Eberlein oligomorphic groups

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When topological dynamics meets model theory

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The Fourier–Stieltjes algebra

Let G be a topological group. We denote by $C(G)$ the algebra of complex-valued bounded continuous functions on G.

A function $f \in C(G)$ is positive definite if

$$\sum_{ij} c_i \overline{c_j} f(g_j^{-1} g_i) \geq 0$$

for every $g_1, \ldots, g_n \in G$ and $c_1, \ldots, c_n \in \mathbb{C}$.

The linear span of the family of positive definite functions on G is denoted by $B(G)$. It is actually a subalgebra of $C(G)$, the Fourier–Stieltjes algebra of G.

The Fourier–Stieltjes algebra

Fact (Gelfand–Naimark–Segal construction)

The following are equivalent:

1. $f \in B(G)$.
2. *There is a continuous unitary representation* $\pi : G \to \mathcal{U}(\mathcal{H})$ *and vectors* $v, w \in \mathcal{H}$ *such that, for every* $g \in G$,

$$f(g) = \langle v, \pi(g)w \rangle.$$

The WAP algebra

A function $f \in C(G)$ is weakly almost periodic if the orbit Gf is weakly precompact in $C(G)$. They form an algebra, $\mathrm{WAP}(G)$.

Fact (Grothendieck's double limit criterion, Megrelishvili's reflexive representation theorem)

The following are equivalent:

1. $f \in WAP(G)$.
2. *For any sequences $g_i, h_j \in G$ we have (whenever both limits exist)*
$$\lim_i \lim_j f(g_i h_j) = \lim_j \lim_i f(g_i h_j).$$
3. *There exists a continuous reflexive representation $\pi : G \to Iso(V)$ and vectors $v \in V$, $w \in V^*$ such that*
$$f(g) = \langle v, \pi(g) w \rangle \text{ for all } g \in G.$$

Eberlein groups

It follows that $B(G) \subset \mathrm{WAP}(G)$. However, $\mathrm{WAP}(G)$ is always closed in the norm topology of $C(G)$, whereas $B(G)$ is almost never closed.

Definition

A topological group G is Eberlein if $\overline{B(G)} = \mathrm{WAP}(G)$.

Examples

- Compact groups are Eberlein (Peter–Weyl).
- The group $\mathbb{Z}$ is *not* Eberlein (Rudin). Neither is any locally compact noncompact nilpotent group (Chou).
- Eberlein groups include $SL_n(\mathbb{R})$ (Veech), $\mathcal{U}(\ell^2)$ (Megrelishvili), $\mathrm{Aut}([0,1],\mu)$ (Glasner) or $S(\mathbb{N})$ (Glasner–Megrelishvili).

The algebra UC(G)

A function $f \in C(G)$ is UC if for every $\epsilon > 0$ there is a neighborhood $1 \in U \subset G$ such that

$$|f(ugu') - f(g)| < \epsilon$$

for every $g \in G$ and $u, u' \in U$. We have $\mathrm{WAP}(G) \subset \mathrm{UC}(G)$.

Definition

We say that G is a WAP group if $\mathrm{WAP}(G) = \mathrm{UC}(G)$, and that it is strongly Eberlein if $\overline{B(G)} = \mathrm{UC}(G)$.

Problem (Glasner–Megrelishvili)

Show a WAP group that is not Eberlein.

Oligomorphic groups

A topological group G is oligomorphic if it can be presented as a closed permutation group $G \leq S(X)$ of a countable set whose orbit spaces X^n/G are finite for every n.

Equivalently: $G = \mathrm{Aut}(M)$ for some $\aleph_0$-categorical classical structure M (Ryll-Nardzewski).

Generalization: closed groups of isometries $G \leq \mathrm{Iso}(X)$ of Polish metric spaces with compact closed-orbit spaces $X^n \,/\!\!/\, G$ are exactly the Roelcke precompact Polish groups (Ben Yaacov–Tsankov, Rosendal).

Equivalently: $G = \mathrm{Aut}(M)$ for some $\aleph_0$-categorical metric structure M.

Motivation

A number of tools are available for oligomorphic groups.

- Unlike many other cases, $B(G)$ is separable.
  ($\mathrm{UC}(G)$ is separable.)
- We have a Classification Theorem for unitary representations of oligomorphic groups (Tsankov).
- We have a model-theoretic interpretation of the WAP semigroup compactification (Ben Yaacov–Tsankov).

