Transparent boundary conditions for the elastic waves in anisotropic media

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Outline

- Problem formulation
- Generation of LRBC operator, main steps
- Numerical tests
- Conclusions

Motivation

Generation of low-reflecting boundary conditions on open boundaries for anisotropic elastodynamics is a challenging problem. The PML method is not stable in this case, see

- *Beaches E., Fauqueux S., Joly P.,* Stability of perfectly matched layers group velocities and anisotropic waves, *JCP, 188, 2003, 399 – 433;*
- *D. Appelö and G. Kreiss,* A New Absorbing Layer for Elastic Waves, *JCP, 215, 2006, 642 – 660.*

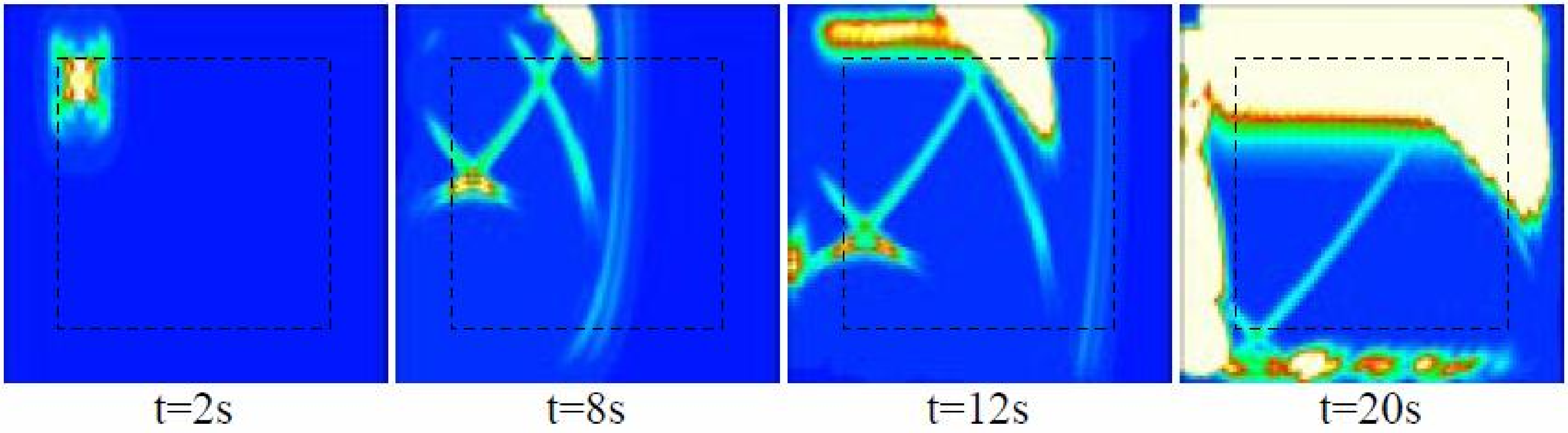
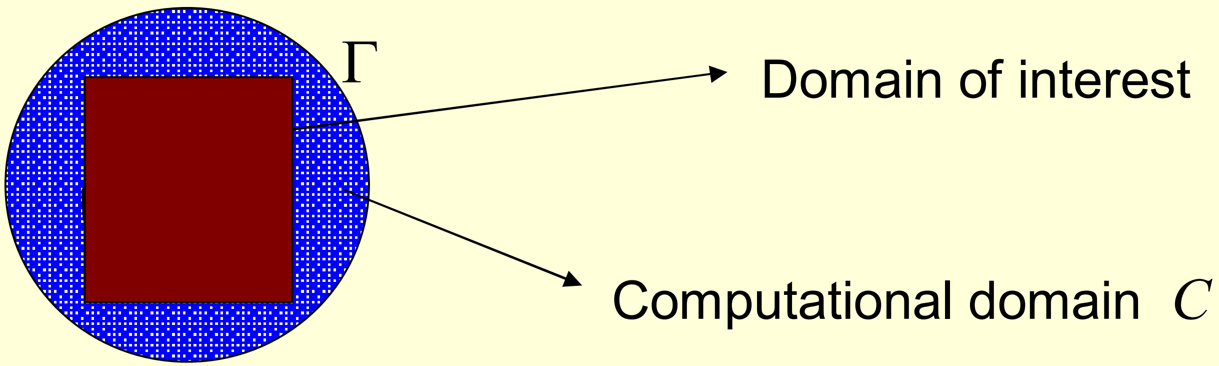


Figure 13: Some snapshots at different times for the orthotropic media (IV).

Low-reflecting boundary conditions (LRBC)

LRBC on “open boundaries” are needed for modeling of wave propagation using a bounded computational domain.

Typical setup:



Generation of LRBC on Γ is an additional problem which is set up by considering auxiliary external Initial Boundary Value Problems (IBVPs) outside C.

Remark: In the domain of interest the governing equations and geometry can be much more complex than outside C.

Anisotropic elasticity, 2D orthotropic media

We consider 2D elastodynamic equations:

$$\rho \frac{\partial^2 v_1}{\partial t^2} = c^{11} \frac{\partial^2 v_1}{\partial x_1^2} + c^{33} \frac{\partial^2 v_1}{\partial x_2^2} + \left(c^{33} + c^{12}\right) \frac{\partial^2 v_2}{\partial x_1 \partial x_2},$$

$$\rho \frac{\partial^2 v_2}{\partial t^2} = c^{33} \frac{\partial^2 v_2}{\partial x_1^2} + c^{22} \frac{\partial^2 v_2}{\partial x_2^2} + \left(c^{33} + c^{12}\right) \frac{\partial^2 v_1}{\partial x_1 \partial x_2}.$$

Here ρ is density; (v_1, v_2) is the Cartesian velocities; $\{c^{nm}\}$ are (constant) elastic coefficients of the Hook's law written in the matrix form, $\sigma^n = c^{nm}\varepsilon_m$.

In polar coordinates for the velocity vector $f = (v_r, v_\theta)$ the system reads:

$$\frac{\partial^2 f}{\partial t^2} = A^{11} \frac{\partial^2 f}{\partial r^2} + A^{22} \frac{\partial^2 f}{\partial \theta^2} + A^{12} \frac{\partial^2 f}{\partial r \partial \theta} + A^1 \frac{\partial f}{\partial r} + A^2 \frac{\partial f}{\partial \theta} + A^0 f$$

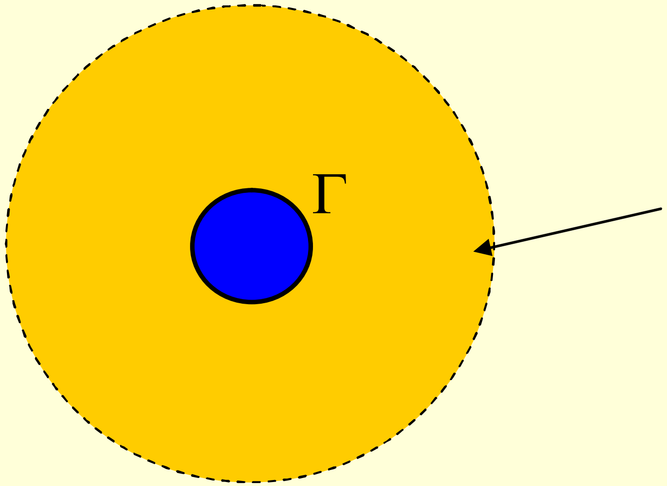
where $A^{ij}(r,\theta)$, $A^i(r,\theta)$ are 2x2 matrices

Generation of LRBC operator, main steps

Governing equation in 2D space: $f_{tt} - Lf = 0$

Main steps:

Stage 1: consider a set of auxiliary external IBVPs outside C (set wrt "*m*"):



$$\begin{cases} \mathcal{E}^m{}_{tt} - L\mathcal{E}^m = 0 \ \text{ in } \ R^2 / C \\ \mathcal{E}^m \big|_{t=0} = 0 \\ \mathcal{E}^m \big|_{\Gamma} = \boldsymbol{\delta}(t)\boldsymbol{\varphi}^m(\boldsymbol{\theta}) \end{cases} \qquad (1)$$

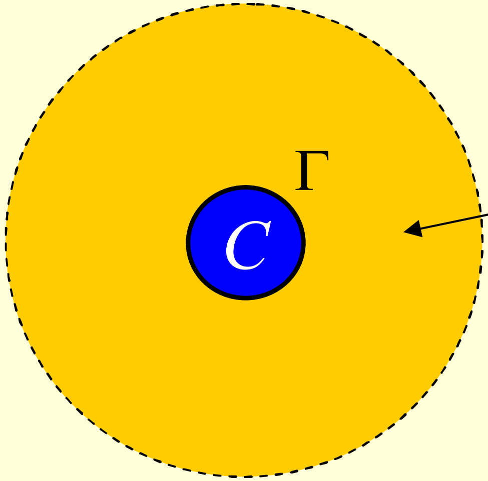
where $\delta(t)$ is Dirac's delta function;

$\{\varphi^m(\theta)\}_{m=0}^{\infty}$ is a basis on Γ, i.e. $f(t,\theta) = \sum_m c^m(t)\varphi^m(\theta)$

(in 2D it is constructed, e.g., by using $\{\sin k\theta, \cos k\theta\}$)

Generation of LRBC operator (cont.)

Stage 2: make Laplace transform and pass to elliptic BVPs (parameterized by s):



$$\begin{cases} s^2\hat{\mathcal{E}}^m - L\hat{\mathcal{E}}^m = 0 \ \text{ in } R^2 / C \\ \hat{\mathcal{E}}^m \,|_{\Gamma} = \varphi^m(\theta) \\ \hat{\mathcal{E}}^m \,|_{r\to\infty} = 0 \end{cases} \qquad (2)$$

Stage 3: solve (numerically) the problems and evaluate $\frac{\partial}{\partial n}\hat{\mathcal{E}}^m(r,\theta)$ on Γ

Thus we obtain the Dirichlet-to-Neumann maps

$$\varphi^m(\theta) \mapsto \psi^m(\theta)[s] \qquad \left(\equiv \frac{\partial}{\partial n}\hat{\mathcal{E}}^m(\theta, r),\ r = R_\Gamma \right)$$

Generation of LRBC operator (cont.)

Stage 4: form matrix of the Poincare-Steklov operator:

we take arbitrary data on Γ

$$\hat{f}(s,\theta) = \sum_m \hat{c}^m(s)\varphi^m(\theta)$$

and write the representation of its normal derivative on Γ

$$\frac{\partial}{\partial n}\hat{f}(s,\theta) = \sum_m \hat{c}^m(s)\psi^m(\theta)[s] = \sum_m \hat{c}^m(s)\sum_n P_n^m(s)\varphi^n(\theta)$$

Thus we obtain the Poincare-Steklov operator in space of Fourier coefficients:

$$\hat{d}^n(s) = \sum_n \hat{P}_n^m(s)\hat{c}^m(s) \qquad \left(\frac{\partial}{\partial n}\hat{f}(s,\theta) = \sum_n \hat{d}^n(s)\varphi^n(\theta)\right)$$

or, in matrix form:

$$\hat{\mathbf{d}}(s) = \hat{\mathbf{P}}(s)\hat{\mathbf{c}}(s), \qquad \hat{\mathbf{c}} = \left\{\hat{c}^0,\, \hat{c}^1,\, \ldots\right\}^T$$

Generation of LRBC operator (cont.)

Stage 5: make inverse Laplace transform for the P-S operator.

First we represent matrix $\hat{\mathbf{P}}(s)$ by sum of three matrices to take into account asymptotic at $s \to \infty$:

$$\hat{\mathbf{P}}(s) = \mathbf{P}_1 s + \mathbf{P}_0 + \hat{\mathbf{K}}(s); \quad \mathbf{P}_1, \mathbf{P}_0 \text{ are } consts, \quad \hat{\mathbf{K}}(s) = o(1)$$

Then we calculate rational approximations to each entry in $\hat{\mathbf{K}}(s)$ such that all poles have negative real parts, i.e.

$$\hat{K}_n^m(s) \approx \hat{\tilde{K}}_n^m(s) \equiv \sum_{l=1}^{L_n^m} \frac{\alpha_{n\,l}^m}{s - \beta_{n\,l}^m}, \quad \mathrm{Re}(\beta_{n\,l}^m) \le \delta < 0$$

Finally the inverse Laplace transform of $\hat{\mathbf{d}}(s) = \hat{\mathbf{P}}(s)\hat{\mathbf{c}}(s)$ gives:

$$\mathbf{d}(t) = \mathbf{P}_1 \frac{\partial \mathbf{c}(t)}{\partial t} + \mathbf{P}_0 \mathbf{c}(t) + \tilde{\mathbf{K}}(t) * \mathbf{c}(t)$$

Generation of LRBC operator (cont.)

