Introduction to model theory

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When topological dynamics meets model theory

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Overview

Model theory in 7 simple steps. (Syntax/Semantics)

1. Choose a signature σ: a list of basic symbols.
   Look at σ-structures: sets and relations interpreting σ.
2. Build a language $\mathcal{L}$: well-formed formulas using σ.
   Look at the definable sets on the structures.
3. Choose axioms (a theory, T): a set of statements from $\mathcal{L}$.
   Restrict to models of T (how many are there?).
4. Look at consistent sets of formulas.
   Finitely satisfiable conditions: types.
5. Invoke a monster (a structure realizing most types).
6. Look at definable groups and/or automorphism groups.

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   Finitely satisfiable conditions: types.
5. Invoke a monster (a structure realizing most types).
6. Look at definable groups and/or automorphism groups.
7. Do dynamics!

Signature

A signature is a list of

- relation symbols (basic predicates)
- and function symbols,

each with a prescribed arity (a natural number). Function symbols of arity 0 are called constants.

In continuous logic (CL), a modulus of uniform continuity is also prescribed.

Examples

- $\sigma_{\text{rings}} = \{+, -, \cdot, 0, 1\}$, where $+$, $-$, $\cdot$ are binary function symbols and 0, 1 are constants.
- $\sigma_{\text{graphs}} = \{R\}$, where R is a binary predicate.
- $\sigma_{\text{MALG}} = \{\mu, \Delta, \cap, \cdot^c, 0, 1\}$, where μ is a 1-Lipschitz unary predicate, Δ, $\cap$ are binary function symbols, $\cdot^c$ is a unary function symbol and 0, 1 are constants.

Structures

Fix a signature σ. A (classical) σ-structure M is a set (which we will also denote by M) together with interpretations for the symbols in σ:

- each n-ary basic predicate P is interpreted as a relation $P^M \subset M^n$;
- each n-ary function symbol f is interpreted as a function $f^M : M^n \to M$.

In CL: a metric σ-structure M is a bounded complete metric space; an n-ary predicate P is interpreted as a continuous function $P^M : M^n \to [0,1]$. Moreover, P^M and f^M must respect the given moduli of uniform continuity.

Examples

- Every ring or field is naturally a σ_{rings}-structure.
- A measure algebra (with the distance given by the measure of the symmetric difference) is naturally a σ_{MALG}-structure.
- Any complete bounded metric space is a structure over $\sigma = \emptyset$.

The first-order language

First-order formulas are well-formed expressions using the symbols of σ and the logical symbols: the equality relation, connectives, variables and quantifiers.

More formally, one starts by defining terms:

- every constant or variable is a term;
- if f is an n-ary function symbol and $t_0, \dots t_{n-1}$ are terms, then $f(t_0, \dots, t_{n-1})$ is a term.

Examples

- $x^2 + 2x - 1$ is a term in σ_{rings} (more formally, replace x^2 by $\cdot(x, x)$, 2 by $+(1, 1)$, etc).
- $x \cap y^c$ is a term in σ_{MALG}.

The first-order language

Then one defines basic formulas:

- if t and t' are terms, $t = t'$ is a basic formula;
- if P is an n-ary basic predicate and t is an n-tuple of terms, $P(t)$ is a basic formula.

In CL, $t = t'$ is replaced by $d(t, t')$.

Finally, the set $\mathcal{L}_\sigma$ of formulas is given as follows:

- basic formulas are formulas;
- if φ and ψ are formulas, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\neg\varphi$;
- if φ is a formula and x is a variable, then $\forall x \varphi$ and $\exists x \varphi$ are formulas.

In CL, connectives are replaced by any continuous combinations $[0,1]^n \to [0,1]$. Quantifiers are suprema and infima: $\sup_x \varphi$, $\inf_x \varphi$. One also considers forced limits of sequences of formulas.

The first-order language

Remark: Formulas may or may not have free variables (i.e. not quantified). Intuitively, in the first case they express properties, in the second they express statements.

Respectively, in CL, they express functions or statements of a numerical nature.

Examples

- $x^2 + 2x - 1 = 0$ ("x is a root of the polynomial $x^2 + 2x - 1$").
- $\exists x\ x^2 + 2x - 1 = 0$ ("the polynomial $x^2 + 2x - 1$ has a root").
- $\forall y_0 \forall y_1 \exists x\ x^2 + y_1 x + y_0 = 0$ ("every monic quadratic polynomial has a root").
- $\frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$ (the inner product in a real Hilbert space, in the language of Banach spaces).
- $\sup_x \inf_y |\mu(x \cap y) - \mu(x \cap y^c)|$ (the measure of the largest atom in a measure algebra).

