THE INCOMPRESSIBLE α–EULER EQUATIONS IN THE EXTERIOR OF A VANISHING DISK

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Abstract. In this article we consider the α–Euler equations in the exterior of a small fixed disk of radius ε. We assume that the initial potential vorticity is compactly supported and independent of ε, and that the circulation of the unfiltered velocity on the boundary of the disk does not depend on ε. We prove that the solution of this problem converges, as $\varepsilon \to 0$, to the solution of a modified α–Euler equation in the full plane where an additional Dirac located at the center of the disk is imposed in the potential vorticity.

1. Introduction

In this work we study the initial-boundary-value problem for the two-dimensional incompressible α–Euler equations, $\alpha > 0$ fixed, in the exterior of the small disk $D(0;\varepsilon) = \{x \in \mathbb{R}^2 \mid |x| \leq \varepsilon\}$.

Let $\Pi_\varepsilon \equiv \{|x| > \varepsilon\}$. The system we are interested in is given by:

$$\begin{cases} \partial_t v_\varepsilon + u_\varepsilon \cdot \nabla v_\varepsilon + \sum_{j=1}^2 (v_\varepsilon)_j \nabla (u_\varepsilon)_j = -\nabla p_\varepsilon, & \text{in } (0,\infty) \times \Pi_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0, & \text{in } [0,\infty) \times \Pi_\varepsilon, \\ v_\varepsilon = u_\varepsilon - \alpha \Delta u_\varepsilon, & \text{in } [0,\infty) \times \Pi_\varepsilon, \\ u_\varepsilon = 0, & \text{on } [0,\infty) \times \{|x| = \varepsilon\}, \\ \lim_{|x|\to\infty} u_\varepsilon(t,x) = 0, & \text{for all } t \geq 0, \\ u_\varepsilon(0,\cdot) = u_{\varepsilon,0}, & \text{at } \{t=0\} \times \Pi_\varepsilon. \end{cases} \tag{1}$$

Above u_ε is called the *filtered* velocity while v_ε is the *unfiltered* velocity.

The α–Euler equations arise in several ways: as the inviscid case of the second-grade fluid model, see [5], averaging the transporting velocity in the Euler equations at scale $\sqrt{\alpha}$, as the equation for geodesics in the group of volume-preserving diffeomorphisms with a natural metric, see [10], or as a variant of the vortex blob method, see [11].

The α-Euler equations are a natural desingularization of the inviscid flow equations, obtained by averaging momentum transport at small scales, away from solid boundaries. In domains with boundary a boundary condition must be imposed; a natural choice is to impose the no-slip condition $u = 0$ on the filtered velocity, something which makes the α-Euler equations into a rough analog of the standard initial-boundary value problem for the Navier-Stokes equations. Recent progress has been obtained in understanding the flow-boundary interaction for this α-model, focusing mainly on the vanishing α limit, see [9, 3, 4].

The present work is part of this program, seeking to identify the limiting behavior of the flow in the exterior of a small obstacle for fixed α. This is inspired by work of Iftimie *et al.* in two space dimensions, where this limit was identified both for the Euler and Navier-Stokes equations, see [7, 8]. The limit is sharply different in the inviscid and viscous cases. For inviscid flow, the small obstacle leads to a modified Euler system, whereas for viscous flow, it is the initial condition that must be adjusted. Given that the α-Euler model is a regularized inviscid system using the standard viscous boundary condition, it is natural to wonder whether the present limit follows the inviscid pattern, the viscous one, or something else altogether. Our main result is that the limit follows the inviscid pattern.

The proof also follows the structure of the corresponding result for the Euler equations, with additional complications coming from potential theory. Dealing with these complications makes up the bulk of the present work.

Let us note that, taking the two-dimensional curl of (1), which corresponds to applying the differential operator $\nabla^{\perp}\cdot$ to the system, gives rise to the *potential vorticity equation*:

$$\begin{cases} \partial_t q_\varepsilon + u_\varepsilon \cdot \nabla q_\varepsilon = 0, & \text{in } (0,\infty)\times\Pi_\varepsilon, \\ \operatorname{div} u_\varepsilon = 0, & \text{in } [0,\infty)\times\Pi_\varepsilon, \\ \operatorname{curl}(1-\alpha\Delta)u_\varepsilon = \operatorname{curl} v_\varepsilon = q_\varepsilon, & \text{in } [0,\infty)\times\Pi_\varepsilon \\ u_\varepsilon = 0, & \text{on } [0,\infty)\times\{|x|=\varepsilon\}, \\ \lim\limits_{|x|\to\infty} u_\varepsilon(t,x) = 0, & \text{for all } t\geq 0, \\ q_\varepsilon(0,\cdot) = q_{\varepsilon,0}, & \text{at } \{0\}\times\Pi_\varepsilon. \end{cases} \tag{2}$$

The scalar quantity q_ε is the *potential vorticity*.

We will work with the vorticity equation rather than the velocity equation. This requires a modified Biot-Savart law expressing the velocity u_ε in terms of the vorticity q_ε. Since the domain we are considering is not simply-connected, we require additional information to determine velocity from vorticity. We will impose a given circulation of the unfiltered velocity v_ε on the boundary; we denote this circulation by γ. This is a conserved quantity for the evolution, as noted in [3, Lemma 2.3]. We will see, in Section 2, that the velocity u_ε is uniquely determined in terms of q_ε and of γ. We will show that the modified Biot-Savart $u_\varepsilon = \mathbf{T}_\varepsilon(q_\varepsilon)$ law, which gives the velocity u_ε in terms of the potential vorticity q_ε and of the circulation γ, is:

$$u_\varepsilon = \mathbf{T}_\varepsilon(q_\varepsilon) \equiv (1+\alpha\mathbb{A}_\varepsilon)^{-1}[\mathbf{K}_\varepsilon(q_\varepsilon) + (\gamma+m)H] \tag{3}$$

where $\mathbb{A}_\varepsilon$ is the Stokes operator, $m = \int_{\Pi_\varepsilon} q_\varepsilon$ (another conserved quantity), H is the following harmonic vector field

$$H = \frac{x^\perp}{2\pi|x|^2}$$

and $\mathbf{K}_\varepsilon(q_\varepsilon)$ is the classical Biot-Savart law in Π_ε:

$$\mathbf{K}_\varepsilon(q_\varepsilon)(x) = \int_{\Pi_\varepsilon} \nabla_x^\perp G_\varepsilon(x,y) q_\varepsilon(y)\,dy. \tag{4}$$

Above G_ε denotes the Green's function for the Laplacian on Π_ε with zero boundary conditions.

The purpose of this article is to prove the following result.

Theorem 1. *Assume that the initial potential vorticity $q_{\varepsilon,0} = q_0 \in L^1\cap L^\infty$ is compactly supported outside the origin and independent of ε. Assume, in addition, that the circulation of the unfiltered velocity v_ε on the boundary of Π_ε is a constant γ independent of ε, so that the velocity u_ε can be expressed from the potential vorticity with the Biot-Savart law* (3)*: $u_\varepsilon = \mathbf{T}_\varepsilon(q_\varepsilon)$. Then*

a) *There exists a unique global solution* q_ε *of* (2) *with* $u_\varepsilon = \mathbf{T}_\varepsilon(q_\varepsilon)$, *such that* $q^\varepsilon \in L^\infty(\mathbb{R}_+; L^1(\Pi_\varepsilon) \cap L^\infty(\Pi_\varepsilon))$.

b) *Let* $\widetilde{q}_\varepsilon$ *be the extension of* q_ε *to* $\mathbb{R}^2$ *which coincides with* q_ε *in* Π_ε *and vanishes in* $|x| \leq \varepsilon$. *Then we have that* $\widetilde{q}_\varepsilon \rightharpoonup q$ *weak-*$*$ *in* $L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$, *as* $\varepsilon \to 0$, *and* q *is a global solution of the following system of PDE in the full plane:*

$$\begin{cases} \partial_t q + u \cdot \nabla q = 0, & \text{in } (0,\infty) \times \mathbb{R}^2, \\ \operatorname{div} u = 0, & \text{in } [0,\infty) \times \mathbb{R}^2, \\ \operatorname{curl}(1 - \alpha\Delta)u = q + \gamma\delta, & \text{in } [0,\infty) \times \mathbb{R}^2, \\ q(0,\cdot) = q_0, & \text{at } \{0\} \times \Pi_\varepsilon, \end{cases} \tag{5}$$

and $u \in L^\infty_{loc}(\mathbb{R}_+; L^p(\mathbb{R}^2))$ *for all* $p > 2$.

c) *The limit system* (5) *has at most one global solution* $q \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$.