Remark: the explicit form of kernels $\tilde{K}_n^m(t)$ is

$$\tilde{K}_n^m(t) = \sum_{l=1}^{L_n^m} \alpha_{n\,l}^m \exp(\beta_{n\,l}^m t), \qquad \mathrm{Re}(\beta_{n\,l}^m) \le \delta < 0$$

that permits to treat convolutions by *stable recurrent* formulas.

Stage 6: compose azimuth modes:

Denote by $\mathbf{Q}$ the operator of Fourier decomposition for $f(t,\theta) = \sum_m c^m(t)\varphi^m(\theta)$

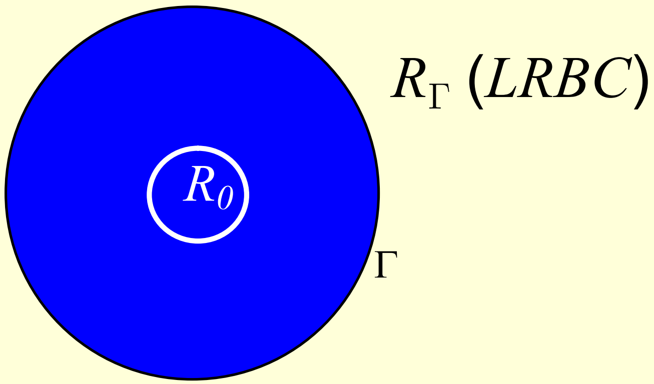
i.e. $\mathbf{Q}: f(t,\theta) \to \left\{c^m(t)\right\}$. Consequently, $\mathbf{Q}^{-1}: \left\{c^m(t)\right\}, \left\{d^m(t)\right\} \to f, \frac{\partial f}{\partial n}$

The LRBC in the physical space reads:

$$\mathbf{Q}^{-1}\mathbf{P}_1\mathbf{Q}\frac{\partial f}{\partial t} - \frac{\partial f}{\partial n} + \mathbf{Q}^{-1}\mathbf{P}_0\mathbf{Q}f + \mathbf{Q}^{-1}\left\{\tilde{\mathbf{K}}(t)*\right\}\mathbf{Q}f = 0$$

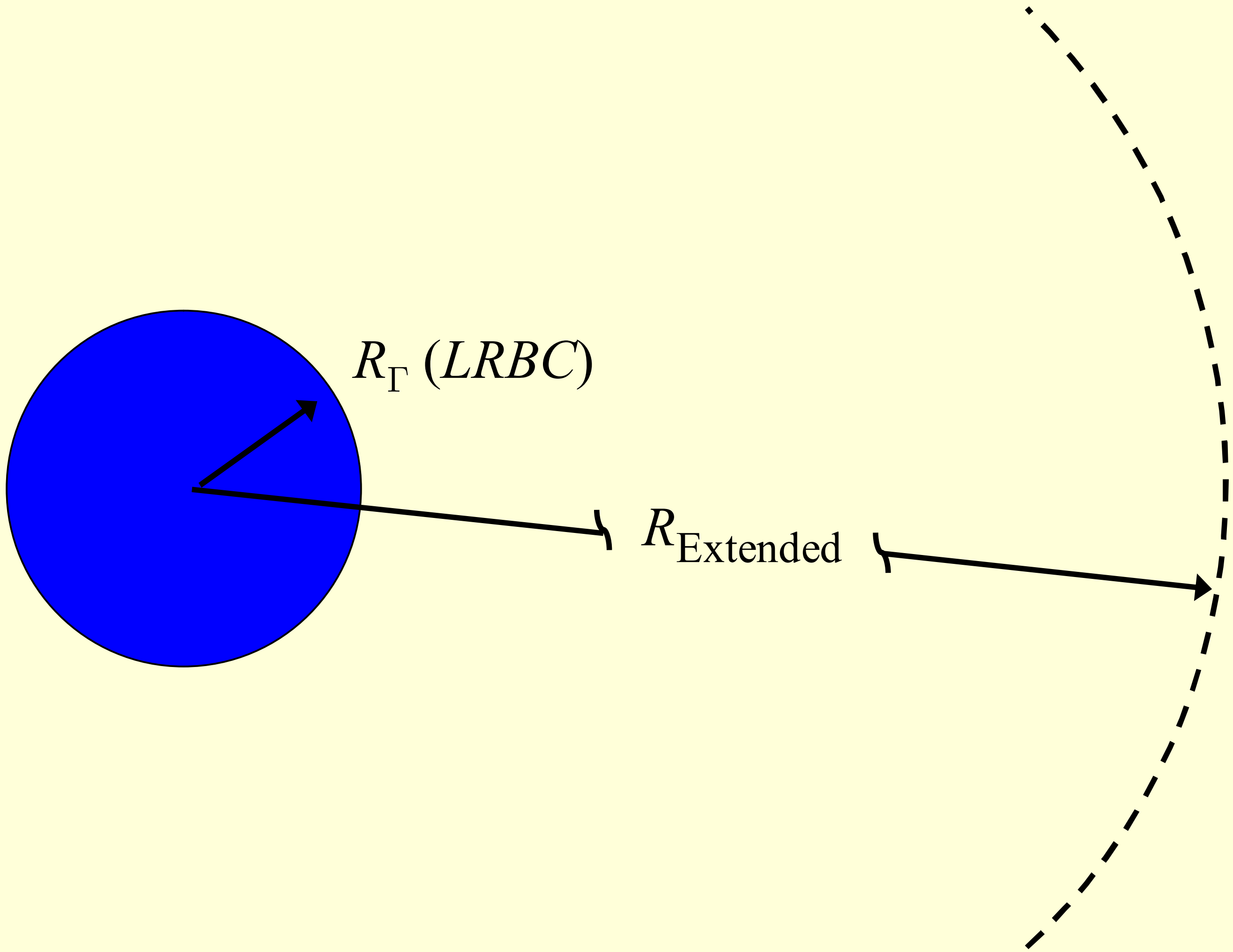
Setup of the test problem

Governing equations are implemented in the polar system of coordinates. We consider the task in a circle:



- At $R_0 = 2$ we prescribe Dirichlet data (pulse) to initiate elastic waves;
- Γ with radius $R_\Gamma = 10$ is the external boundary where we put LRBC (with 36 azimuth harmonics).

Test problem: reference solution



The verification of LRBC is made by comparing with the *reference solution* of a second problem having $8x$ bigger external radius (extended domain).

Comparison of two solutions in C-norm is made at $r < R_\Gamma$.

Test calculations

Parameters of anisotropic media are taken from

Beaches E., Fauqueux S., Joly P., Stability of perfectly matched layers group velocities and anisotropic waves, *JCP, 188, 2003, 399 – 433:*

anisotropic medium, case IV: $c_{11} = 4, \quad c_{22} = 20, \quad c_{33} = 2, \quad c_{12} = 7.5$

$$\rho \frac{\partial^2 v_1}{\partial t^2} = c^{11} \frac{\partial^2 v_1}{\partial x_1^2} + c^{33} \frac{\partial^2 v_1}{\partial x_2^2} + \left(c^{33} + c^{12}\right) \frac{\partial^2 v_2}{\partial x_1 \partial x_2},$$

$$\rho \frac{\partial^2 v_2}{\partial t^2} = c^{33} \frac{\partial^2 v_2}{\partial x_1^2} + c^{22} \frac{\partial^2 v_2}{\partial x_2^2} + \left(c^{33} + c^{12}\right) \frac{\partial^2 v_1}{\partial x_1 \partial x_2}.$$

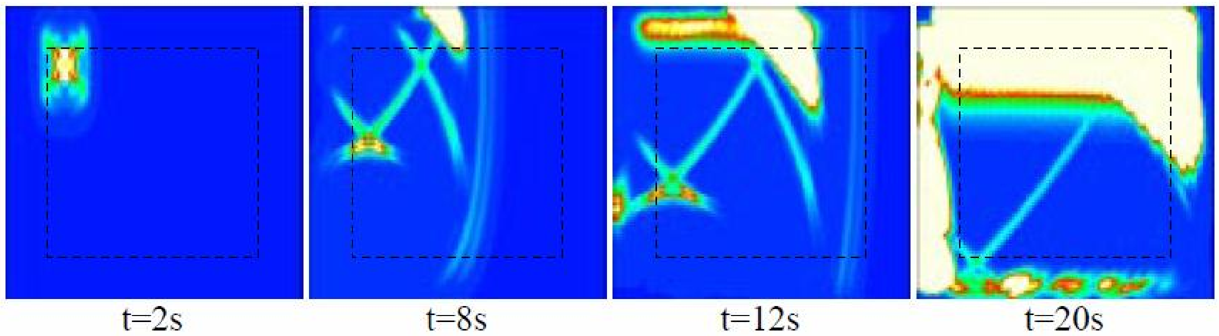
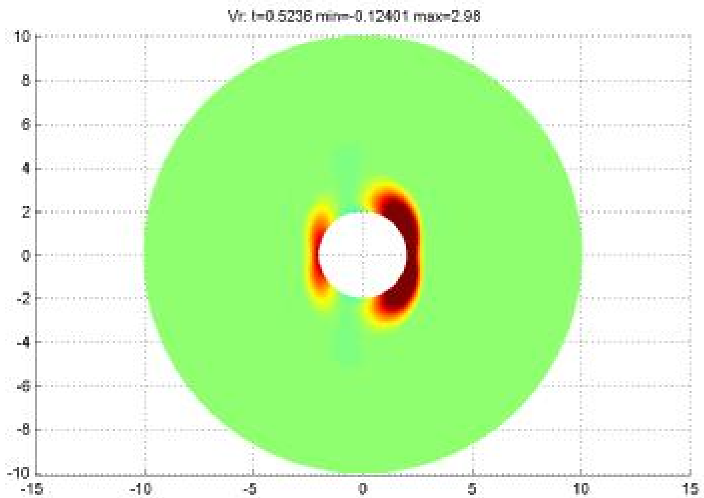


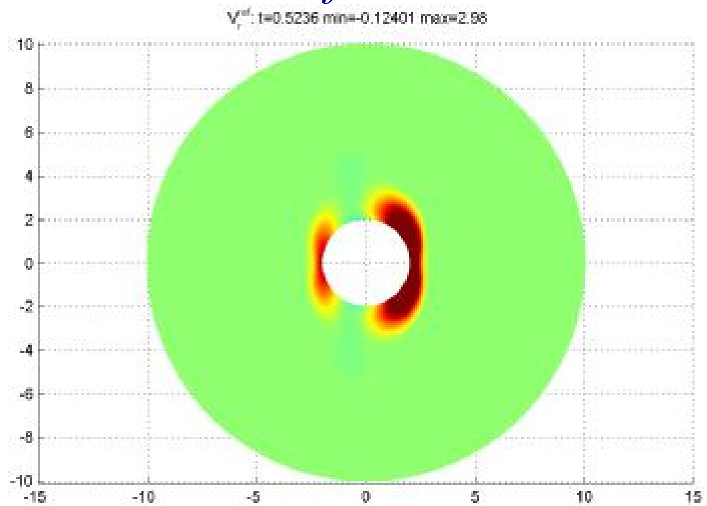
Figure 13: Some snapshots at different times for the orthotropic media (IV).

