz & N. IMA Appl. Math. 2006

Evaluating the **horizontal velocity** at an INTERMEDIATE depth $\zeta = Z_0 \in [0, 1]$

$$\phi_\xi(\xi, Z_0, t) \equiv u(\xi, t) = f_\xi - \frac{\beta}{2} {Z_0}^2 f_{\xi\xi\xi} + O(\beta^2)$$

FREE SURFACE CONDITIONS reduce to...

...the BOUSSINESQ-family of equations

$$M(\xi)\eta_t + \left[\left(1 + \frac{\alpha\ \eta}{M(\xi)}\right) u\right]_\xi + \frac{\beta}{2}\left[\left(\mathbf{Z_0}^2 - \frac{1}{3}\right) u_{\xi\xi}\right]_\xi = 0$$

$$u_t + \eta_\xi + \alpha\left(\frac{u^2}{2M^2(\xi)}\right)_\xi + \frac{\beta}{2}(\mathbf{Z_0}^2 - 1)u_{\xi\xi t} = 0$$

BOUSSINESQ-family of equations

$$M(\xi)\eta_t + \left[\left(1 + \frac{\alpha\ \eta}{M(\xi)}\right) u\right]_\xi + \frac{\beta}{2}\left[\left(\mathbf{Z_0}^2 - \frac{1}{3}\right) u_{\xi\xi}\right]_\xi = 0$$

$$u_t + \eta_\xi + \alpha\left(\frac{u^2}{2M^2(\xi)}\right)_\xi + \frac{\beta}{2}(\mathbf{Z_0}^2 - 1)u_{\xi\xi t} = 0$$

$$C^2 = \frac{\omega^2}{k^2} = \frac{1 - (\beta/2)(Z_0{}^2 - \frac{1}{3})k^2}{1 - (\beta/2)(Z_0{}^2 - 1)k^2}$$

The underlying Riemann invariants satisfy, up to order α,

$$
\begin{aligned}
&A_t - A_\xi + \frac{\alpha}{4}(3A+B)A_\xi - \frac{\beta}{6}A_{\xi\xi t} = \frac{\beta}{2}(\frac{2}{3} - Z_0^2)B_{\xi\xi t} \\
&+\frac{1}{2}(\frac{1}{M} - 1)(A_\xi - B_\xi) + \frac{1}{2}(\frac{1}{M})_\xi(A-B) \\
&+\alpha A A_\xi(1 - \frac{1}{M^2}) + \frac{\alpha}{8}(\frac{2}{M^2} - \frac{1}{M} - 1)(A-B)(A_\xi - B_\xi) \\
&-\frac{\alpha}{16}(\frac{1}{M})_\xi\left[(A-B)^2 + \frac{4}{M}(3A^2 + 2AB - B^2)\right],
\end{aligned}
$$

$$
\begin{aligned}
&B_t + B_\xi + \frac{\alpha}{4}(3B+A)B_\xi + \frac{\beta}{6}B_{\xi\xi t} = \frac{\beta}{2}(\frac{2}{3} - Z_0^2)A_{\xi\xi t} \\
&+\frac{1}{2}(\frac{1}{M} - 1)(A_\xi - B_\xi) + \frac{1}{2}(\frac{1}{M})_\xi(A-B) \\
&+\alpha B B_\xi(1 - \frac{1}{M^2}) + \frac{\alpha}{8}(\frac{2}{M^2} - \frac{1}{M} - 1)(A-B)(A_\xi - B_\xi) \\
&-\frac{\alpha}{16}(\frac{1}{M})_\xi\left[(A-B)^2 + \frac{4}{M}(-A^2 + 2AB + 3B^2)\right].
\end{aligned}
$$

Coupled KdV equations for the Riemann-Invariants:

In absence of random perturbations ($M \equiv 1$):

$$\mathbf{A_t} - \mathbf{A}_\xi + \frac{\alpha}{\mathbf{4}}(\mathbf{3A} + \mathbf{B})\mathbf{A}_\xi - \frac{\beta}{\mathbf{6}}\mathbf{A}_{\xi\xi\xi} = \frac{\beta}{2}(\frac{2}{3} - {Z_0}^2)B_{\xi\xi t}$$

$$\mathbf{B_t} + \mathbf{B}_\xi + \frac{\alpha}{\mathbf{4}}(\mathbf{3B} + \mathbf{A})\mathbf{B}_\xi + \frac{\beta}{\mathbf{6}}\mathbf{B}_{\xi\xi\xi} = \frac{\beta}{2}(\frac{2}{3} - {Z_0}^2)A_{\xi\xi t}$$

By choosing

$${Z_0}^2 = \frac{2}{3}$$

the system then supports pure left- and right-going waves satisfying a **KdV-like equation**.