Intepretation of formulas

Let φ be a σ-formula. We usually write $\varphi(x)$ to indicate that the free variables of φ are contained in x (a tuple of distinct variables).

Let M be a σ-structure and let $a \in M^{|x|}$. We write

$$\varphi^M(a)$$

for the truth value of $\varphi(x)$ on M when x is interpreted to denote the tuple a. Of course, quantifiers are interpreted as ranging over elements of M.

We omit the formal (recursive, natural) definition.

In CL, $\varphi^M(a)$ is a real number.

Satisfaction, definable sets

We write

$$M \models \varphi(a)$$

to say that $\varphi^M(a)$ is true.

A subset $D \subset M^n$ is definable if there is a formula $\varphi(x)$, $|x| = n$, such that

$$D = \{a \in M^n : M \models \varphi(a)\}.$$

Sometimes this set is denoted by $\varphi(M)$.

In CL one can think of truth as given by the value zero, then write

$$M \models \varphi(a)$$

to mean that $\varphi^M(a) = 0$. A function $P : M^n \to [0,1]$ is a definable predicate if there is a formula $\varphi(x)$ such that $\varphi^M = P$ as functions on M^n.

Definability with parameters

It is useful to admit parameters: if M is a σ-structure and $B \subset M$ is any subset, then $D \subset M^n$ is B-definable if there is a formula $\varphi(x, y)$ and a tuple $b \in B^m$ such that

$$D = \{a \in M^n : M \models \varphi(a, b)\}.$$

Equivalently: D is definable in the σ_B-structure M_B, where we have expanded σ to a signature σ_B with constants c_b for each $b \in B$, and M_B is just M with the obvious interpretation of this constants.

We denote the set of σ_B-formulas by $\mathcal{L}_\sigma(B)$. Thus, with a small abuse of notation, $\varphi(x, b) \in \mathcal{L}_\sigma(B)$.

Theories

A theory (on a given signature) is a set of statements (formulas with no free variables). A structure M is a model of a theory T, denoted $M \models T$, if each $\varphi \in T$ is true in M.

A theory T implies a statement φ if φ is true in every model of T:

$$\text{if } M \models T, \text{ then } M \models \varphi.$$

Each structure M induces a theory,

$$\mathrm{Th}(M) = \{\varphi : M \models \varphi\},$$

which is complete in the sense that, for every statement φ, either $T \models \varphi$ or $T \models \neg\varphi$.

Theories

Examples

- The theory of infinite sets is axiomatized by the statements

$$\varphi_n : \ \exists x_0 \ldots \exists x_{n-1} \bigwedge_{0 \leq i < j < n} x_i \neq x_j .$$

- The usual axioms of fields can be written in the first-order language of σ_{rings}.
- By adding the (infinitely many) axioms saying that 0 is different from 1, $1+1$, $1+1+1$, etc, and that every monic polynomial of degree $n \geq 2$ has a root, we obtain the theory of algebraically closed fields of characteristic 0, denoted by ACF_0.
- The theory of measure algebras is also first-order axiomatizable. Moreover, we have

$$M \models \sup_x \inf_y |\mu(x \cap y) - \mu(x \cap y^c)|$$

if and only if M is atomless.

Elementary extensions

Let M and N be two σ-structures such that $M \subset N$ as sets. Then N is an extension of M (or M is a substructure of N) if we have

$$P^M(a) = P^N(a),\ f^M(a) = f^N(a)$$

for every basic predicate P and function symbol f, and every tuple a from M.

In CL, M must be a metric subspace of N.

If moreover

$$\varphi^M(a) = \varphi^N(a)$$

for every $\varphi(x) \in \mathcal{L}_\sigma$, then N is an elementary extension of M, denoted $M \prec N$. In particular, if $M \prec N$ then $\mathrm{Th}(M) = \mathrm{Th}(N)$.

E.g.: as linear orders, $\mathbb{Q}$ is an extension of $\mathbb{Z}$ but $\mathbb{Z} \not\prec \mathbb{Q}$. Instead, $\mathbb{Q} \prec \mathbb{R}$.

Compactness

Let $\Gamma(x)$ be a set of σ-formulas with free variables from x.

$\Gamma(x)$ is satisfiable if there is an x-tuple a in some σ-structure M such that

$$M \models \Gamma(a).$$

We also say that $\Gamma(x)$ is realized by a.

$\Gamma(x)$ is finitely realized (in M) if every finite $\Delta(x) \subset \Gamma(x)$ is realized (by some tuple of M).

Theorem

If $\Gamma(x)$ is finitely realized (in M) then it is satisfiable (realized in some elementary extension of M, e.g. in an ultrapower of M).

Compactness

In CL, the same definition says that $\Gamma(x)$ is satisfiable (or realized in M) if for some a in some structure (resp., in M) we have $\varphi(a) = 0$ for every $\varphi \in \Gamma(x)$.