Anisotropic case-IV,   t=0.52

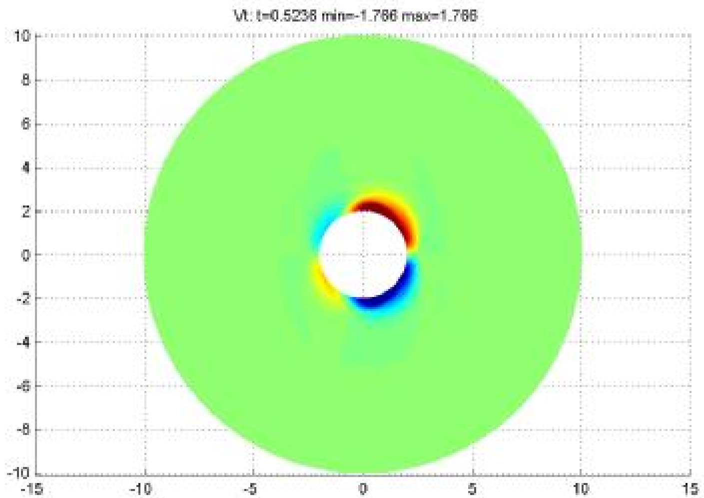
LRBC *Reference*

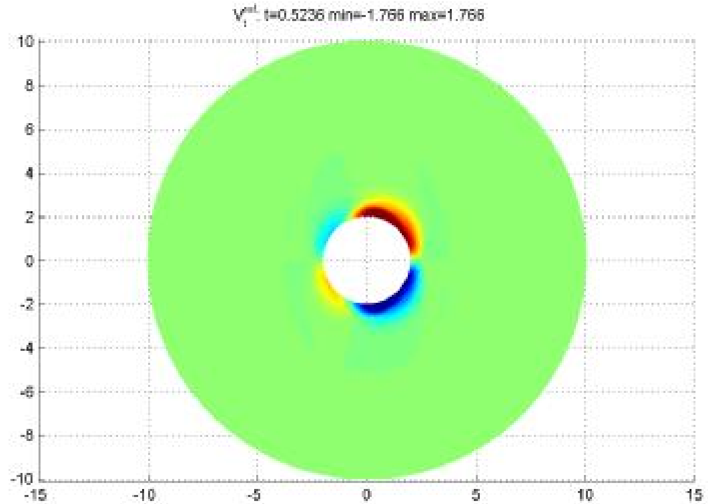
Ur



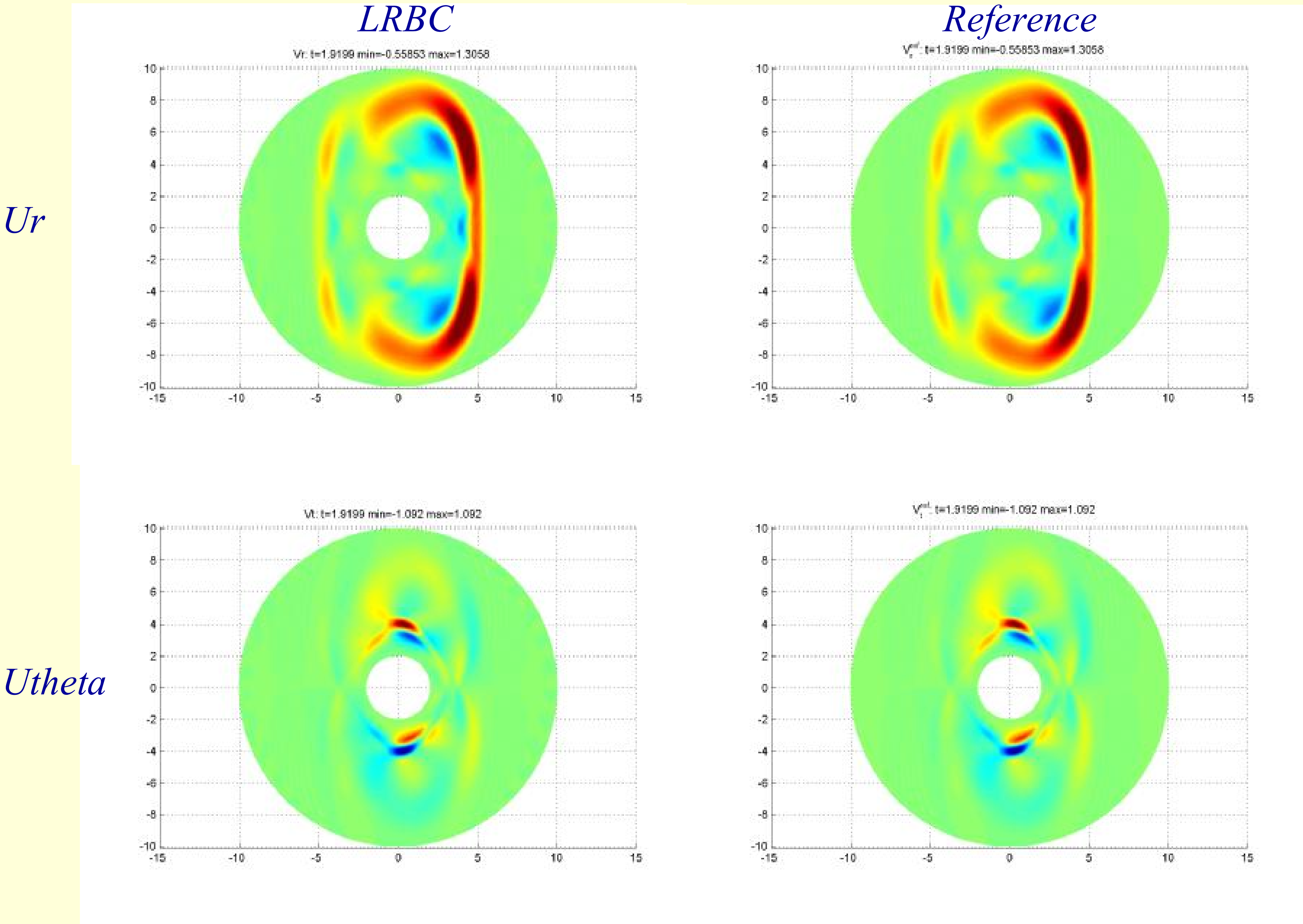


Utheta

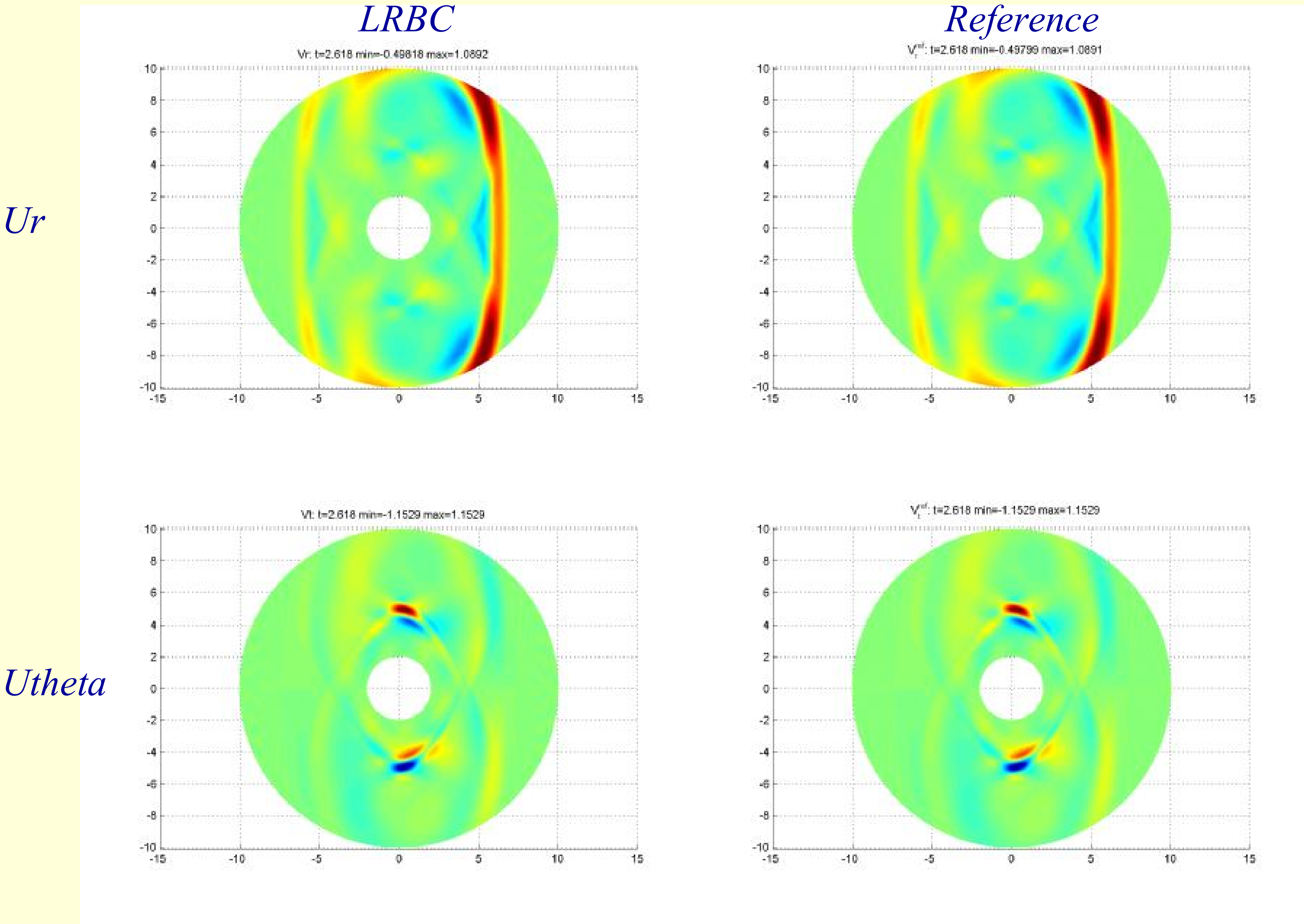




Anisotropic case-IV,   t=1.92



Anisotropic case-IV, t=2.62

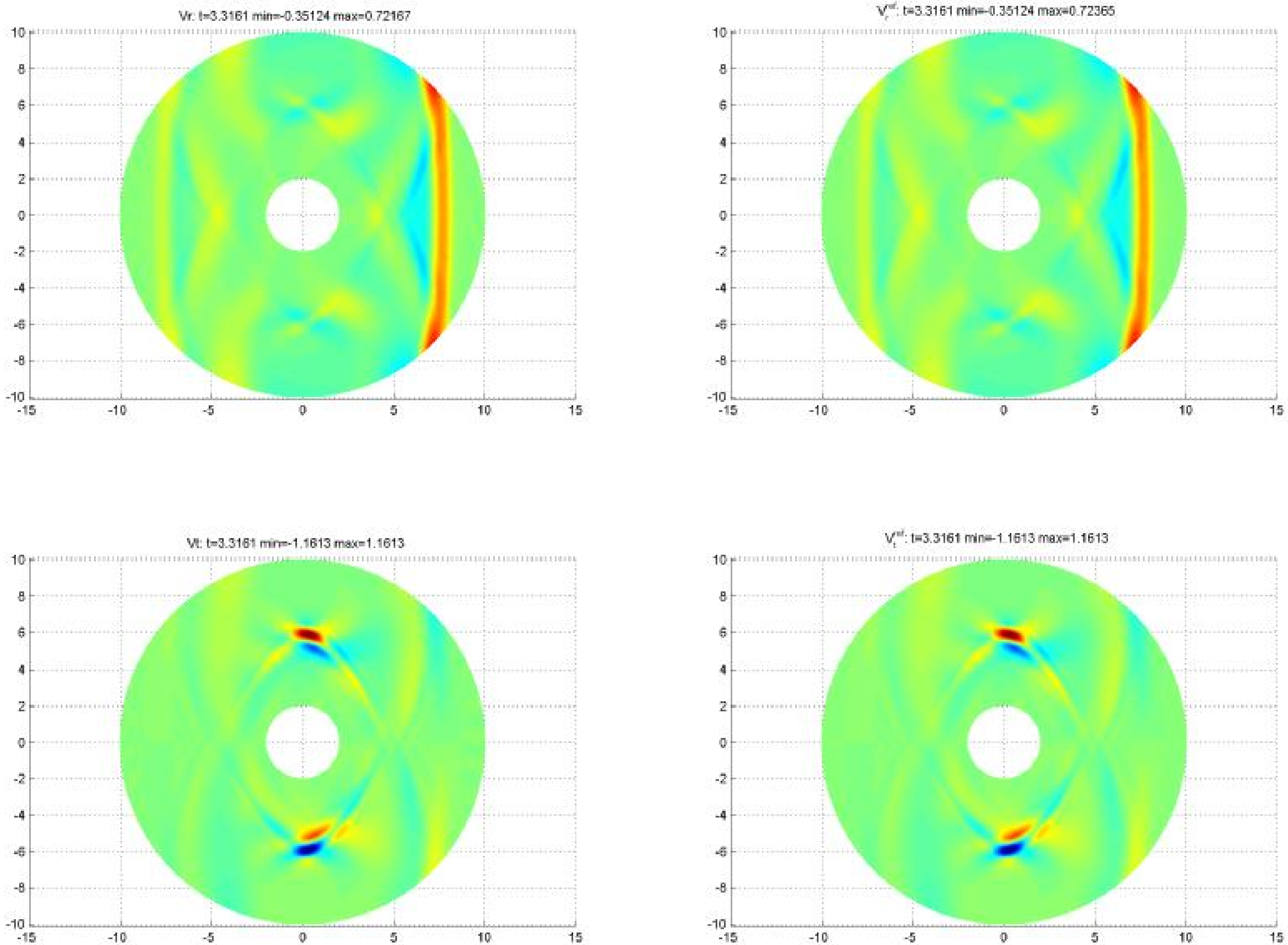


Anisotropic case-IV,   t=3.32

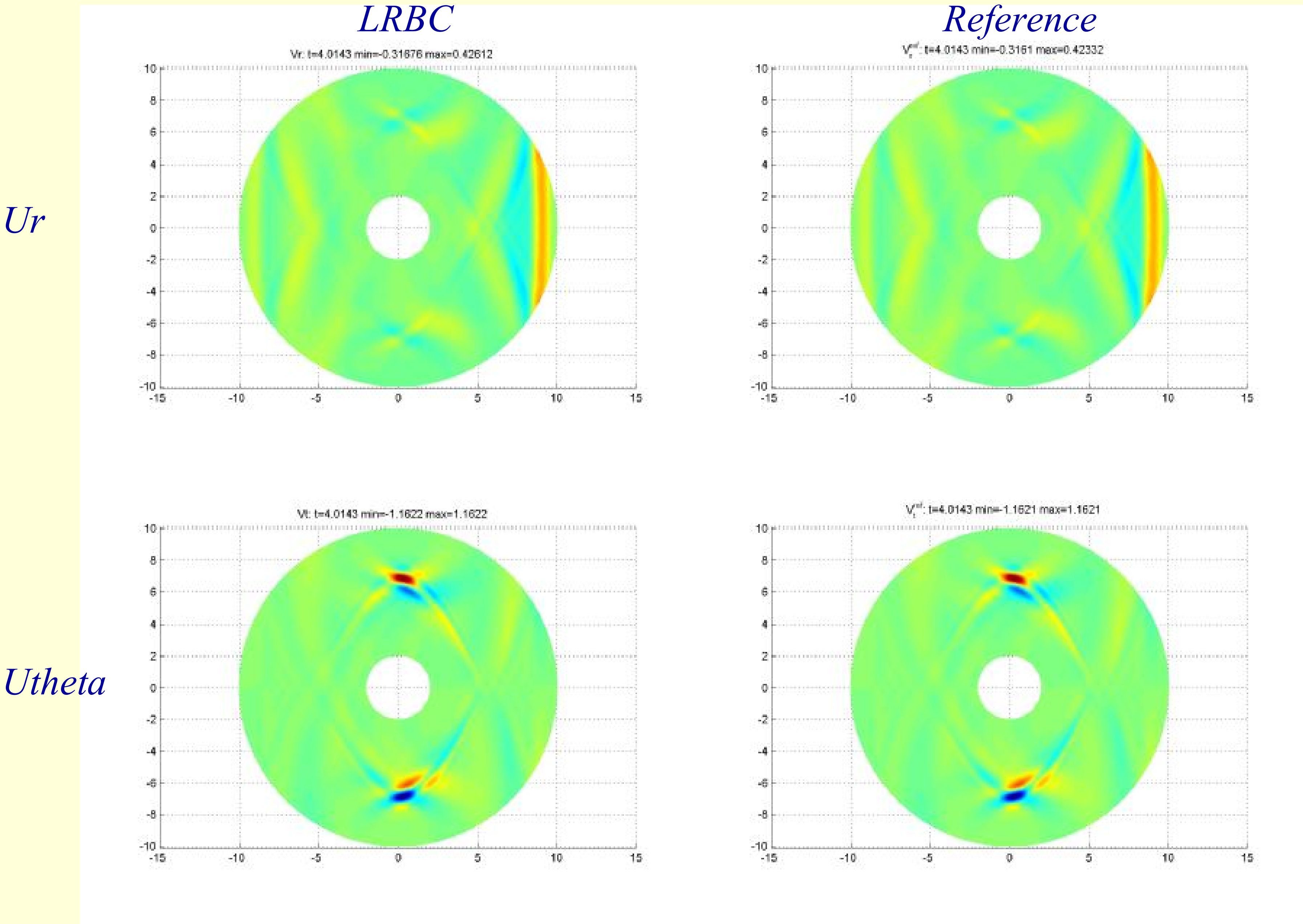
LRBC   *Reference*

Ur

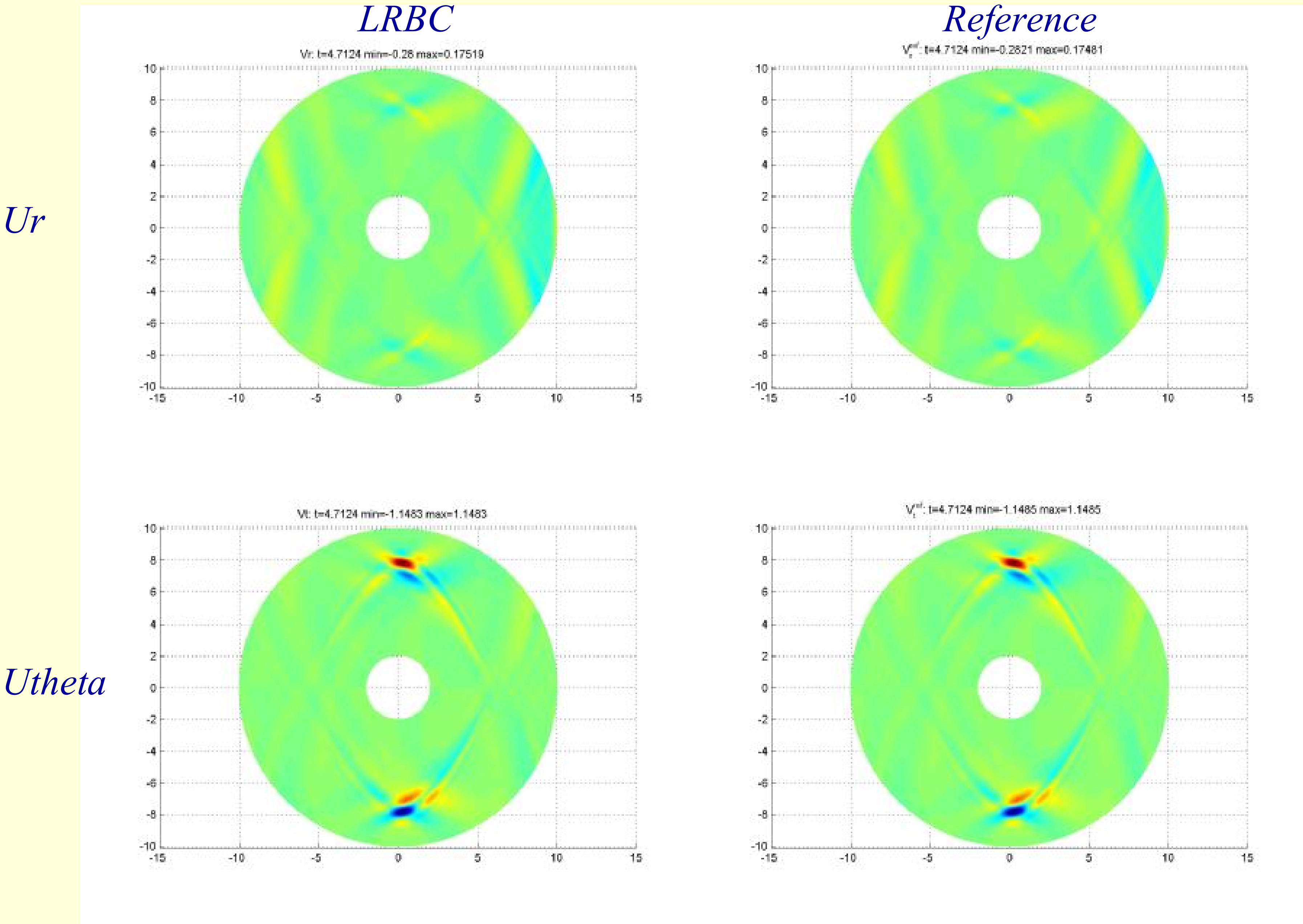