We note that the limit system (5) above is not the α–Euler equations in $\mathbb{R}^2$. Indeed, one could consider the α–Euler equations in $\mathbb{R}^2$ with initial potential vorticity $\overline{q}_0 = q_0 + \gamma\delta$. Such an initial data is a bounded measure so it produces a unique global solution $\overline{q} = q + \gamma\delta_{z(t)}$ with $z(0) = 0$, see [11]. It is easy to see that the PDE for the regular part q is the same as (5) except that one has to write $\delta_{z(t)}$ instead of δ. In contrast with (5), for the α–Euler equations the position $z(t)$ of the discrete part is no longer constant and must evolve along the trajectories of the velocity associated to the regular part q. So, although these two equations are very similar, they are not the same.

The plan of this paper is the following. In Section 2 we introduce notation, we deduce the modified Biot-Savart law (3) and we establish some ε-dependent estimates. Global existence and uniqueness for system (2), for fixed ε, is shown in Section 3. Next, we prove estimates uniform in ε in Section 4. The convergence result is the subject of Section 5. Finally, the uniqueness of solutions of (5) is shown in Section 6.

2. Notation, modified Biot-Savart law and preliminary estimates

We begin by introducing basic notation. Given a Banach space X of vector fields on Π_ε we denote by X_σ the subspace of X consisting of divergence-free vector fields in X which are tangent to the boundary.

Recall that Π_ε is the exterior of the disk of radius ε. We use the subscript ε to denote the dependence of solutions of the α–Euler equations on the domain, as in u_ε, v_ε, q_ε. We will assume that ε is small.

We choose a smooth radial function $\eta \in C^\infty(\mathbb{R}^2; [0,1])$ such that $\eta(x) = 0$ for all $|x| \leq 1$ and $\eta(x) = 1$ for all $|x| \geq 2$. We define

$$H = \frac{x^\perp}{2\pi|x|^2} \tag{6}$$

and

$$H_\infty \equiv \eta(x)H.$$

The radial symmetry of η implies that H_∞ is divergence free.

We denote by $\mathbb{P}_\varepsilon$ the *Leray projector* in Π_ε, i.e. the orthogonal projection from $L^2(\Pi_\varepsilon)$ to $L^2_\sigma(\Pi_\varepsilon)$. It is well known that the Leray projector can be extended by density to a continuous projection from $L^p(\Pi_\varepsilon)$ to $L^p_\sigma(\Pi_\varepsilon)$, for all $1 < p < \infty$.

The Stokes operator on Π_ε with homogeneous Dirichlet boundary conditions is denoted by $\mathbb{A}_\varepsilon = -\mathbb{P}_\varepsilon\Delta$. When we apply these operators, or their inverses, to functions defined on the whole $\mathbb{R}^2$ we mean to apply them to the restrictions of these functions to Π_ε.

We will use the following result about the inverse of the operator $1 + \alpha\mathbb{A}_\varepsilon$.

Proposition 2. *The operator $(1 + \alpha\mathbb{A}_\varepsilon)^{-1}$ is bounded from $L^p_\sigma(\Pi_\varepsilon)$ to $W^{2,p}(\Pi_\varepsilon) \cap W^{1,p}_0(\Pi_\varepsilon) \cap L^p_\sigma(\Pi_\varepsilon)$ for any $1 < p < \infty$ and from $L^\infty_\sigma(\Pi_\varepsilon)$ to $W^{1,\infty}(\Pi_\varepsilon)$. In addition, there exists a universal constant C_0 such that*

$$\|(1 + \alpha\mathbb{A}_\varepsilon)^{-1} f\|_{L^\infty(\Pi_\varepsilon)} \leq C_0\|f\|_{L^\infty(\Pi_\varepsilon)} \quad \text{for all } f \in L^\infty_\sigma(\Pi_\varepsilon). \tag{7}$$

Proof. The statement regarding L^p_σ can be found in [6, Corollary 5] and the boundedness from L^∞_σ to $W^{1,\infty}$ is proved in [1].

To prove the bound (7) we observe first that it was already proved in [1] that (7) holds true with a constant $C_0 = C_0(\varepsilon)$ which may depend on ε but not on α. An immediate scaling argument shows that it is also independent of ε. Indeed, changing variables $x = \varepsilon x'$ the operator $1 + \alpha\mathbb{A}_\varepsilon$ becomes $1 + \frac{\alpha}{\varepsilon^2}\mathbb{A}_1$. So we can apply the result of [1] to the operator $(1 + \frac{\alpha}{\varepsilon^2}\mathbb{A}_1)^{-1}$ on $L^\infty_\sigma(\Pi_1)$ and find a universal constant C_0 as an upper bound for its norm as a bounded operator on $L^\infty_\sigma(\Pi_1)$. By the rescaling performed, this universal constant C_0 is also an upper bound for the norm of $(1 + \alpha\mathbb{A}_\varepsilon)^{-1}$ in $L^\infty_\sigma(\Pi_\varepsilon)$. □

The integral of the potential vorticity is a constant of motion. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Pi_\varepsilon} q_\varepsilon(t,x)\,\mathrm{d}x = \int_{\Pi_\varepsilon} \partial_t q_\varepsilon\,\mathrm{d}x = -\int_{\Pi_\varepsilon} \mathrm{div}(u_\varepsilon q_\varepsilon)\,\mathrm{d}x = 0,$$

since u_ε vanishes at $|x| = \varepsilon$. We denote

$$m \equiv \int_{\Pi_\varepsilon} q_\varepsilon = \int_{\Pi_\varepsilon} q_0.$$

We will later need some detailed information on the operator $(1 - \alpha\Delta)^{-1}$ in the full plane. This is a convolution operator against a kernel, denoted $\mathcal{G}_\alpha$, which is a re-scaled *Bessel potential*. The classical Bessel potential from Harmonic Analysis, $\mathcal{J}_2$, is the kernel for $(1 - \Delta)^{-1}$, and we have

$$\mathcal{G}_\alpha(x) = \frac{1}{\alpha}\mathcal{J}_2\left(\frac{x}{\sqrt{\alpha}}\right).$$

We will make use of the following properties of $\mathcal{G}_\alpha$, deduced from those satisfied by $\mathcal{J}_2$; the first three can be found in Chapter V.3.1 of [12], and, for the fourth property, see [13], page 80, relation (14).