Motivation

Let M be an $\aleph_0$-categorical metric structure, $G = \mathrm{Aut}(M)$.

Algebras	Matrix coefficients	Formulas
$f \in A$	$f(g) = \langle v, \pi(g)w \rangle$, $\pi : G \to \mathrm{Iso}(V)$	$f(g) = \varphi(a, gb)$

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Formulas generating $B(G)$ for oligomorphic G

Let M be a classical $\aleph_0$-categorical structure, $G = \mathrm{Aut}(M)$.
The first basic observation is the following.

Lemma

If a formula $\varphi(x, y)$ defines an equivalence relation on M^n (more generally, if $\varphi(x, b)$ defines a weakly normal set), then $g \mapsto \varphi(a, gb)$ is in $B(G)$.

Proof.

We have $\varphi(a, gb) = \langle e_{[a]_\varphi}, \pi(g) e_{[b]_\varphi} \rangle$ for the natural map $\pi : G \to \mathcal{U}(\ell^2(M^n/\varphi))$. $\square$

Before we give a converse to this statement we recall the general form of the unitary representations of G.

Classification theorem for unitary representations of oligomorphic groups

Fact (Tsankov)

Let G be an oligomorphic group.

- *Every unitary representation of G is a direct sum of irreducible representations.*
- *Every irreducible unitary representation is a subrepresentation of the quasi-regular representation $\pi_V : G \to \mathcal{U}(\ell^2(G/V))$ for some open subgroup $V \leq G$.*

Remark: the matrix coefficients induced by π_V are generated by the basic ones

$$g \mapsto \langle e_{h_0 V}, \pi_V(g) e_{h_1 V} \rangle \quad \big(= \langle e_{h_0 V}, e_{g h_1 V} \rangle \big).$$

Formulas generating $B(G)$ for oligomorphic G

Now, every open subgroup $V' \leq G$ is the stabilizer of an imaginary element of M: there is a definable equivalence relation $\varphi(x, y)$ and a tuple $b \in M^n$ such that $V' = \{g \in G : M \models \varphi(b, gb)\}$.

Applying this to $V' = h_1 V h_1^{-1}$ and taking $a = h_0 h_1^{-1} b$, we have

$$\langle e_{h_0 V}, \pi_V(g) e_{h_1 V} \rangle = \varphi(a, gb).$$

We obtain the following:

Proposition

$\overline{B(G)}$ is the closed algebra generated by the functions $g \mapsto \varphi(a, gb)$ where $\varphi(x, y)$ is a definable equivalence relation on M.

Semitopological semigroup compactifications

A semitopological semigroup compactification of G is a compact semitopological semigroup S together with a continuous homomorphism $\alpha : G \to S$ with dense image.

There is a one-to-one correspondence:

closed G-bi-invariant subalgebras of $\mathrm{WAP}(G)$	$\leftrightarrow$	semitopological semigroup compactifications of G
$A \subset \mathrm{WAP}(G)$	$\mapsto$	maximal ideal space of A
functions $f \in C(G)$ that factor through α	$\leftarrow\!\shortmid$	$\alpha : G \to S$
inclusions	$\leftrightarrow$	quotients

The WAP and Hilbert compactifications

In particular, the compactifications $G \to W$ and $G \to H$ corresponding to $\mathrm{WAP}(G)$ and $\overline{B(G)}$ have the structure of semitopological semigroups. We have a continuous surjective commuting homomorphism $W \to H$.

Moreover, they are semitopological *-semigroup compactifications, that is, they admit continuous involutions

$$^* : W \to W \text{ and } ^* : H \to H$$

extending the inverse function on the image of G.

Representations of semigroups

If V is a reflexive Banach space, we denote by $\Theta(V)$ the compact semitopological semigroup of linear contractions of V:

$$\Theta(V) = \{T \in L(V) : \|T\| \leq 1\}.$$

Fact (Shtern)

Every compact semitopological semigroup can be embedded in $\Theta(V)$ for some reflexive Banach space V.

Definition

A semitopological semigroup S is Hilbert representable if it can be embedded in $\Theta(\mathcal{H})$ for a Hilbert space $\mathcal{H}$.

Representations of semigroup compactifications

Fact

- *H is Hilbert representable.*
- *If S is a Hilbert representable semitopological semigroup compactification of G, then it is a quotient of H.*
- *G is Eberlein if and only if W is Hilbert representable.*

Question (Glasner–Megrelishvili)

Conversely, if a S is a semigroup quotient of H, is it Hilbert representable?