Utheta



Anisotropic case-IV,   t=4.01



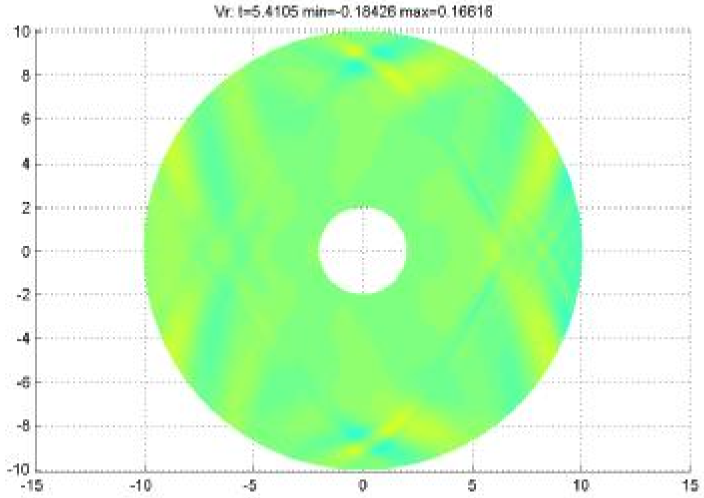
Anisotropic case-IV,   t=4.71

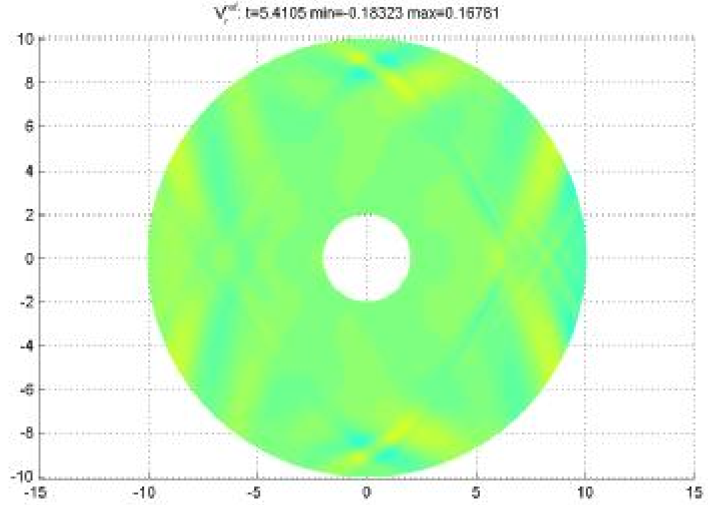


Anisotropic case-IV,   t=5.41

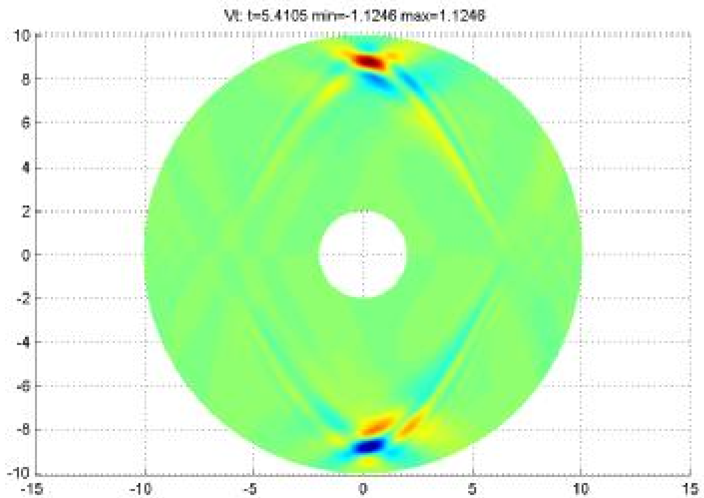
LRBC   *Reference*

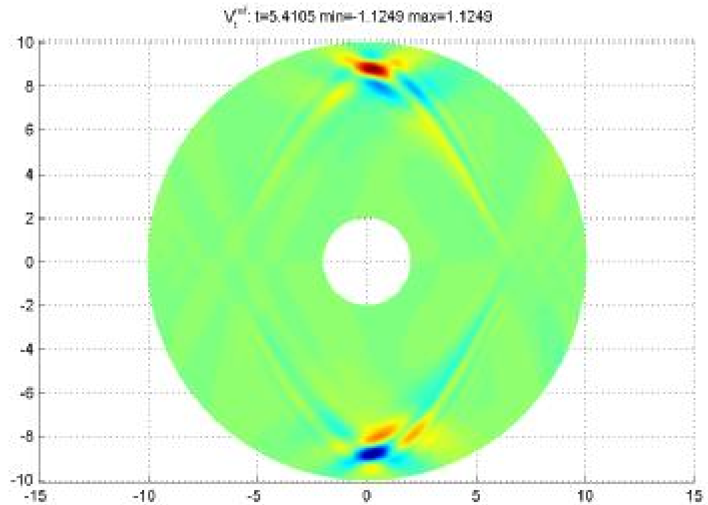
Ur





Utheta



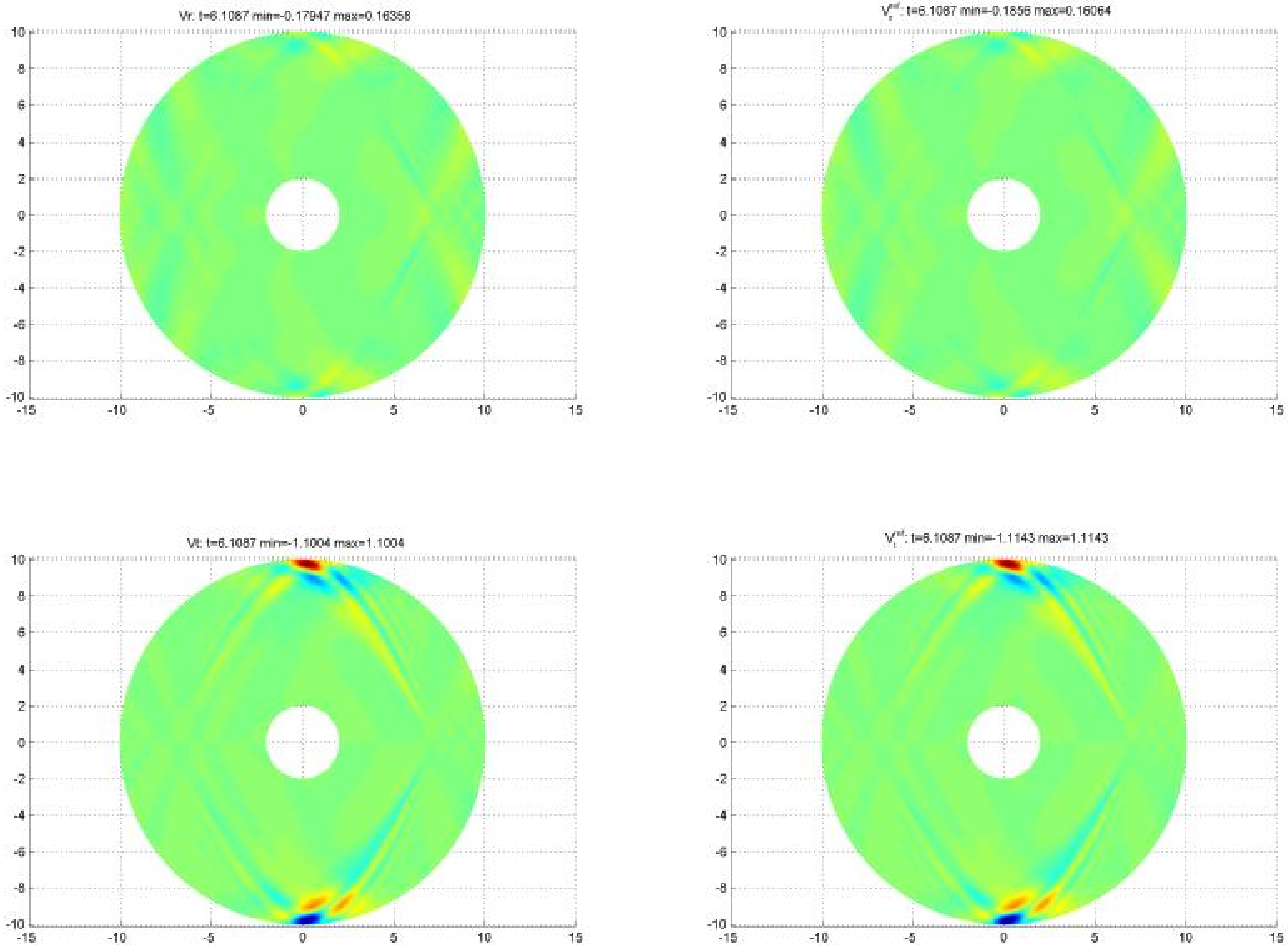


Anisotropic case-IV, t=6.11

LRBC *Reference*

Ur

Utheta

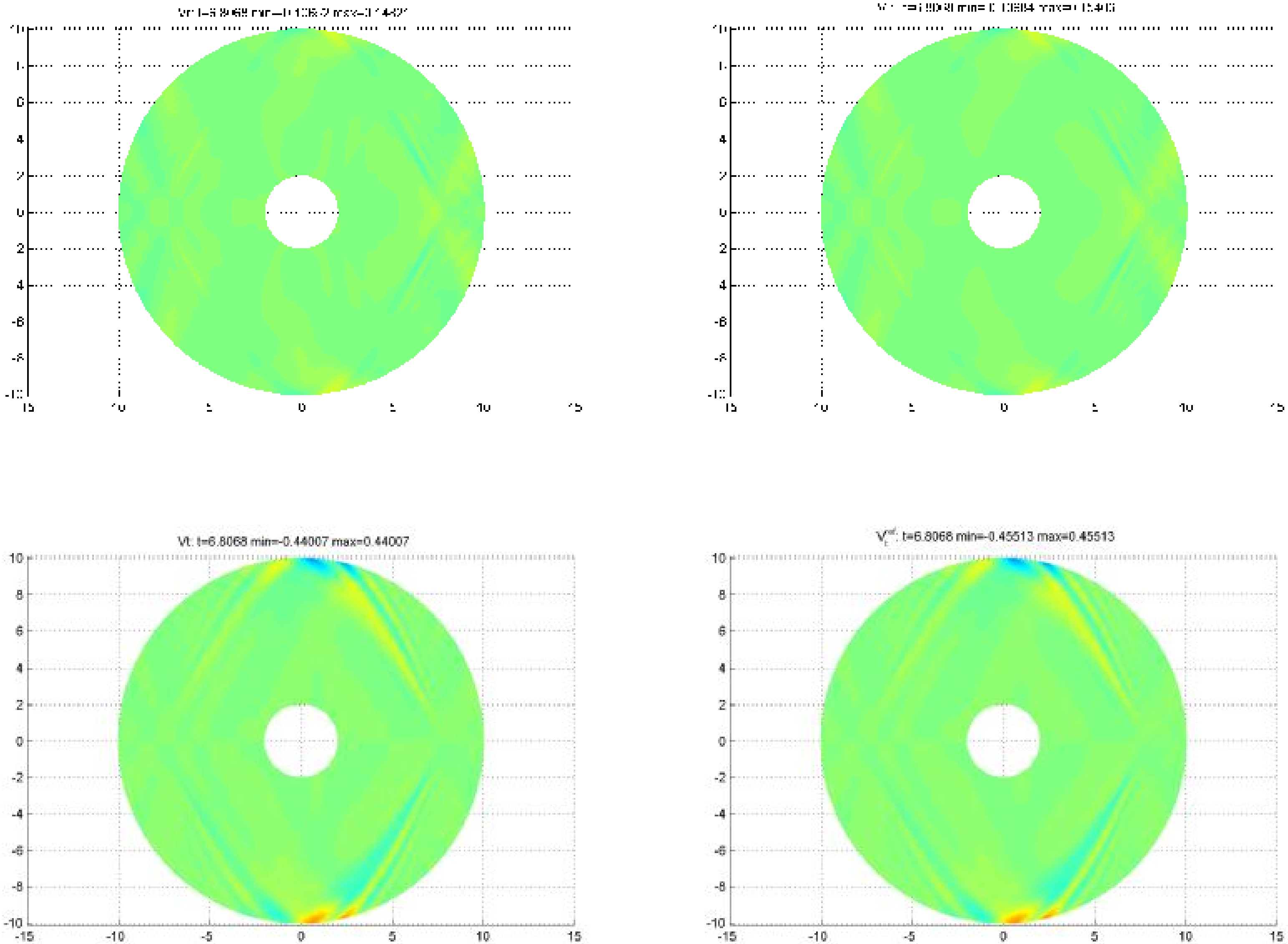