(P1) $\mathcal{G}_\alpha$ is radially symmetric; i.e. $\mathcal{G}_\alpha(x) = g_\alpha(|x|)$ for some $g_\alpha = g_\alpha(r)$;

(P2) g_α is positive and

$$\int_{\mathbb{R}^2} \mathcal{G}_\alpha = 2\pi\int_0^\infty s g_\alpha(s)\,ds = 1; \tag{8}$$

(P3) $\mathcal{G}_\alpha$ decays exponentially at infinity, i.e. for any $M > 0$, there exist positive constants c_1 and c_2 such that

$$g_\alpha(|x|) \leq c_1 e^{-c_2|x|}, \text{ whenever } |x| > M;$$

(P4) $\mathcal{G}_\alpha$ has a logarithmic singularity at 0, i.e., there exists $c_3 \in \mathbb{R}$, such that

$$g_\alpha(|x|) = c_3 \log|x| + \mathcal{O}(1), \text{ as } |x| \to 0. \tag{9}$$

Additionally, g_α is bounded for $|x| \geq 1/2$.

Let us introduce now the following kernel

$$K^\alpha = \mathcal{G}_\alpha * H. \tag{10}$$

We will need, later, the following estimates for K^α.

Lemma 3. *There exists a constant $C > 0$, which depends only on α, such that:*

a) *If $|x| < 1/2$ we have:*

$$|K^\alpha(x)| \leq C|x|\big|\log|x|\big| \quad \textit{and} \quad |\nabla K^\alpha(x)| \leq C\big|\log|x|\big|.$$

b) *For all $x \in \mathbb{R}^2$, we have that $|K^\alpha(x)| \leq C/(1+|x|)$.*

c) *We have that*

$$|\partial_2 K_1^\alpha(x) + \partial_1 K_2^\alpha(x)| \leq C \quad \textit{and} \quad |\partial_1 K_1^\alpha(x) - \partial_2 K_2^\alpha(x)| \leq C \tag{11}$$

for all $x \in \mathbb{R}^2$.

Proof. There is a simple way to express $K^\alpha(x)$ in terms of g_α, see for instance [2, page 5476], namely:

$$K^\alpha(x) = \frac{x^\perp}{|x|^2}\int_0^{|x|} s g_\alpha(s)\,ds. \tag{12}$$

Let x be such that $|x| < 1/2$. It follows from property (P4), (9), that

$$|K^\alpha(x)| \leq \frac{C}{|x|}\int_0^{|x|} s|\log s|\,ds \leq C|x|\big|\log|x|\big|,$$

for some constant $C > c_3$.

Differentiating (12) and estimating the result as above gives the gradient estimate and proves part a).

The bound $C/(1+|x|)$ in part b) follows from the estimates given in [2, page 5476].

To prove part c) we compute $\partial_2 K_1^\alpha + \partial_1 K_2^\alpha$ using (12). We find, after some calculations, that:

$$\partial_2 K_1^\alpha(x) + \partial_1 K_2^\alpha(x) = \frac{x_1^2 - x_2^2}{|x|^2}\left[g_\alpha(|x|) - \frac{2}{|x|^2}\int_0^{|x|} s g_\alpha(s)\,ds\right].$$

We use again (P4), (9), to obtain that

$$\begin{aligned} g_\alpha(|x|) - \frac{2}{|x|^2}\int_0^{|x|} s g_\alpha(s)\,ds &\\ = c_3\log(|x|) - c_3\frac{2}{|x|^2}\int_0^{|x|} s\log(s)\,ds + O(1) &\\ = O(1) \text{ as } |x| \to 0. & \end{aligned}$$

It follows that $\partial_2 K_1^\alpha + \partial_1 K_2^\alpha$ is bounded for small x. The global bound is a consequence of property (P3) and the boundedness of g_α for $|x| \geq 1/2$. The corresponding estimate for $\partial_1 K_1^\alpha - \partial_2 K_2^\alpha$ follows in the same manner. □

We will now deduce the modified Biot-Savart law, which expresses the velocity u_ε in terms of the potential vorticity q_ε and the circulation of v_ε around the boundary of Π_ε. Our point of departure is the following elliptic system, which relates potential vorticity to the unfiltered velocity:

$$\begin{cases} \operatorname{div} v_\varepsilon = 0, & \text{in } [0,\infty) \times \Pi_\varepsilon, \\ \operatorname{curl} v_\varepsilon = q_\varepsilon, & \text{in } [0,\infty) \times \Pi_\varepsilon. \end{cases}$$

We recall that, above, $v_\varepsilon = (1 - \alpha\Delta)u_\varepsilon$, and u_ε satisfies the boundary condition

$$u_\varepsilon = 0, \text{ on } [0,\infty) \times \{|x| = \varepsilon\}.$$

Then, since $\mathbb{P}_\varepsilon v_\varepsilon$ and v_ε differ by a gradient, it follows easily that

$$\begin{cases} \operatorname{div}(\mathbb{P}_\varepsilon v_\varepsilon) = 0, & \text{in } [0,\infty) \times \Pi_\varepsilon, \\ \operatorname{curl}(\mathbb{P}_\varepsilon v_\varepsilon) = q_\varepsilon, & \text{in } [0,\infty) \times \Pi_\varepsilon, \\ \mathbb{P}_\varepsilon v_\varepsilon \cdot \widehat{\mathbf{n}} = 0 & \text{on } [0,\infty) \times \{|x| = \varepsilon\}. \end{cases}$$

The system above was studied in detail in [7]. It was shown, see [7, page 358], that there exists $\beta_\varepsilon = \beta_\varepsilon(t) \in \mathbb{R}$ for which

$$\mathbb{P}_\varepsilon v_\varepsilon = \mathbf{K}_\varepsilon(q_\varepsilon) + \beta_\varepsilon(t) H,$$

where the operator $\mathbf{K}_\varepsilon(q_\varepsilon)$ is the classical Biot-Savart law defined in (4), and H is the generator of the harmonic vector fields in Π_ε with unit circulation defined in (6).

We now argue that $\beta_\varepsilon = \gamma + m$, where γ is the circulation of the unfiltered velocity v_ε on the boundary of Π_ε and m is the mass of vorticity: $m = \int_{\Pi_\varepsilon} q_\varepsilon$. Following the proof of [7, Lemma 3.1] we have that

$$\beta_\varepsilon = \int_{\{|x|=\varepsilon\}} \mathbb{P}_\varepsilon v_\varepsilon \cdot \mathrm{d}s + \int_{\Pi_\varepsilon} q_\varepsilon \,\mathrm{d}x.$$

First observe that $v_\varepsilon = \mathbb{P}_\varepsilon v_\varepsilon + \nabla Q$, for some smooth function Q, and that

$$\int_{\{|x|=\varepsilon\}} \nabla Q \cdot \mathrm{d}s = 0,$$

since $\{|x| = \varepsilon\}$ is a closed curve. Therefore

$$\int_{\{|x|=\varepsilon\}} \mathbb{P}_\varepsilon v_\varepsilon \cdot \mathrm{d}s = \int_{\{|x|=\varepsilon\}} v_\varepsilon \cdot \mathrm{d}s \equiv \gamma,$$

which is a conserved quantity, see [3, Lemma 2.3].

In summary, we have shown that

$$\mathbb{P}_\varepsilon v_\varepsilon = \mathbf{K}_\varepsilon(q_\varepsilon) + (\gamma + m)H.$$

We regard the expression on the right-hand-side above as an operator acting on q_ε, which we denote by $\mathbf{S}_\varepsilon$:

$$q_\varepsilon \mapsto \mathbf{S}_\varepsilon(q_\varepsilon) \equiv \mathbf{K}_\varepsilon(q_\varepsilon) + (\gamma + m)H.$$

Then

$$\mathbf{S}_\varepsilon(q_\varepsilon) = \mathbb{P}_\varepsilon v_\varepsilon = \mathbb{P}_\varepsilon(u_\varepsilon - \alpha\Delta u_\varepsilon) = u_\varepsilon + \alpha \mathbb{A}_\varepsilon u_\varepsilon.$$

Inverting the operator $1 + \alpha\mathbb{A}_\varepsilon$ allows to deduce the following modified Biot-Savart law

$$u_\varepsilon = \mathbf{T}_\varepsilon(q_\varepsilon) = (1 + \alpha\mathbb{A}_\varepsilon)^{-1}\, \mathbf{S}_\varepsilon(q_\varepsilon) = (1 + \alpha\mathbb{A}_\varepsilon)^{-1}[\mathbf{K}_\varepsilon(q_\varepsilon) + (\gamma + m)H]. \tag{13}$$

In the following Proposition we collect some estimates related to this modified Biot-Savart law.