Theorem

Yes if G is oligomorphic.

Regular elements, inverse semigroups

Let S be a semigroup. An element $p \in S$ is regular if there is $q \in S$ such that $p = pqp$. If moreover $q = qpq$, then q is an inverse of p. S is an inverse semigroup if every element has a unique inverse.

E.g. the inverse semigroup of partial bijections of a set.

Fact

- *S is an inverse semigroup if and only if every element is regular and the idempotents commute.*
- *Let $G \to S$ be a semitopological *-semigroup compactification of G. The following are equivalent for any element $p \in S$.*
  1. *p is regular.*
  2. *p has a unique inverse.*
  3. *$p = pp^*p$.*

Stable independence, one-based structures

Let M be a saturated structure. A formula $\varphi(x, y)$ is stable if for every type $t \in S(M)$, the function $d_t\varphi : M^n \to \mathbb{C}$,

$$d_t\varphi(b) = \varphi(x, b)^t,$$

is M-definable.

Given sets $A, B, C \subset M^{\mathrm{eq}}$ (we fix an enumeration of A), we say that A is stably independent from C over B,

$$A \underset{B}{\downarrow} C,$$

if for every stable formula φ the type $\mathrm{tp}_\varphi(A/BC)$ extends to a type $t \in S(M)$ such that $d_t\varphi(y)$ is definable over $\mathrm{acl}^{\mathrm{eq}}(B)$.

We say that M is one-based for stable independence if for any algebraically closed sets $A, B \subset M^{\mathrm{eq}}$ we have

$$A \underset{A \cap B}{\downarrow} B.$$

Characterization

Theorem

Let G be an oligomorphic group, say $G = Aut(M)$ for an $\aleph_0$-categorical classical structure M.

- *H is the semigroup of partial elementary maps of M^{eq} with algebraically closed domain.*
  Equivalently, H is the closure of G in $\Theta(\ell^2(M^{eq}))$.
  In particular, H is an inverse semigroup (and so are all of its semigroup quotients).
- *The following are equivalent:*
  1. *W is an inverse semigroup.*
  2. *The idempotents of W commute.*
  3. *M is one-based for stable independence.*
  4. *G is Eberlein.*
- *G is strongly Eberlein if and only if M is $\aleph_0$-stable.*

Examples

Remark: $\Theta(\ell^2)$ is not an inverse semigroup (but it is the WAP compactification of the Eberlein Roelcke precompact group $\mathcal{U}(\ell^2)$).

Examples

- The groups $S(\mathbb{N})$, $\mathrm{Aut}(\mathbb{Q}, <)$, $\mathrm{Homeo}(2^\omega)$ and $\mathrm{Aut}(RG)$ are Eberlein oligomorphic groups.
- The automorphism group of Hrushovski's $\aleph_0$-categorical stable pseudoplane is a WAP group that is not Eberlein.

A model-theoretic description of W

Let $G = \mathrm{Aut}(M)$ where M is an $\aleph_0$-categorical metric structure. The left-completion $E = \widehat{G}_L$ is the semigroup of elementary embeddings $M \to M$. The UC-compactification coincides with $R = (E \times E) /\!\!/ G$. Then R can be seen as the space of types $[x, y]_R$ of pairs of embeddings.

The WAP-compactification is the quotient formed by the types $[x, y]$ restricted to stable formulas.

The *-semigroup structure of W is as follows:

- $[x, y]^* = [y, x]$.
- $[x, y][y, z] = [x, z]$ if $x \mathop{\smile}_y z$.

Restriction to equivalence relations

By our characterization of $\overline{B(G)}$ we have that H is the quotient formed by the types $[x, y]_H$ restricted to definable equivalence relations.

Then the map

$$[x, y]_H \mapsto x^{-1} \circ y$$

gives the identification of H with the semigroup of partial elementary maps $M^{\mathrm{eq}} \to M^{\mathrm{eq}}$ with algebraically closed domain.

Characterization of idempotents and regular elements

The key to the equivalences of the main theorem is the following description of idempotents and regular elements.

Lemma

Let $p = [x, y] \in W$.

- *p is an idempotent if and only if $x \equiv_{x \cap y} y$ and $x \mathop{\smile\!\!\!\!\mid}_{x \cap y} y$.*
- *p is regular if and only if $x \mathop{\smile\!\!\!\!\mid}_{x \cap y} y$.*

Merci beaucoup.