Anisotropic case-IV,   t=6.81

LRBC *Reference*

Ur

Utheta

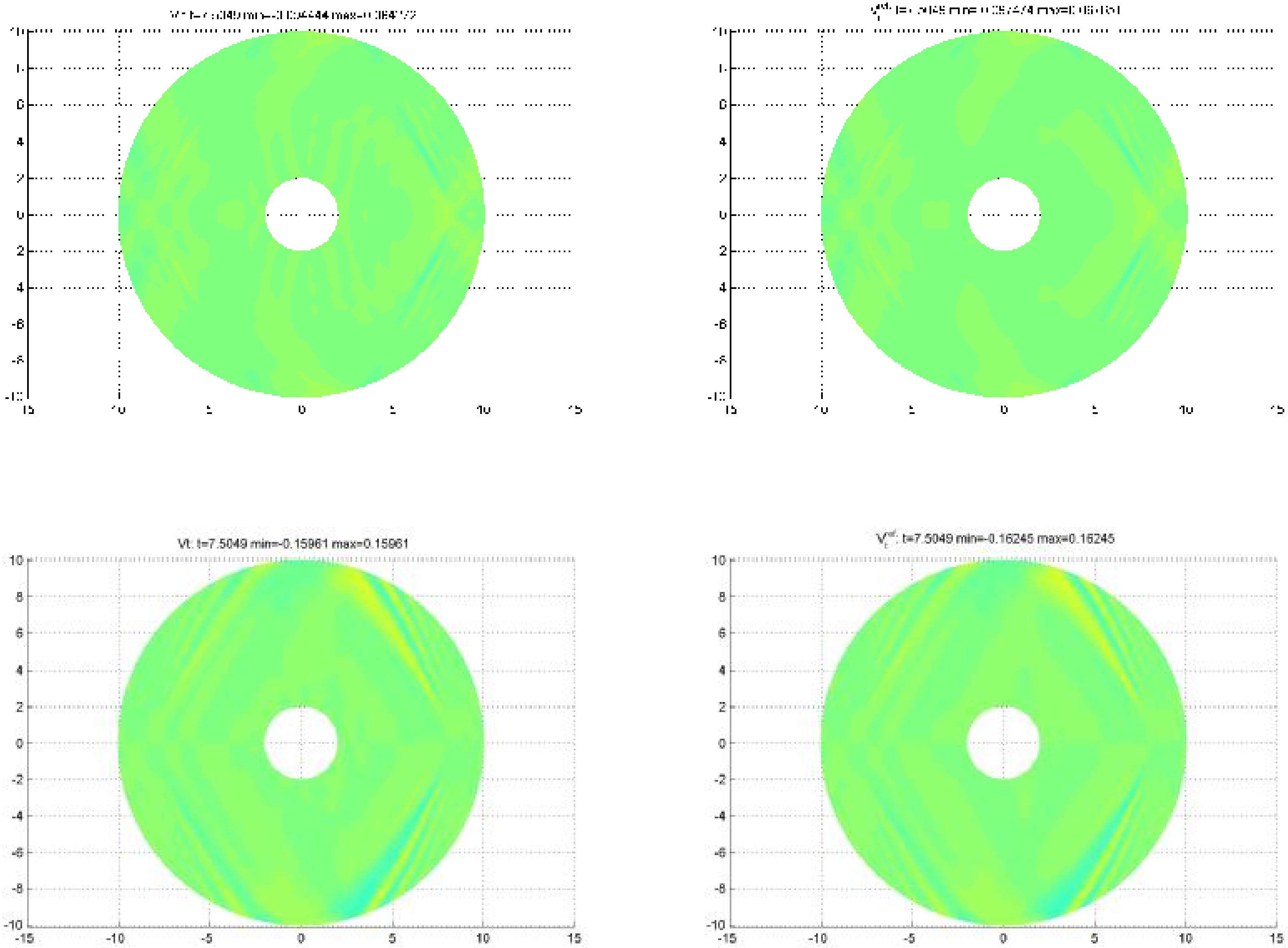


Anisotropic case-IV, t=7.50

LRBC *Reference*

Ur

Utheta

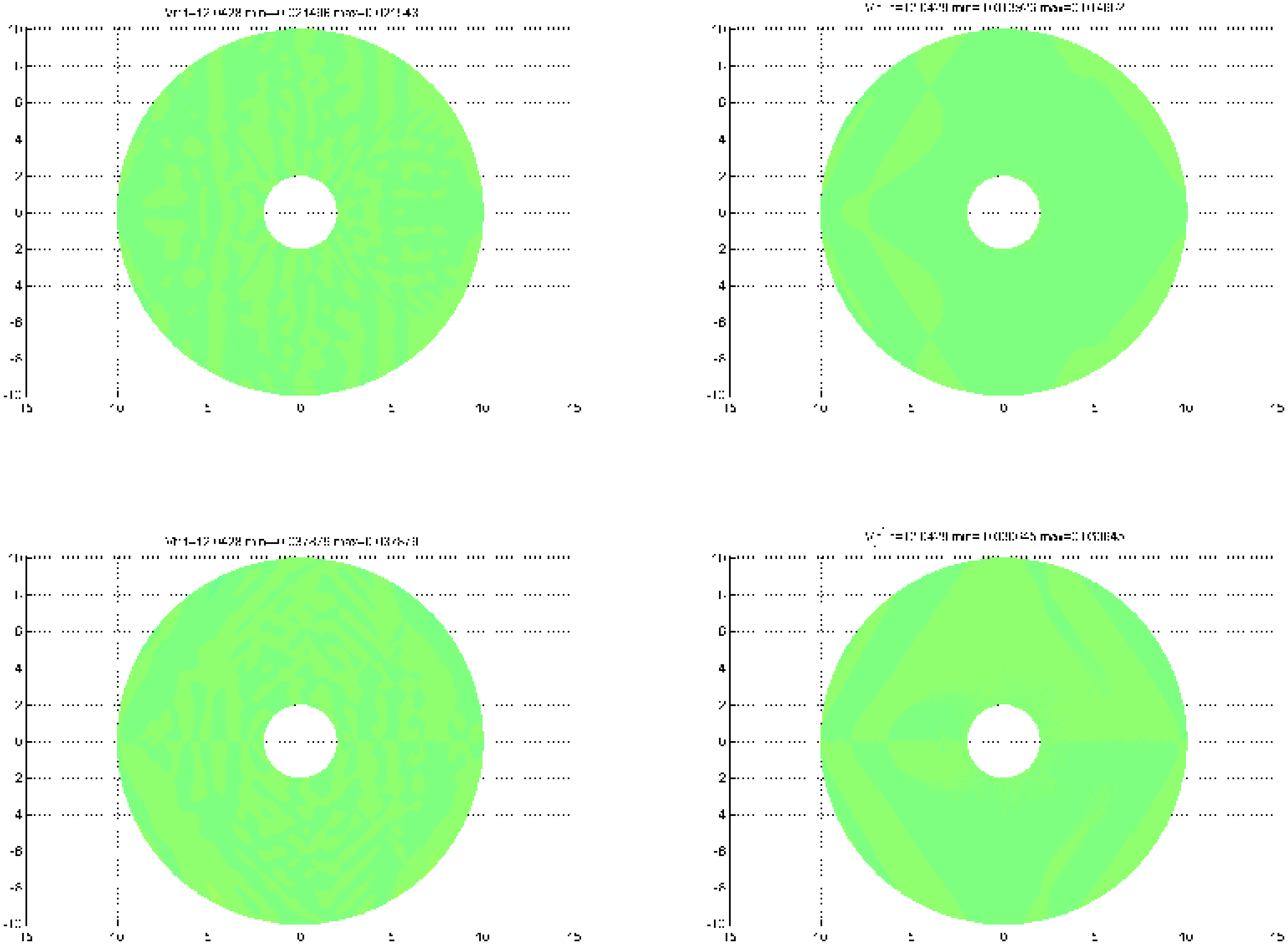


Anisotropic case-IV, t=12.0

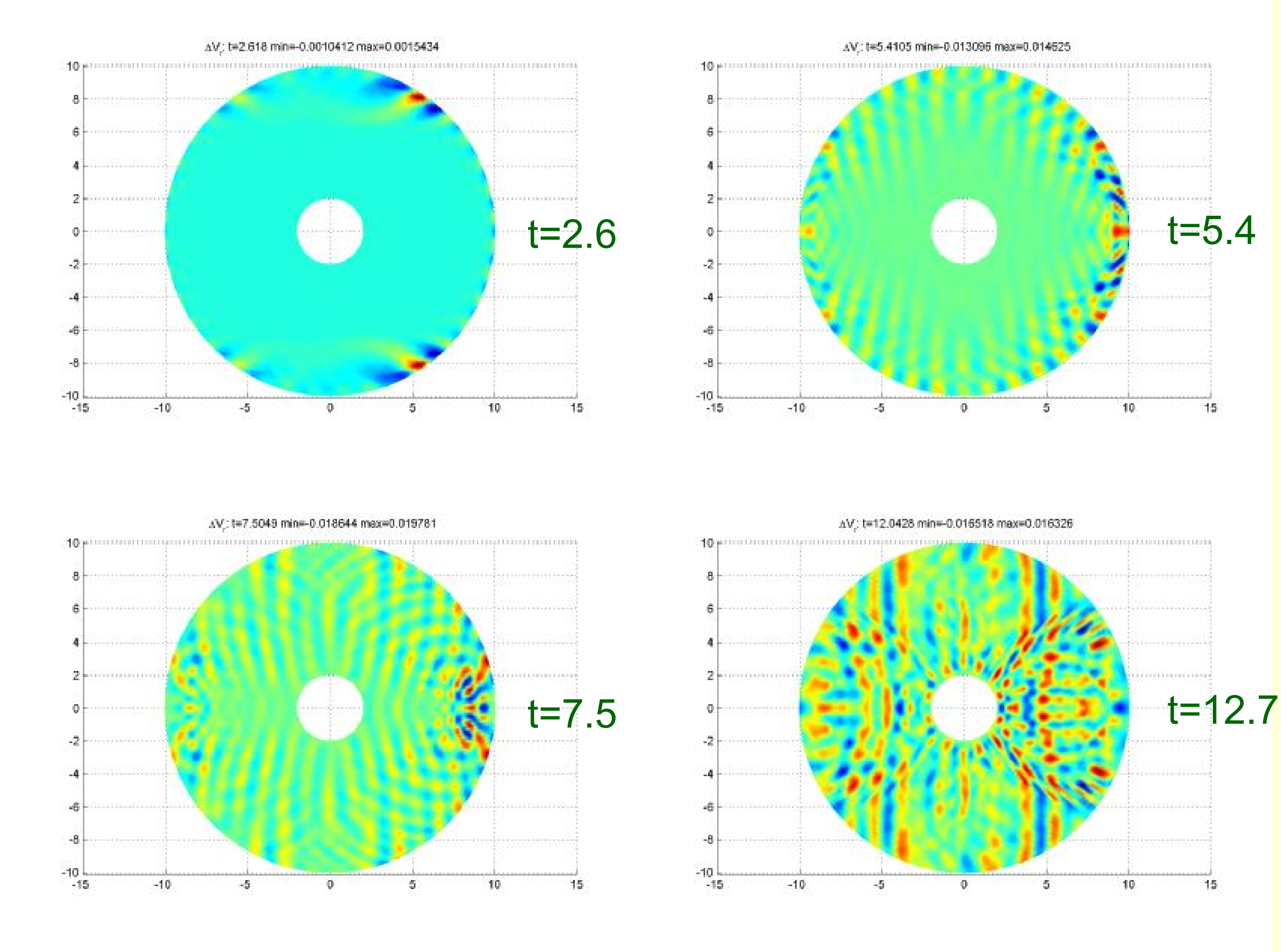
LRBC *Reference*

Ur

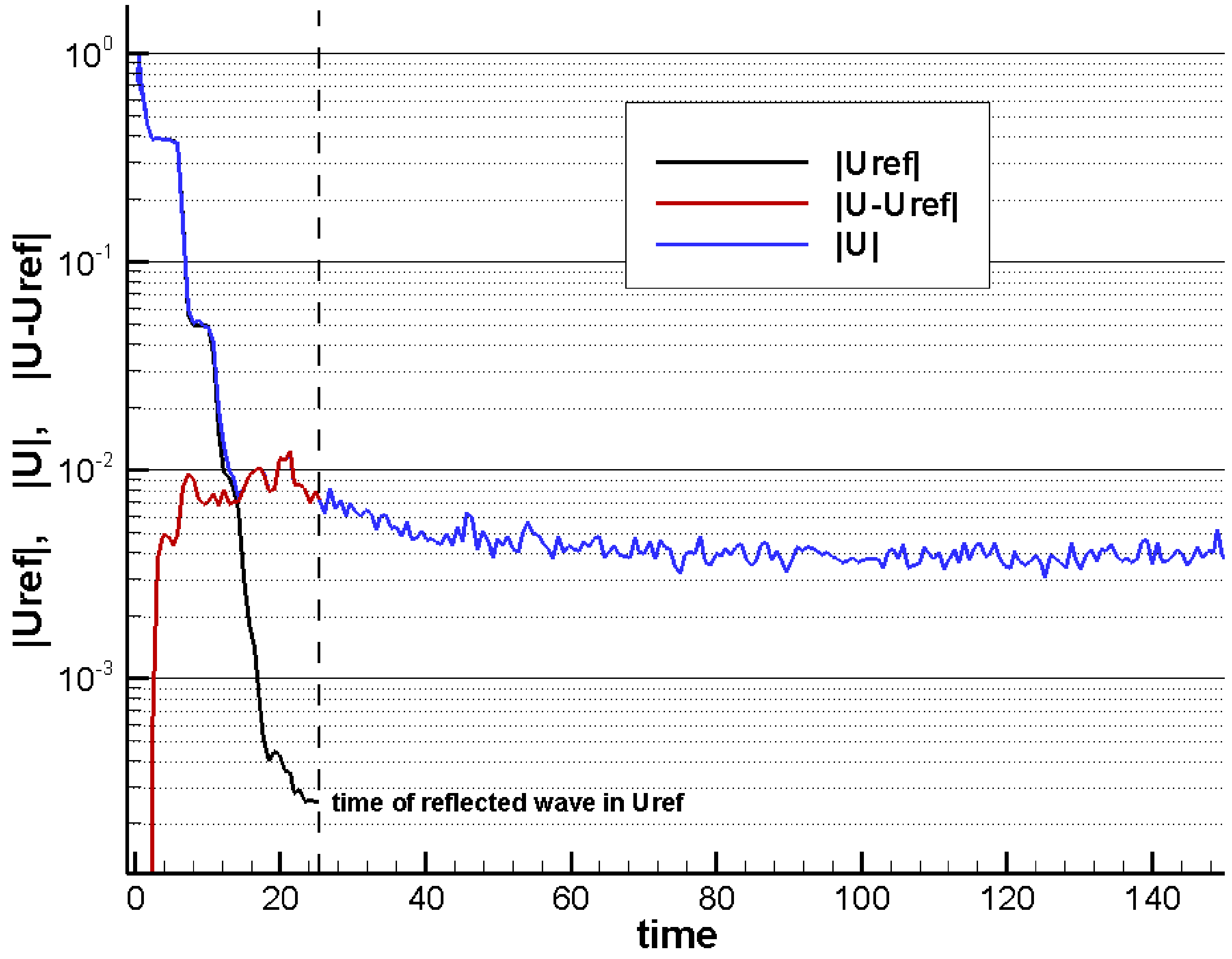
Utheta



Difference between LRBC and Reference solution



LRBC stability at large times



Conclusions

- A novel approach to generate numerical LRBCs for anisotropic time-domain wave propagation problems is proposed and implemented in 2D

- LRBCs are efficient in both isotropic and anisotropic media, including the case where PML approach fails (case IV)

- LRBC operator does not depend on the meshing inside the computational domain

- LRBC efficiency (amplitude of reflected waves) can be automatically controlled during generation of the operator