Proposition 4. *For all $q \in L^1(\Pi_\varepsilon) \cap L^\infty(\Pi_\varepsilon)$ such that $\int_{\Pi_\varepsilon} q = m$ we have that $\mathbf{T}_\varepsilon(q) \in W^{1,\infty}(\Pi_\varepsilon)$ and*

$$\| \mathbf{T}_\varepsilon(q)\|_{W^{1,\infty}(\Pi_\varepsilon)} \leq C_1(\|q\|_{L^1(\Pi_\varepsilon)\cap L^\infty(\Pi_\varepsilon)} + |\gamma|), \tag{14}$$

where the constant $C_1 > 0$ depends only on α and ε. If we assume, in addition, that $\operatorname{supp} q \subset \{|x| \leq R\}$ for some finite R, then we also have $\mathbf{T}_\varepsilon(q) - (\gamma + m)H \in W^{2,p}(\Pi_\varepsilon)$ for all $1 < p < \infty$ and

$$\| \mathbf{T}_\varepsilon(q) - (\gamma + m)H\|_{W^{2,p}(\Pi_\varepsilon)} \leq C_2(\|q\|_{L^1(\Pi_\varepsilon)\cap L^\infty(\Pi_\varepsilon)} + |\gamma|),$$

where the constant $C_2 > 0$ depends only on α, ε, R and p.

Proof. It follows from [7, Theorem 4.1] that the quantity $\mathbf{K}_\varepsilon(q) + mH$ is bounded in Π_ε, and the L^∞-norm may be estimated by $C\|q\|_{L^1(\Pi_\varepsilon)\cap L^\infty(\Pi_\varepsilon)}$, where C is a universal constant. Clearly

$$\| \mathbf{S}_\varepsilon(q)\|_{L^\infty} \leq \| \mathbf{K}_\varepsilon(q) + mH\|_{L^\infty} + \|\gamma H\|_{L^\infty} \leq C\|q\|_{L^1(\Pi_\varepsilon)\cap L^\infty(\Pi_\varepsilon)} + \frac{|\gamma|}{2\pi\varepsilon}.$$

Since $\mathbf{T}_\varepsilon(q) = (1 + \alpha\mathbb{A}_\varepsilon)^{-1}\, \mathbf{S}_\varepsilon(q)$, relation (14) follows from Proposition 2.

Assume now that $\operatorname{supp} q \subset \{|x| \leq R\}$. Then we know, from [7, relation (2.8)], that $\mathbf{K}_\varepsilon(q)$ is bounded by $C(\varepsilon, R)/|x|^2$, so it belongs to $L^p(\Pi_\varepsilon)$ for all $p > 1$. Therefore $\mathbf{S}_\varepsilon(q) - (\gamma + m)H = \mathbf{K}_\varepsilon(q) \in L^p(\Pi_\varepsilon)$ for all $p > 1$. Then Proposition 2 implies that

$$\mathbf{T}_\varepsilon(q) - (\gamma + m)(1 + \alpha\mathbb{A}_\varepsilon)^{-1}H = (1 + \alpha\mathbb{A}_\varepsilon)^{-1}(\mathbf{S}_\varepsilon(q) - (\gamma + m)H) \in W^{2,p}(\Pi_\varepsilon)$$

for all $1 < p < \infty$. Finally, we observe that $(1 + \alpha\mathbb{A}_\varepsilon)^{-1}(H_\infty - \alpha\Delta H_\infty) = H_\infty$ and we write

$$\begin{aligned}\mathbf{T}_\varepsilon(q) - (\gamma + m)H &= \mathbf{T}_\varepsilon(q) - (\gamma + m)(1 + \alpha\mathbb{A}_\varepsilon)^{-1}H + (\gamma + m)[(1 + \alpha\mathbb{A}_\varepsilon)^{-1}H - H] \\ &= \mathbf{T}_\varepsilon(q) - (\gamma + m)(1 + \alpha\mathbb{A}_\varepsilon)^{-1}H \\ &\qquad + (\gamma + m)[H_\infty - H + (1 + \alpha\mathbb{A}_\varepsilon)^{-1}(H - H_\infty + \alpha\Delta H_\infty)] \\ &\in W^{2,p}(\Pi_\varepsilon)\end{aligned}$$

for all $1 < p < \infty$. We used above that $H_\infty - H \in W^{2,p}(\Pi_\varepsilon)$, that $H - H_\infty + \alpha\Delta H_\infty \in L^p_\sigma(\Pi_\varepsilon)$ and Proposition 2. This completes the proof. □

3. Existence of the solution for fixed ε

In this section we prove part a) of Theorem 1. A similar result, requiring more regularity and with a different proof can be found in [14].

We want to solve the following problem

$$\begin{cases} \partial_t q + \mathbf{T}_\varepsilon(q) \cdot \nabla q = 0, & t > 0, |x| > \varepsilon \\ q\big|_{t=0} = q_0, & |x| > \varepsilon \end{cases} \tag{15}$$

where $q_0 \in L^1(\Pi_\varepsilon) \cap L^\infty(\Pi_\varepsilon)$.

We construct a recursive sequence of approximate solutions in the following manner. We set $q^0(t, x) = q_0(x)$ and $u^0(t, x) = \mathbf{T}_\varepsilon(q^0)(x)$. We define $q^1 = q^1(t, x)$,

$q^1 \in L^\infty((0,+\infty); L^1 \cap L^\infty(\Pi_\varepsilon))$, to be the unique weak solution of the following transport equation

$$\begin{cases} \partial_t q^1 + u^0 \cdot \nabla q^1 = 0, & t > 0, |x| > \varepsilon \\ \qquad\qquad q^1\big|_{t=0} = q_0, & |x| > \varepsilon. \end{cases}$$

The global existence of q^1 follows from Picard's theorem since, by Proposition 4, we have that u^0 is a Lipschitz vector field so its flow map is globally well-defined. Uniqueness can be established by energy estimates.

Next, given $q^n \in L^\infty((0,+\infty); L^1 \cap L^\infty(\Pi_\varepsilon))$, we define $q^{n+1} \in L^\infty((0,+\infty); L^1 \cap L^\infty(\Pi_\varepsilon))$ recursively as the unique global weak solution of the transport equation

$$\begin{cases} \partial_t q^{n+1} + u^n \cdot \nabla q^{n+1} = 0, & t > 0, |x| > \varepsilon \\ \qquad\qquad q^{n+1}\big|_{t=0} = q_0, & |x| > \varepsilon, \end{cases} \tag{16}$$

where $u^n = \mathbf{T}_\varepsilon(q^n)$. We will pass to the limit in the above problem and find a solution of (15).

We first observe that, since q^n satisfies a transport equation by a divergence-free vector field, the integral of q^n is conserved:

$$\int_{\Pi_\varepsilon} q^n(t,x)\,dx = \int_{\Pi_\varepsilon} q^n(0,x)\,dx = \int_{\Pi_\varepsilon} q_0(x)\,dx = m.$$

We have that the circulation of $\mathbf{K}_\varepsilon(q^n)$ on the boundary is equal to the quantity $\left(-\int_{\Pi_\varepsilon} q^n(t,x)\,dx\right)$ (see the proof of [7, Lemma 3.1]) and we recall that H has unit circulation on the boundary. With this we infer that the circulation of $\mathbf{S}_\varepsilon(q^n)$ on the boundary is γ.