$\Gamma(x)$ is approximately finitely realized (in M) if for any $\epsilon > 0$ and finitely many formulas $\varphi_i(x) \in \Gamma(x)$, $i < n$, there is a tuple a (in M) such that

$$|\varphi_i(a)| < \epsilon$$

for every $i < n$.

(*Equivalently:* the closed ideal generated by $\{\varphi^M : \varphi \in \Gamma(x)\}$ in the space of real-valued continuous bounded functions $C(M^{|x|})$ is proper.)

Theorem

If $\Gamma(x)$ is approximately finitely realized (in M) then it is satisfiable (realized in some elementary extension of M, e.g. in an ultrapower of M).

Types

Fix a σ-structure M, $B \subset M$. A (partial) type in x over B in M is a set $\pi(x) \subset \mathcal{L}_\sigma(B)$ that is (approximately) finitely realized in M. When $|x| = n$, $\pi(x)$ is also called an n-type over B.

Given an x-tuple a in M, we define the type of a over B by

$$\mathrm{tp}(a/B) = \{\varphi \in \mathcal{L}_\sigma(B) : M \models \varphi(a)\}.$$

These are complete types: maximal for inclusion. That is, complete types over A are ultrafilters in the algebra of B-definable sets.

If $B \subset M \prec N$, then any set $\Gamma(x) \subset \mathcal{L}_\sigma(B)$ is (app.) finitely realized in M if and only if it is (app.) finitely realized in N. In particular, types over B in M or in N coincide.

By the compactness theorem, every type over $B \subset M$ is realized in some elementary extension of M.

Types, quantifier elimination

A theory has quantifier elimination if $\text{tp}(a/B)$ is determined by the basic formulas in $\mathcal{L}_\sigma(B)$ satisfied by a. Using that this is true for dense linear orders and for pure sets, we see that:

Examples

- There is only one 1-type over $\emptyset$ in $(\mathbb{Q}, <)$, only three 2-types over $\emptyset$, etc: a type over $\emptyset$ is determined by the order isomorphism type of a tuple that realizes it.
- The type $\{x \neq b : b \in \mathbb{N}\}$ is the only non-realized 1-type over $B = \mathbb{N}$ in the pure set $M = \mathbb{N}$.
- There as many non-realized 1-types over $B = \mathbb{Q}$ in $(\mathbb{Q}, <)$ as there are partitions $\mathbb{Q} = C \sqcup D$ with $c < d$ for every $c \in C$, $d \in D$.

Space of types

Fix $B \subset M$ as before. We denote by $S_x(B)$ the space of all complete types over B in the variable x, or alternatively $S_n(B)$ if $|x| = n$. It is a compact Hausdorff totally disconnected space with basic clopen sets

$$[\varphi] = \{p \in S_x(B) : \varphi \in p\}$$

for each $\varphi(x) \in \mathcal{L}_\sigma(B)$.

In CL, the space of complete types $S_x(B)$ can be seen as the maximal ideal space of the algebra of B-definable predicates on M, with its usual Gelfand topology (of course, here it need not be totally disconnected). In other words, $S_x(B)$ is the minimal compactification of M^n through which every function φ^M $(\varphi(x) \in \mathcal{L}_\sigma(B))$ factors.

Saturation

Let κ be an infinite cardinal. A structure M is κ-saturated if, for any $B \subset M$ of cardinality $|B| < \kappa$, every type in $S_1(B)$ is realized in M (*equivalently:* any n-type over B).

Examples

1. Every $\aleph_0$-categorical structure is $\aleph_0$-saturated.
2. A model of ACF_0 is $\aleph_0$-saturated if and only if it has infinite transcendence degree.

The monster

Fix a theory T. It is usual and convenient to work inside a fixed very saturated, homogeneous model of T containing all models of interest as elementary substructures.

More precisely, for an arbitrarily large cardinal κ one can find a model $\mathbb{M}$ (a monster model) such that:

- (call a set B small if $|B| < \kappa$)
- all small models of T are elementary embeddable in $\mathbb{M}$;
- every type over a small subset of $\mathbb{M}$ is realized in $\mathbb{M}$;
- every elementary map between small subsets of $\mathbb{M}$ can be extended to an automorphism of $\mathbb{M}$.

Definable groups

Let M be a structure. A definable group in M is given by definable sets $G \subset M^n$ and $\cdot \subset M^n \times M^n \times M^n$ such that

$$M \models \text{“}(G, \cdot) \text{ is a group”}.$$

We may abuse notation and identify G and $\cdot$ with the formulas defining them.