- It is possible to generate in advance a library of LRBC matrices for given media parameters and geometry of computational domain

- The algorithm is highly parallelized

Further research

- The approach is still computationally expensive and must be revised and improved to enhance the performance

- Real*16 accuracy

- 3D problems

LRBC operator, numerical aspects

Main features of the algorithm for *discretized system:*

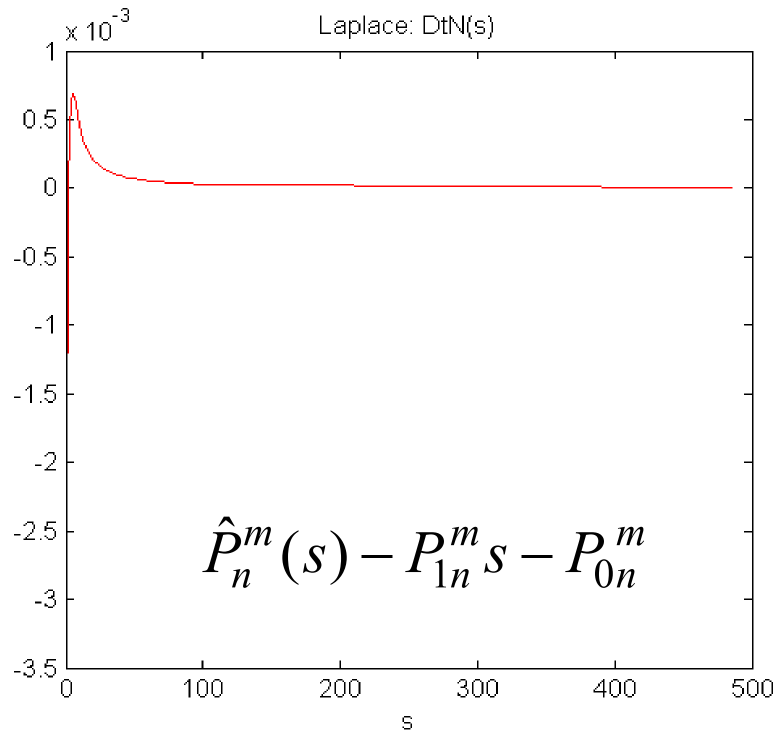
Stages 1–6: we take discrete azimuth basis $\{\sin k\theta, \cos k\theta\}, k = 0,1,\ldots,M;\ (M = 20);$

Stage 2 [Laplace image]: we take a finite interval $[0,\ S_{\max}]$ and choose set of knots $\{s_j\} \in [0, S_{\max}]$ which are representative enough to consider discrete counterparts of kernels $\hat{P}_n^m(s_j)$ for rational approximation (Stage 5).

In particular, the Chebyshev's nodes are used:

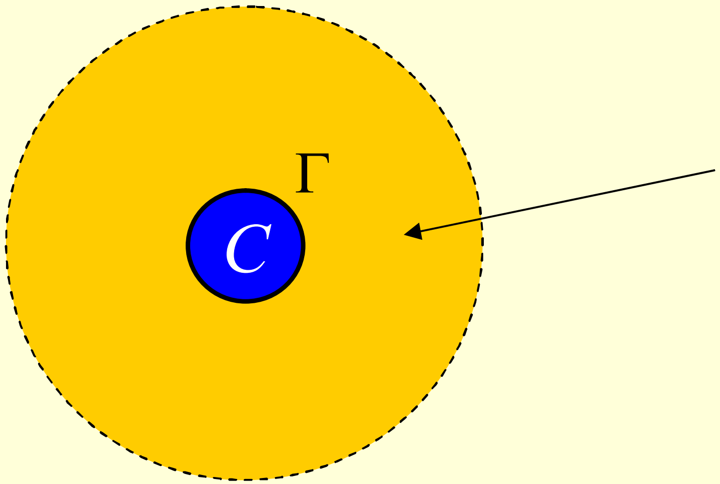
$$s_j = \frac{S_{\max}}{2}\left(1 - \cos\left(\pi \frac{j - 0.5}{J}\right)\right), \quad j = 1, \ldots, J$$

$J \sim 100$



LRBC operator, numerical aspects (cont.)

Stages 3-4 [P-S operator]: we discretise governing equations by *pseudospectral* derivatives in azimuth direction (uniform grid) and *finite differences* in radial direction (exponential grid) and make massive calculations to evaluate each $\hat{P}_n^m(s_j)$ with a guarantied accuracy, $\varepsilon_{PS} \sim 10^{-10}$ (controlled by mesh convergence). Number of separate tasks is $4*(M+1)*J;$



$$\begin{cases} s^2\hat{\mathcal{E}}^m - L\hat{\mathcal{E}}^m = 0 \ \text{ in } R^2/C \\ \hat{\mathcal{E}}^m\,|_\Gamma = \varphi^m(\theta) \\ \hat{\mathcal{E}}^m\,|_{r\to\infty} = 0 \end{cases}$$

Thus we form the matrix $\hat{\mathbf{P}}(s_j).$

LRBC operator, numerical aspects (cont.)

Stage 5 [inverse Laplace transform]:

Constant matrices $\mathbf{P}_1,\ \mathbf{P}_0$ are estimated from the rational approximations

$$R_n^m(s_j) \approx \hat{P}_n^m(s_j)$$

on the interval $[0, S_{\max}]$ $\left(R_n^m(s)\right.$ is calculated by Chebyshev-Pade algorithm$\left.\right)$,

and the reminder matrix is formed:

$$\hat{\mathbf{K}}(s_j) := \hat{\mathbf{P}}(s_j) - \mathbf{P}_1 s_j - \mathbf{P}_0$$

Rational approximations are calculated for each $\hat{K}_n^m(s_j)$:

$$\left\| \sum_{l=1}^{L} \frac{\alpha_l}{s_j - \beta_l} - \hat{K}(s_j) \right\|_{\{s_j\}}^2 \to \min_{\alpha_l, \beta_l} ,\ \ \mathrm{Re}(\beta_l) \le \delta < 0$$

(indices $\substack{m \\ n}$ at $\hat{K}, L, \boldsymbol{\alpha}, \boldsymbol{\beta}$ are omitted); *L=8* .

The minimization is made by an optimization algorithm.

LRBC operator, numerical aspects (cont.)

As a result we obtain the rational functions $\hat{\tilde{K}}_n^m(s)$ satisfying

$$|\hat{K}_n^m(s_j) - \hat{\tilde{K}}_n^m(s_j)| < \varepsilon_R \qquad (\varepsilon_R \sim 10^{-8})$$

They are explicitly inverted from the Laplace space:

$$\hat{\tilde{K}}_n^m(s) \mapsto \tilde{K}_n^m(t) \equiv \sum_{l=1}^{L_n^m} \alpha_{n\ l}^{m} \exp(\beta_{n\ l}^{m} t), \qquad \mathrm{Re}(\beta_{n\ l}^{m}) \le \delta < 0$$

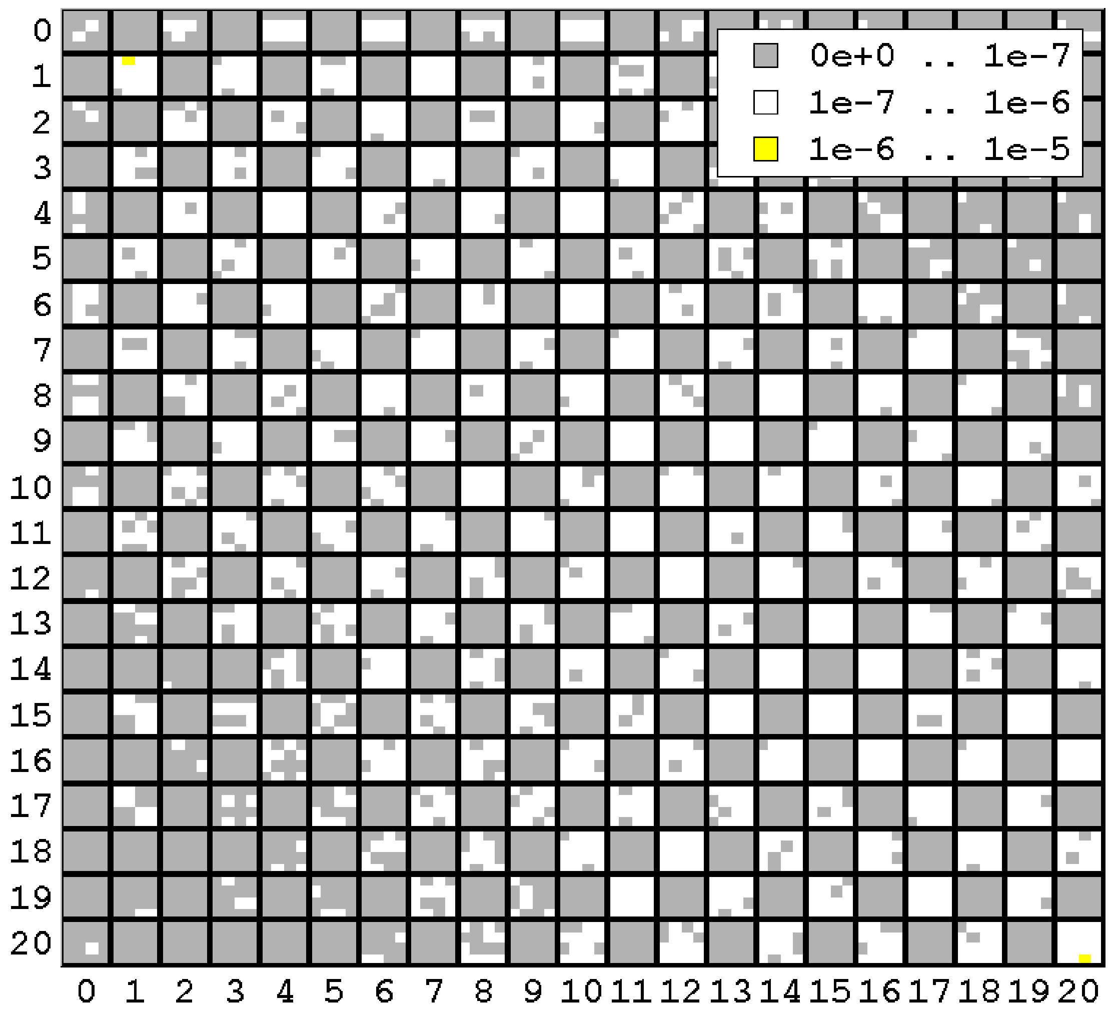
Stage 6 [discrete NRBC]: introducing matrices $\mathbf{Q}_M^{-1}, \mathbf{Q}_M$ of the discrete Fourier transform for vector-functions $f = (v_r, v_\theta)$ in *sin-cos* basis, we write out our discrete BC:

$$\mathbf{Q}_M^{-1}\mathbf{P}_1\mathbf{Q}_M \frac{\partial f}{\partial t} - \frac{\partial f}{\partial n} + \mathbf{Q}_M^{-1}\mathbf{P}_0\mathbf{Q}_M f + \mathbf{Q}_M^{-1}\left\{\tilde{\mathbf{K}}(t) *\right\}\mathbf{Q}_M f = 0$$

Remark: in isotropic case $\mathbf{P}_1 = p_1\mathbf{I}$, $\mathbf{P}_0 = p_0\mathbf{I}$, $\tilde{\mathbf{K}}(t)$ is diagonal