Using, again, that q^n satisfies a transport equation, we have that q^n is bounded in $L^\infty(\mathbb{R}_+; L^1 \cap L^\infty)$ uniformly in n. Proposition 4 implies that u^n is bounded in $L^\infty(\mathbb{R}_+; W^{1,\infty}(\Pi_\varepsilon))$ uniformly in n. Indeed:

$$\begin{aligned} \|u^n(t,\cdot)\|_{W^{1,\infty}(\Pi_\varepsilon)} &\le C_1(\alpha,\varepsilon)(\|q^n\|_{L^1(\Pi_\varepsilon)\cap L^\infty(\Pi_\varepsilon)} + |\gamma|) \\ &= C_1(\alpha,\varepsilon)(\|q_0\|_{L^1(\Pi_\varepsilon)\cap L^\infty(\Pi_\varepsilon)} + |\gamma|) \\ &\equiv C_3. \end{aligned} \tag{17}$$

Since q^{n+1} is transported by the flow of u^n, we deduce that $\operatorname{supp} q^{n+1}(t,\cdot) \subset \{|x| \le R_0 + C_3 t\}$ where $\operatorname{supp} q_0 \subset \{|x| \le R_0\}$. We use the second part of Proposition 4 to obtain that, for all $1 < p < \infty$, the quantity $u^n - (\gamma+m)H$ is bounded in $L^\infty_{loc}([0,\infty); W^{2,p}(\Pi_\varepsilon))$, uniformly in n. We will prove that it actually converges in H^2.

Let us introduce the notation $v^{n+1} = (1-\alpha\Delta)u^{n+1}$ and

$$\overline{v}^n = \mathbb{P}_\varepsilon v^n = \mathbf{S}_\varepsilon(q^n) = (1+\alpha\mathbb{A}_\varepsilon)u^n.$$

We also define $\xi^n = \Delta^{-1}_{\Pi_\varepsilon} q^n = \mathbf{G}_\varepsilon(q^n)$, where $\mathbf{G}_\varepsilon$ is the operator with kernel G_ε, the Green function on Π_ε. Let us observe that

$$\overline{v}^{n+1} - \overline{v}^n = \mathbf{S}_\varepsilon(q^{n+1}) - \mathbf{S}_\varepsilon(q^n) = \mathbf{K}_\varepsilon(q^{n+1} - q^n) = \nabla^\perp(\xi^{n+1} - \xi^n).$$

In addition, since q^n and q^{n+1} are compactly supported and $\nabla^\perp G_\varepsilon$, the kernel of $\mathbf{K}_\varepsilon$, decays like $1/|x|^2$ at infinity (see [7, relation (2.8)]) we have that $\overline{v}^{n+1} - \overline{v}^n$ also decays like $1/|x|^2$ at infinity. In particular, it belongs to $L^2(\Pi_\varepsilon)$.

We subtract the equation for q^n from the equation for q^{n+1} (see relation (16)), we multiply it by $\xi^{n+1}-\xi^n$ and integrate in space (which is possible because q^n and q^{n+1} are compactly supported). We then obtain

$$\int_{\Pi_\varepsilon}\partial_t(q^{n+1}-q^n)(\xi^{n+1}-\xi^n)+\int_{\Pi_\varepsilon}(u^n\cdot\nabla q^{n+1}-u^{n-1}\cdot\nabla q^n)(\xi^{n+1}-\xi^n)=0.$$

The integrals above should be interpreted as duality pairings between $W^{-1,p}$ and $W^{1,p'}$, where $p'=p/(p-1)$ and $p\in(1,\infty)$.

Clearly

$$\begin{aligned}\int_{\Pi_\varepsilon}\partial_t(q^{n+1}-q^n)(\xi^{n+1}-\xi^n)&=\int_{\Pi_\varepsilon}\Delta[\partial_t(\xi^{n+1}-\xi^n)](\xi^{n+1}-\xi^n)\\&=-\int_{\Pi_\varepsilon}[\partial_t\nabla(\xi^{n+1}-\xi^n)]\nabla(\xi^{n+1}-\xi^n)\\&=-\frac12\frac{\mathrm d}{\mathrm dt}\|\nabla(\xi^{n+1}-\xi^n)\|^2_{L^2(\Pi_\varepsilon)}\\&=-\frac12\frac{\mathrm d}{\mathrm dt}\|\overline v^{n+1}-\overline v^n\|^2_{L^2(\Pi_\varepsilon)}.\end{aligned}$$

Therefore

$$\begin{aligned}\frac12\frac{\mathrm d}{\mathrm dt}\|\overline v^{n+1}-\overline v^n\|^2_{L^2(\Pi_\varepsilon)}&=\int_{\Pi_\varepsilon}(u^n\cdot\nabla q^{n+1}-u^{n-1}\cdot\nabla q^n)(\xi^{n+1}-\xi^n)\\&=\int_{\Pi_\varepsilon}u^n\cdot\nabla(q^{n+1}-q^n)(\xi^{n+1}-\xi^n)+\int_{\Pi_\varepsilon}(u^n-u^{n-1})\cdot\nabla q^n(\xi^{n+1}-\xi^n)\\&=\int_{\Pi_\varepsilon}u^n\cdot(\overline v^{n+1}-\overline v^n)^\perp(q^{n+1}-q^n)+\int_{\Pi_\varepsilon}(u^n-u^{n-1})\cdot(\overline v^{n+1}-\overline v^n)^\perp q^n\\&\equiv I_1+I_2,\end{aligned}$$

where we integrated by parts and used that $\nabla(\xi^{n+1}-\xi^n)=-(\overline v^{n+1}-\overline v^n)^\perp$. We will now estimate these two terms.

Observe first that for a divergence free vector field h we have the identity

$$h^\perp\operatorname{curl}h=\begin{pmatrix}\partial_2\\\partial_1\end{pmatrix}(h_1h_2)+\begin{pmatrix}-\partial_1\\\partial_2\end{pmatrix}\frac{h_2^2-h_1^2}{2}.$$

Recalling that $q^{n+1}-q^n=\operatorname{curl}(\overline v^{n+1}-\overline v^n)$ and integrating by parts we find:

$$\begin{aligned}|I_1|&=\left|\int_{\Pi_\varepsilon}u^n\cdot(\overline v^{n+1}-\overline v^n)^\perp(q^{n+1}-q^n)\right|\\&=\left|\int_{\Pi_\varepsilon}(\partial_2u_1^n+\partial_1u_2^n)(\overline v_1^{n+1}-\overline v_1^n)(\overline v_2^{n+1}-\overline v_2^n)\right.\\&\qquad\left.+\frac12\int_{\Pi_\varepsilon}(-\partial_1u_1^n+\partial_2u_2^n)\big[(\overline v_2^{n+1}-\overline v_2^n)^2-(\overline v_1^{n+1}-\overline v_1^n)^2\big]\right|\\&\le C\|\nabla u^n\|_{L^\infty}\|\overline v^{n+1}-\overline v^n\|^2_{L^2}\\&\le C\|\overline v^{n+1}-\overline v^n\|^2_{L^2},\end{aligned}$$

where we used (17). We estimate now I_2:

$$\begin{aligned}|I_2| &\leq \|u^n - u^{n-1}\|_{L^2}\|\overline{v}^{n+1} - \overline{v}^n\|_{L^2}\|q^n\|_{L^\infty}\\ &\leq C\|(1+\alpha\mathbb{A}_\varepsilon)^{-1}(\overline{v}^n - \overline{v}^{n-1})\|_{L^2}\|\overline{v}^{n+1} - \overline{v}^n\|_{L^2}\\ &\leq C\|\overline{v}^n - \overline{v}^{n-1}\|_{L^2}\|\overline{v}^{n+1} - \overline{v}^n\|_{L^2}\\ &\leq C\|\overline{v}^n - \overline{v}^{n-1}\|^2_{L^2} + C\|\overline{v}^{n+1} - \overline{v}^n\|^2_{L^2}.\end{aligned}$$