Then for any elementary extension $M \prec N$ we have that $(G^N, \cdot^N)$ is also a group. In fact it contains $(G, \cdot)$ as a subgroup, since for any $a, b, c \in G$ we have

$$M \models a \cdot b = c \text{ if and only if } N \models a \cdot b = c.$$

Definable groups

Now let $S_G(M)$ be the space of types over M containing the formula G. That is, the closure of the image of the set G in the natural embedding $\mathrm{tp} : M^n \to S_n(M)$. By saturation we have

$$S_G(M) = \{\mathrm{tp}(\tilde{g}/M) : \tilde{g} \in G^{\mathbb{M}}\}.$$

But G is a subgroup of $G^{\mathbb{M}}$, and this induces an action of G on $S_G(M)$:

$$g.\mathrm{tp}(\tilde{g}/M) = \mathrm{tp}(g \cdot \tilde{g}/M)$$

for $g \in G \subset M^n$ and $\tilde{g} \in G^{\mathbb{M}} \subset \mathbb{M}^n$. Since the product is definable, this is a well-defined action by homeomorphisms. That is, $S_G(M)$ is a point-transitive G-flow.

Automorphism groups

Let M be a structure. We denote by $\mathrm{Aut}(M)$ the group of automorphisms of M. Then $\mathrm{Aut}(M)$ is a topological group under the topology of pointwise convergence. If M is countable (separable) then $\mathrm{Aut}(M)$ is a Polish group.

In fact, automorphism groups of classical countable structures are precisely the closed subgroups of S_∞: if $G \leq S(X)$, one can define basic predicates on X to turn it into a structure with $G = \mathrm{Aut}(X)$.

Similarly, any Polish group can be seen as the automorphism group of a separable metric structure: one chooses a left-invariant metric on G, takes $X = \widehat{G_L}$ its completion and defines appropriate predicates on X to turn it into a metric structure with $G = \mathrm{Aut}(M)$.

Automorphism groups

$\mathrm{Aut}(M)$ acts continuously (by isometries) on M. It also acts continuously on $S_x(M)$. If $g \in \mathrm{Aut}(M)$, $p \in S_x(M)$ then gp is defined by

$$\varphi(x, m)^{gp} = \varphi(x, g^{-1}m)^p,$$

where $\varphi(x, y)$ ranges over σ-formulas, $m \in M^{|y|}$, and $\varphi(x, b)^q$ denotes the value of $\varphi(a, b)$ for any a realizing $q \in S_x(M)$.

Categoricity

Let κ be a cardinal. A theory T is κ-categorical if there is only one model of cardinal κ up to isomorphism.

In CL: if there is only one model of density character κ.

Examples

- The theory of infinite sets is κ-categorical for every infinite κ.
- ACF_0 is κ-categorical for every $\kappa \geq \aleph_1$ but not for $\kappa = \aleph_0$.
- $\mathrm{Th}(\mathbb{Q}, <)$ is κ-categorical for $\kappa = \aleph_0$ but not for any $\kappa \geq \aleph_1$.
- The theory of infinite dimensional Hilbert spaces is categorical in every infinite cardinal.
- The theory of atomless measure algebras is $\aleph_0$-categorical but not κ-categorical for larger κ.

$\aleph_0$-categorical structures

Theorem

Let T be a complete theory in a countable signature. The following are equivalent.

1. *T is $\aleph_0$-categorical.*
2. *$S_n(\emptyset)$ is finite for every n.*

Theorem

Let M be a countable structure such that $Th(M)$ is $\aleph_0$-categorical. Then:

- *M is homogeneous: if a, b are finite tuples with $tp(a/\emptyset) = tp(b/\emptyset)$ then there is $g \in Aut(M)$ with $ga = b$.*
- *A set $D \subset M^n$ is definable if and only if it is $Aut(M)$-invariant.*
- *It follows that $S_n(\emptyset)$ can be identified with M^n/G.*

Hence the theory of M is $\aleph_0$-categorical if and only if the action of $\mathrm{Aut}(M)$ on M is oligomorphic.

$\aleph_0$-categorical structures

Analogous continuous/approximate statements hold for $\aleph_0$-categorical structures in CL. Among them:

- $S_n(T)$ can be identified with the metric quotient $M^n \mathbin{/\!\!/} \mathrm{Aut}(M)$ (in particular these quotients are compact for all n, and this is equivalent to $\aleph_0$-categoricity).
- A predicate $P : M^n \to \mathbb{R}$ is definable if and only if it is uniformly continuous and $\mathrm{Aut}(M)$-invariant.

Suppose M is $\aleph_0$-categorical and denote by E the set of endomorphisms of M, which is a topological semigroup under the topology of pointwise convergence. Then by (approximate) homogeneity we have the following:

Theorem

E is exactly the pointwise closure of G in M^M, and it can be identified with the left-completion $\widehat{G_L}$.

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