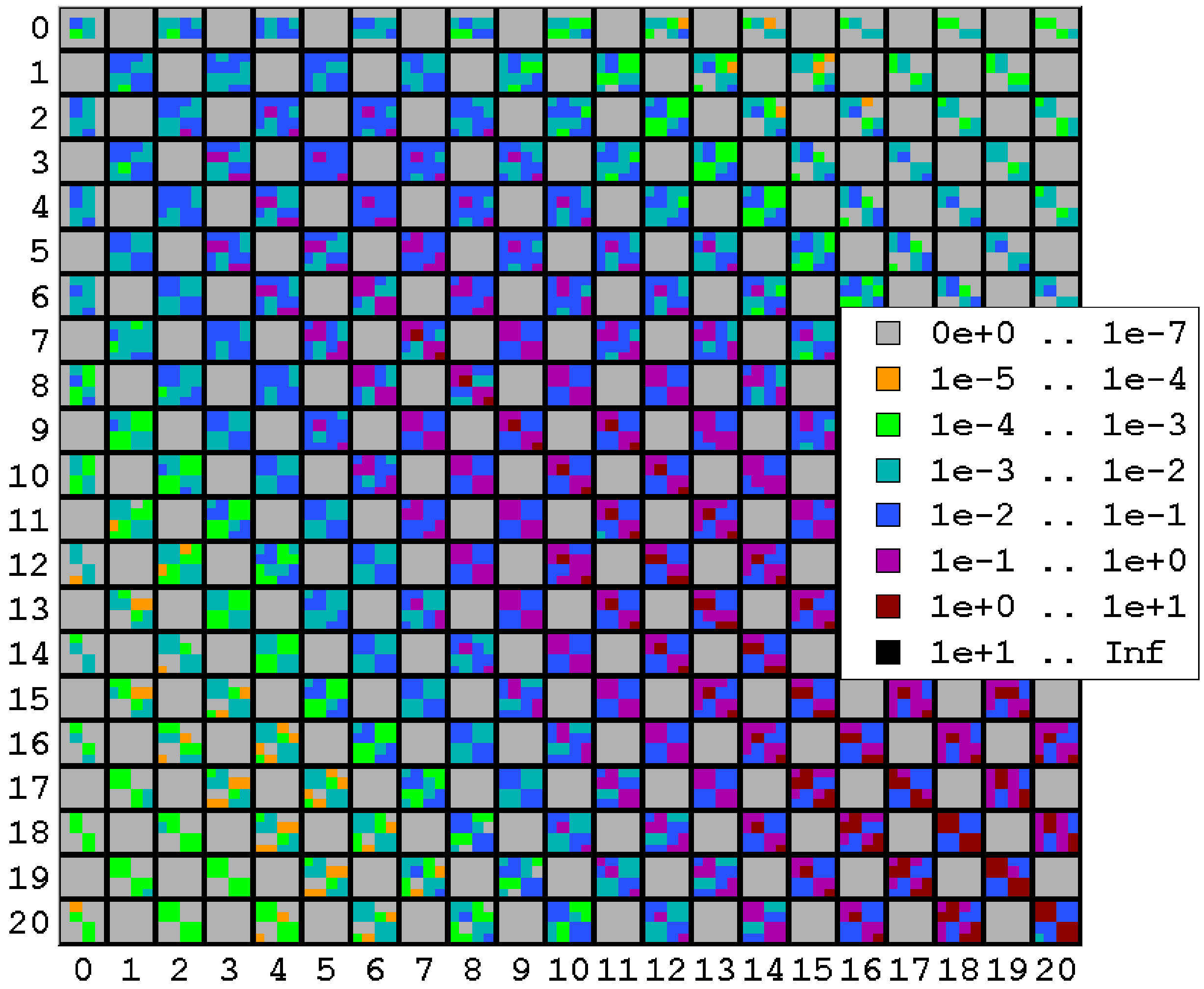
LRBC matrix, residual $\max_{s_j} | \hat{K}_n^m(s_j) - \hat{\tilde{K}}_n^m(s_j) |$

$$\mathbf{Q}_M^{-1}\mathbf{P}_1\mathbf{Q}_M \frac{\partial f}{\partial t} - \frac{\partial f}{\partial n} + \mathbf{Q}_M^{-1}\mathbf{P}_0\mathbf{Q}_M f + \mathbf{Q}_M^{-1}\left\{\tilde{\mathbf{K}}(t)*\right\}\mathbf{Q}_M f = 0$$



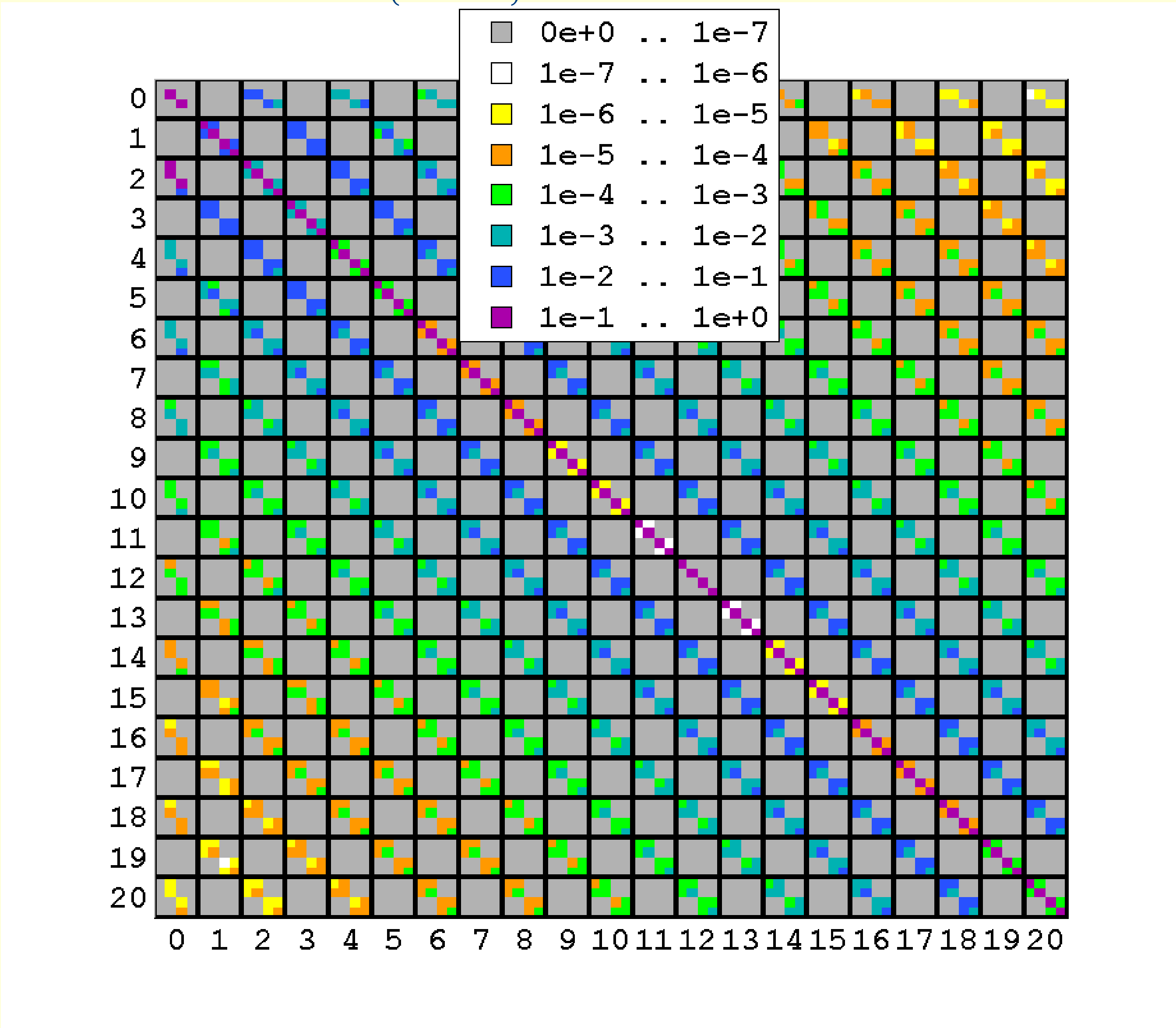
LRBC matrix, amplitude $\max_t |\tilde{K}_n^m(t)|$

$$\mathbf{Q}_M^{-1}\mathbf{P}_1\mathbf{Q}_M \frac{\partial f}{\partial t} - \frac{\partial f}{\partial n} + \mathbf{Q}_M^{-1}\mathbf{P}_0\mathbf{Q}_M f + \mathbf{Q}_M^{-1}\left\{\tilde{\mathbf{K}}(t) *\right\}\mathbf{Q}_M f = 0$$



LRBC matrix, amplitude $|P_{1n}^{m}|$

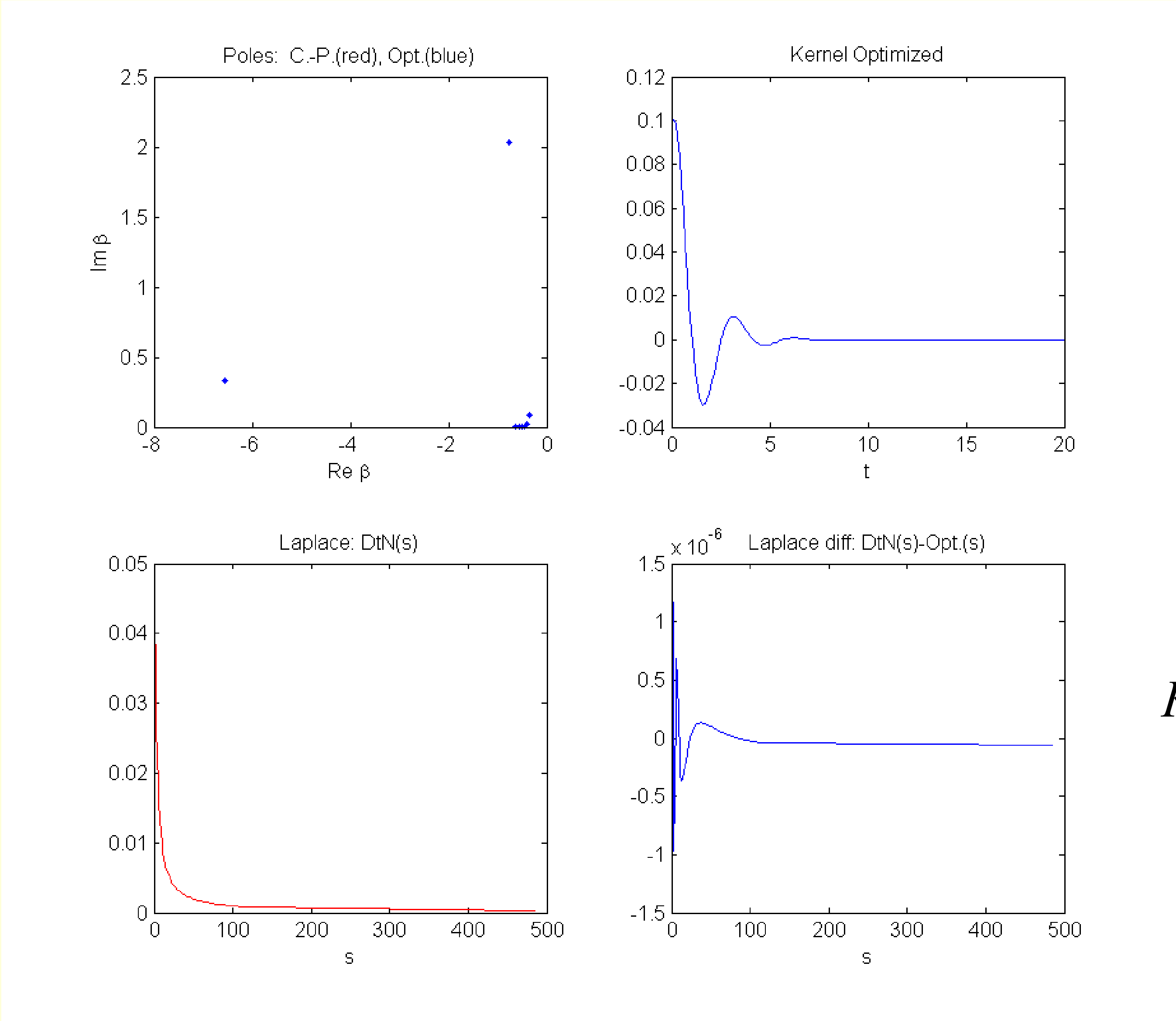
$$\mathbf{Q}_M^{-1}\mathbf{P}_1\mathbf{Q}_M \frac{\partial f}{\partial t} - \frac{\partial f}{\partial n} + \mathbf{Q}_M^{-1}\mathbf{P}_0\mathbf{Q}_M f + \mathbf{Q}_M^{-1}\left\{\tilde{\mathbf{K}}(t)*\right\}\mathbf{Q}_M f = 0$$



LRBC matrix, entry $n=(14,1)\rightarrow m=(18,3)$

poles of $\hat{\tilde{K}}_n^m(s)$

$\tilde{K}_n^m(t)$



$\hat{K}_n^m(s)$

$\hat{K}_n^m(s)-\hat{\tilde{K}}_n^m(s)$