We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{v}^{n+1} - \overline{v}^n\|^2_{L^2(\Pi_\varepsilon)} \leq C_4\|\overline{v}^n - \overline{v}^{n-1}\|^2_{L^2} + C_4\|\overline{v}^{n+1} - \overline{v}^n\|^2_{L^2},$$

for some constant C_4 independent of n and t. Given that $\overline{v}^{n+1} - \overline{v}^n$ vanishes at the initial time, the Gronwall lemma implies the following bound:

$$\sup_{[0,T]}\|\overline{v}^{n+1} - \overline{v}^n\|_{L^2(\Pi_\varepsilon)} \leq \sup_{[0,T]}\|\overline{v}^n - \overline{v}^{n-1}\|_{L^2(\Pi_\varepsilon)}\sqrt{e^{C_4T} - 1}.$$

We choose the time T_0 such that $\sqrt{e^{C_4T_0} - 1} = \frac{1}{2}$. Then, using the estimate above recursively we find, by induction, that

$$\sup_{[0,T_0]}\|\overline{v}^{n+1} - \overline{v}^n\|_{L^2(\Pi_\varepsilon)} \leq 2^{-n}\sup_{[0,T_0]}\|\overline{v}^1 - \overline{v}^0\|_{L^2(\Pi_\varepsilon)}.$$

It then follows, from Proposition 2, that

$$\sup_{[0,T_0]}\|u^{n+1} - u^n\|_{H^2(\Pi_\varepsilon)} \leq C2^{-n}.$$

This means that the sequence $u^n - (\gamma + m)H$ is a Cauchy sequence in the space $C^0([0,T_0]; H^2(\Pi_\varepsilon))$ and, therefore, that it is convergent. Hence there exists some u such that $u - (\gamma + m)H \in C^0([0,T_0]; H^2(\Pi_\varepsilon))$ and

$$u^n - u \to 0 \quad \text{strongly in } C^0([0,T_0]; H^2(\Pi_\varepsilon)). \tag{18}$$

Denoting $q = \operatorname{curl}(u - \alpha\Delta u)$ we further obtain that

$$q^n - q \to 0 \quad \text{strongly in } C^0([0,T_0]; H^{-1}(\Pi_\varepsilon)).$$

In particular $q(0,x) = q_0(x)$.

The sequence q^n being bounded in $L^\infty(\mathbb{R}_+; L^1(\Pi_\varepsilon) \cap L^\infty(\Pi_\varepsilon))$ implies that $q \in L^\infty(\mathbb{R}_+; L^1(\Pi_\varepsilon) \cap L^\infty(\Pi_\varepsilon))$. Moreover, there exists a subsequence q^{n_k} such that

$$q^{n_k} \rightharpoonup q \quad \text{weak}* \text{ in } L^\infty(\mathbb{R}_+; L^1 \cap L^\infty). \tag{19}$$

To complete the proof of the existence part of Theorem 1, it remains to prove that $u = \mathbf{T}_\varepsilon(q)$. We know that $u - (\gamma + m)H \in L^\infty([0,T_0]; H^2(\Pi_\varepsilon))$. From Proposition 4 we also know that $\mathbf{T}_\varepsilon(q) - (\gamma + m)H \in L^\infty([0,T_0]; H^2(\Pi_\varepsilon))$ so we must have that $u - \mathbf{T}_\varepsilon(q) \in L^\infty([0,T_0]; H^2(\Pi_\varepsilon))$. Moreover, $u - \mathbf{T}_\varepsilon(q)$ is divergence free and vanishes at the boundary.

Let $\varphi \in C^\infty_{c,\sigma}([0,T_0] \times \Pi_\varepsilon)$. There exists $\psi \in C^\infty_c([0,T_0] \times \overline{\Pi}_\varepsilon)$ such that $\varphi = \nabla^\perp\psi$. Then $\nabla\psi = 0$ in a neighborhood of the boundary of Π_ε, so ψ must be constant in the same neighborhood. We denote by $\overline{\psi}(t)$ this constant, so that $\psi(t,x) \equiv \overline{\psi}(t)$ in a neighborhood of the boundary of Π_ε.

By self-adjointness of the Stokes operator we have that

$$\begin{aligned}\int_0^{T_0}\int_{\Pi_\varepsilon} u^{n_k}\cdot(1+\alpha\mathbb{A}_\varepsilon)\varphi &= \int_0^{T_0}\int_{\Pi_\varepsilon}(1+\alpha\mathbb{A}_\varepsilon)u^{n_k}\cdot\varphi \\ &= \int_0^{T_0}\int_{\Pi_\varepsilon}\mathbf{S}_\varepsilon(q^{n_k})\cdot\varphi \\ &= \int_0^{T_0}\int_{\Pi_\varepsilon}\mathbf{S}_\varepsilon(q^{n_k})\cdot\nabla^\perp\psi \\ &= -\int_0^{T_0}\int_{\Pi_\varepsilon}\operatorname{curl}\mathbf{S}_\varepsilon(q^{n_k})\ \psi - \int_0^{T_0}\int_{\partial\Pi_\varepsilon}\frac{x^\perp}{|x|}\cdot\mathbf{S}_\varepsilon(q^{n_k})\ \psi \\ &= -\int_0^{T_0}\int_{\Pi_\varepsilon} q^{n_k}\psi - \gamma\int_0^{T_0}\overline{\psi}(t)\,dt,\end{aligned} \tag{20}$$

where we used that the circulation of $\mathbf{S}_\varepsilon(q^{n_k})$ on the boundary is γ.

Since $\Delta\varphi$ is divergence free and tangent to the boundary (actually compactly supported), we have that $\mathbb{P}_\varepsilon\Delta\varphi = \Delta\varphi$. So $(1+\alpha\mathbb{A}_\varepsilon)\varphi = \varphi - \alpha\Delta\varphi \in C^\infty_{c,\sigma}([0,T_0]\times\Pi_\varepsilon)$. The convergence properties expressed in relations (18) and (19) allow to pass to the limit $k\to\infty$ in (20) to obtain that

$$\int_0^{T_0}\int_{\Pi_\varepsilon} u\cdot(1+\alpha\mathbb{A}_\varepsilon)\varphi = -\int_0^{T_0}\int_{\Pi_\varepsilon} q\psi - \gamma\int_0^{T_0}\overline{\psi}(t)\,dt.$$

But the same calculations as in (20) show that

$$\int_0^{T_0}\int_{\Pi_\varepsilon}\mathbf{T}_\varepsilon(q)\cdot(1+\alpha\mathbb{A}_\varepsilon)\varphi = -\int_0^{T_0}\int_{\Pi_\varepsilon} q\psi - \gamma\int_0^{T_0}\overline{\psi}(t)\,dt.$$

We deduce that

$$\int_0^{T_0}\int_{\Pi_\varepsilon} u\cdot(1+\alpha\mathbb{A}_\varepsilon)\varphi = \int_0^{T_0}\int_{\Pi_\varepsilon}\mathbf{T}_\varepsilon(q)\cdot(1+\alpha\mathbb{A}_\varepsilon)\varphi$$

which means that

$$\int_0^{T_0}\int_{\Pi_\varepsilon}(1+\alpha\mathbb{A}_\varepsilon)u\cdot\varphi = \int_0^{T_0}\int_{\Pi_\varepsilon}(1+\alpha\mathbb{A}_\varepsilon)\,\mathbf{T}_\varepsilon(q)\cdot\varphi.$$

This implies that $(1+\alpha\mathbb{A}_\varepsilon)(u-\mathbf{T}_\varepsilon(q)) = 0$, so necessarily $u = \mathbf{T}_\varepsilon(q)$.