$\tilde{B}_0$ is the solution of the deterministic equation

$$\frac{\partial \tilde{B}_0}{\partial \xi} = \mathcal{L}\tilde{B}_0 + \frac{3\alpha_0}{4}\tilde{B}_0 \frac{\partial \tilde{B}_0}{\partial \tau} + \frac{\beta_0}{6}\frac{\partial^3 \tilde{B}_0}{\partial \tau^3}, \qquad (1)$$

$$\tilde{B}_0(0,\tau) = f(\tau), \qquad (2)$$

where the operator $\mathcal{L}$ can be written explicitly in the Fourier domain as

$$\int_{-\infty}^{\infty} \mathcal{L}B(\tau)e^{i\omega\tau}d\tau = -\frac{b_0(2\omega)\omega^2}{4}\int_{-\infty}^{\infty} B(\tau)e^{i\omega\tau}d\tau$$

$\mathcal{L}$ results from the action of the effective pseudo-viscosity

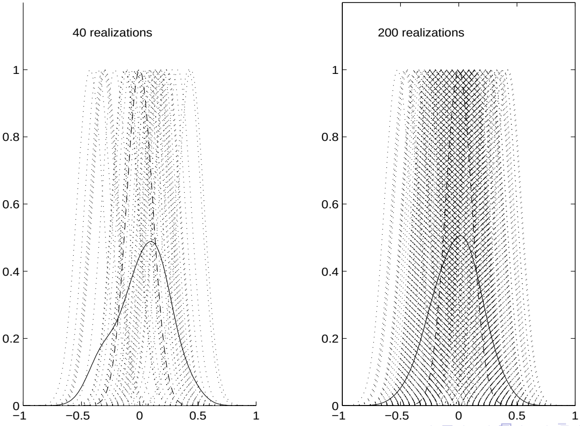
Thank you for your attention.

IMPA, Rio de Janeiro, Brazil.

Limit Theorem versus Mean Field Theory: Over-estimation of attenuation

HYPERBOLIC problem: advection with a random speed

Gaussian pulse (initial data) with a normally distributed speed.



Let $Z_0 = \sqrt{2/3}$ and $u_\xi(\xi, t) = -M(\xi)\eta_t + O(\alpha, \beta)$:

$$(M(\xi)\eta)_t + \left[\left(1 + \frac{\alpha\ \eta}{M(\xi)}\right) u\right]_\xi - \frac{\beta}{6}(M(\xi)\eta)_{\xi\xi t} = 0$$

$$u_t + \eta_\xi + \alpha\left(\frac{u^2}{2M^2(\xi)}\right)_\xi - \frac{\beta}{6}u_{\xi\xi t} = 0$$

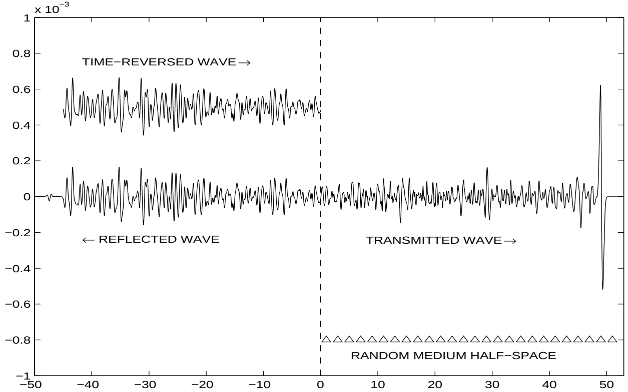
Quintero and Muñoz (Meth.Appl.Anal. '04) proved existence, uniqueness etc... by finding a conserved quantity. Main tool: Bona & Chen '98

$$\left(\mathrm{I} - \frac{\beta}{6}\partial_{\xi\xi}\right)^{-1}[U] = K_\beta * U, \qquad K_\beta(s) \equiv -\frac{1}{2}\sqrt{\frac{6}{\beta}}sign(s)e^{-\sqrt{6/\beta}|s|}$$

$$E(t) \equiv \frac{1}{2}\int_{\Re}\left[\left(1 + \alpha\frac{\eta(\xi, t)}{M(\xi)}\right)[M(\xi)\eta(\xi, t)]^2 + M(\xi)\eta^2(\xi, t)\right]\ d\xi$$

TIME-REVERSAL REFOCUSING: WAVEFORM inversion

- linear hyperbolic ⇒ **Statistical Stability**
- complete refocusing ⇒ recover **original** profile
- **Solitary wave**: TR in reflection and transmission.



Statistical stability: 10 realizations

Alfaro et al., submitted ’06