We proved that there exists a solution of (2) on the time interval $[0,T_0]$. Repeating the argument starting from T_0 we can extend this solution up to time $2T_0$. Continuing like this we construct a global solution. Its uniqueness is classical. It can be proved by estimating the H^2 norm of the difference of two solutions, with exactly the same argument as in the estimate of $\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{v}^{n+1}-\overline{v}^n\|^2_{L^2(\Pi_\varepsilon)}$ so we omit it. Part a) of Theorem 1 is now proved.

4. H^1 ESTIMATES FOR u_ε

The aim of this section is to derive estimates for u_ε which are independent of ε.

We start by noting that, since q_ε satisfies a transport equation by a divergence-free vector field, we have

$$\begin{aligned}\|q_\varepsilon\|_{L^\infty(\mathbb{R}_+;L^1(\Pi_\varepsilon)\cap L^\infty(\Pi_\varepsilon))} &\le \|q_{\varepsilon,0}\|_{L^\infty(\mathbb{R}_+;L^1(\Pi_\varepsilon)\cap L^\infty(\Pi_\varepsilon))} \\ &= \|q_0\|_{L^\infty(\mathbb{R}_+;L^1(\Pi_\varepsilon)\cap L^\infty(\Pi_\varepsilon))},\end{aligned}$$

which is bounded independently of ε.

Let us now make the following observation. If f is a scalar radial function, decaying sufficiently fast at infinity, then $(1+\alpha\mathbb{A}_\varepsilon)^{-1}(x^\perp f)$ is of the form $x^\perp g$, where g is a scalar radial function. Indeed, let $h=(1-\alpha\Delta_{\Pi_\varepsilon})^{-1}(x^\perp f)$. Then, for any special orthogonal matrix M (rotation matrix), the vector field $P(x)=x^\perp f$ is invariant under the transformation $P(x)\to M^tP(Mx)$. The rotational invariance of the Laplacian allows to write the following sequence of computations:

$$\begin{aligned}(1-\alpha\Delta)[M^th(Mx)] &= M^t(1-\alpha\Delta)[h(Mx)]\\ &= M^t[(1-\alpha\Delta)h](Mx)\\ &= M^tP(Mx)\\ &= P(x)\\ &= (1-\alpha\Delta)h(x).\end{aligned}$$

We infer that $M^th(Mx)=h(x)$, so $h(Mx)=Mh(x)$ for every special orthogonal matrix M. This means that h must be of the form $h=x^\perp g_1+xg_2$ with g_1,g_2 scalar radial functions. But one can check that $(1-\alpha\Delta)(xg_2)$ is proportional to x and $(1-\alpha\Delta)(x^\perp g_1)$ is proportional to $x^\perp$. Since the sum must be $x^\perp f$ we infer that $xg_2=0$, so $h=x^\perp g_1$. Recalling that g_1 is radial, we observe that $\operatorname{div} h=\operatorname{div}(x^\perp g_1)=0$. We conclude that $h-\alpha\Delta h=x^\perp f$, that h vanishes at $|x|=\varepsilon$ and at infinity and that h is divergence free. Therefore $h=(1+\alpha\mathbb{A}_\varepsilon)^{-1}(x^\perp f)$. We proved, in this paragraph, that if f is a scalar radial function then $(1+\alpha\mathbb{A}_\varepsilon)^{-1}(x^\perp f)=(1-\alpha\Delta_{\Pi_\varepsilon})^{-1}(x^\perp f)$ is of the form $x^\perp$ times a scalar radial function.

We will now proceed to obtain the required *a priori* estimates for u_ε. We start with the following lemma.

Lemma 5. *We have that $\mathbf{K}_\varepsilon(q_\varepsilon)+mH$ is bounded in $L^\infty(\mathbb{R}_+\times\Pi_\varepsilon)$ and $\mathbf{K}_\varepsilon(q_\varepsilon)+m(H-H_\infty)$ is bounded in $L^\infty_{loc}([0,\infty);L^2(\Pi_\varepsilon))$, independently of ε.*

Proof. We recall the result in [7, Theorem 4.1], which reads, in our notation:

$$\|\mathbf{K}_\varepsilon(q_\varepsilon)+mH\|_{L^\infty(\Pi_\varepsilon)}\le C\|q_\varepsilon\|_{L^\infty}^{1/2}\|q_\varepsilon\|_{L^1}^{1/2}\le C\|q_0\|_{L^\infty}^{1/2}\|q_0\|_{L^1}^{1/2},$$

where $C>0$ is independent of ε. This shows the first bound.

Next, we need to obtain bounds on the support of q_ε which are independent of ε. We write, thanks to the modified Biot-Savart law (13),

$$u_\varepsilon=(1+\alpha\mathbb{A}_\varepsilon)^{-1}[\mathbf{K}_\varepsilon(q_\varepsilon)+mH]+\gamma(1+\alpha\mathbb{A}_\varepsilon)^{-1}H\equiv u_\varepsilon^1+u_\varepsilon^2.$$

We bound u_ε^1 by using relation (7):

$$\begin{aligned}\|u_\varepsilon^1\|_{L^\infty(\Pi_\varepsilon)} &= \|(1+\alpha\mathbb{A}_\varepsilon)^{-1}[\mathbf{K}_\varepsilon(q_\varepsilon)+mH]\|_{L^\infty(\Pi_\varepsilon)}\\ &\le C_0\|\mathbf{K}_\varepsilon(q_\varepsilon)+mH\|_{L^\infty(\Pi_\varepsilon)}\\ &\le C_0C\|q_0\|_{L^\infty}^{1/2}\|q_0\|_{L^1}^{1/2}\\ &\equiv M.\end{aligned}$$

We observe now that since H is of the form $x^\perp$ times a radial function, then $u_\varepsilon^2=(1+\alpha\mathbb{A}_\varepsilon)^{-1}H$ is of the same form. So the trajectories of u_ε^2 are circles centered in the origin. Recall that q_ε is transported by the velocity field $u_\varepsilon=u_\varepsilon^1+u_\varepsilon^2$. If we want to estimate how far from the origin the support of q_ε can go, we can ignore the term u_ε^2. Since we bounded u_ε^1 by M, we infer that

$$\operatorname{supp} q_\varepsilon(t,\cdot)\subset\{|x|\le R_0+Mt\} \tag{21}$$

where R_0 is such that $\operatorname{supp} q_0 \subset \{|x| \leq R_0\}$.

Next we remark that $\mathbf{K}_\varepsilon(q_\varepsilon) + m(H - H_\infty) = (\mathbf{K}_\varepsilon(q_\varepsilon) + mH) - mH_\infty$ is a sum of uniformly bounded vector fields, hence bounded in $L^2_{loc}(\Pi_\varepsilon)$ independently of ε. Furthermore, using the expressions for the kernel $K_\varepsilon(x,y) \equiv \nabla_x^\perp G_\varepsilon(x,y)$ of the Biot-Savart law $\mathbf{K}_\varepsilon$ given in [7, (2.5)] with the conformal mapping given by $T(x) = x/\varepsilon$ implies the following formula

$$K_\varepsilon(x,y) = \frac{(x-y)^\perp}{2\pi|x-y|^2} - \frac{(x-\varepsilon^2 y/|y|^2)^\perp}{2\pi|x-\varepsilon^2 y/|y|^2|^2}.$$

Using the relation $\left|\frac{a}{|a|^2} - \frac{b}{|b|^2}\right| = \frac{|a-b|}{|a||b|}$ and observing that $|\varepsilon^2 y/|y|^2| \leq |y|$ we get for $|x| > 2|y|$ the following upper bound for the kernel K_ε:

$$|K_\varepsilon(x,y)| = \frac{|y-\varepsilon^2 y/|y|^2|}{2\pi|x-y||x-\varepsilon^2 y/|y|^2|} \leq \frac{|y| + |\varepsilon^2 y/|y|^2|}{2\pi(|x|-|y|)(|x|-|\varepsilon^2 y/|y|^2|)} \leq \frac{4|y|}{\pi|x|^2}.$$

Recalling the bound on the support of q_ε given in (21), we can now estimate for any $|x| > 2(R_0 + Mt)$:

$$\begin{aligned} |\,\mathbf{K}_\varepsilon(q_\varepsilon)(x)| &\leq \int_{\Pi_\varepsilon} |K_\varepsilon(x,y)|\,|q_\varepsilon(y)|\,dy \\ &\leq \int_{\Pi_\varepsilon} \frac{4|y|}{\pi|x|^2}\,|q_\varepsilon(y)|\,dy \\ &\leq \frac{4(R_0+Mt)\|q_\varepsilon\|_{L^1(\Pi_\varepsilon)}}{\pi|x|^2}. \end{aligned}$$

Observing that $H - H_\infty$ vanishes for $|x| > 2$ establishes the second bound in $L^2(\Pi_\varepsilon)$ as desired. □

Recalling the modified Biot-Savart law (13) we can rewrite u_ε in the following way:

$$u_\varepsilon = (1+\alpha\mathbb{A}_\varepsilon)^{-1}[\mathbf{K}_\varepsilon(q_\varepsilon) + m(H-H_\infty)] + \gamma(1+\alpha\mathbb{A}_\varepsilon)^{-1}H + m(1+\alpha\mathbb{A}_\varepsilon)^{-1}H_\infty. \quad (22)$$

We will now proceed to derive H^1 estimates for each of these terms.

We start with the first one.

Lemma 6. *We have that* $(1+\alpha\mathbb{A}_\varepsilon)^{-1}(\mathbf{K}_\varepsilon(q_\varepsilon) + m(H - H_\infty))$ *is bounded in the space* $L^\infty_{loc}([0,\infty); H^1(\Pi_\varepsilon))$ *independently of* ε.

Proof. Let us denote $b_\varepsilon = \mathbf{K}_\varepsilon(q_\varepsilon) + m(H - H_\infty)$ and $a_\varepsilon = (1+\alpha\mathbb{A}_\varepsilon)^{-1} b_\varepsilon$. Then $(1+\alpha\mathbb{A}_\varepsilon)a_\varepsilon = b_\varepsilon$. We do a classical energy estimate in which we multiply this relation by a_ε. We get

$$\int_{\Pi_\varepsilon} a_\varepsilon \cdot b_\varepsilon = \int_{\Pi_\varepsilon} a_\varepsilon \cdot (1+\alpha\mathbb{A}_\varepsilon)a_\varepsilon = \int_{\Pi_\varepsilon} (|a_\varepsilon|^2 - \alpha a_\varepsilon \cdot \Delta a_\varepsilon) = \int_{\Pi_\varepsilon} (|a_\varepsilon|^2 + \alpha|\nabla a_\varepsilon|^2).$$

But

$$\int_{\Pi_\varepsilon} a_\varepsilon \cdot b_\varepsilon \leq \|a_\varepsilon\|^{\frac12}_{L^2(\Pi_\varepsilon)} \|b_\varepsilon\|^{\frac12}_{L^2(\Pi_\varepsilon)} \leq \Big(\int_{\Pi_\varepsilon} (|a_\varepsilon|^2 + \alpha|\nabla a_\varepsilon|^2)\Big)^{\frac12} \|b_\varepsilon\|^{\frac12}_{L^2(\Pi_\varepsilon)}$$

so

$$\int_{\Pi_\varepsilon} (|a_\varepsilon|^2 + \alpha|\nabla a_\varepsilon|^2) \leq \int_{\Pi_\varepsilon} |b_\varepsilon|^2.$$

From the previous lemma we know that b_ε is bounded in L^2 independently of ε, so the conclusion follows. □

Next we will discuss the third term in (22). To this end we perform an energy estimate for the term $(1+\alpha\mathbb{A}_\varepsilon)^{-1}H_\infty$.

Lemma 7. *There exists a constant $C = C(\alpha)$ which depends only on α such that*
$$\|(1+\alpha\mathbb{A}_\varepsilon)^{-1}H_\infty - H_\infty\|_{H^1(\Pi_\varepsilon)} \le C(\alpha).$$

Proof. Since H_∞ is of the form $x^\perp$ times a radial function, we have that $w_1 \equiv (1+\alpha\mathbb{A}_\varepsilon)^{-1}H_\infty = (1-\alpha\Delta_{\Pi_\varepsilon})^{-1}H_\infty$ solves the following boundary value problem:
$$\begin{cases} w_1 - \alpha\Delta w_1 = H_\infty \text{ for } |x| > \varepsilon \\ w_1\big|_{|x|=\varepsilon} = 0. \end{cases}$$
Then $w_2 \equiv w_1 - H_\infty$ satisfies
$$\begin{cases} w_2 - \alpha\Delta w_2 = \alpha\Delta H_\infty \text{ for } |x| > \varepsilon \\ w_2\big|_{|x|=\varepsilon} = 0. \end{cases}$$
The same L^2 estimate as in the proof of Lemma 6 shows that
$$\|w_2\|^2_{L^2(\Pi_\varepsilon)} + \alpha\|\nabla w_2\|^2_{L^2(\Pi_\varepsilon)} \le \alpha^2\|\Delta H_\infty\|^2_{L^2(\Pi_\varepsilon)}.$$
Since $H_\infty = \eta H$ and H is harmonic we have that
$$\Delta H_\infty = \Delta\eta\, H + 2\nabla\eta\cdot\nabla H$$
is a C^∞ function compactly supported in the annulus $\{1 \le |x| \le 2\}$ which does not depend on ε. The conclusion follows. □

To complete the estimate of u_ε, it remains only to bound the term $(1+\alpha\mathbb{A}_\varepsilon)^{-1}H$. The strategy in deriving this estimate is to compare $(1+\alpha\mathbb{A}_\varepsilon)^{-1}$, in Π_ε, to $(1-\alpha\Delta_{\mathbb{R}^2})^{-1}$ in all of $\mathbb{R}^2$. We recall that the vector field K^α was defined in relation (10).

Proposition 8. *There exists a constant $C > 0$, which depends only on α, such that*
$$\|(1+\alpha\mathbb{A}_\varepsilon)^{-1}H - K^\alpha\|_{H^1(\Pi_\varepsilon)} \le C\varepsilon|\log\varepsilon|.$$

Proof. Since H is of the form $x^\perp$ times a radial function, we have that $w_3 \equiv (1+\alpha\mathbb{A}_\varepsilon)^{-1}H = (1-\alpha\Delta_{\Pi_\varepsilon})^{-1}H$ is of the same form and solves the following boundary value problem:
$$\begin{cases} (1-\alpha\Delta)w_3 = H, & \text{in } \Pi_\varepsilon \\ \operatorname{div} w_3 = 0, & \text{in } \Pi_\varepsilon \\ w_3 = 0, & \text{on } \partial\Pi_\varepsilon. \end{cases}$$
Since H is divergence free in the whole of $\mathbb{R}^2$, we have that $K^\alpha = \mathcal{G}_\alpha * H$ is also divergence free everywhere. Then $w_4 \equiv w_3 - K^\alpha$ satisfies the following boundary-value problem:
$$\begin{cases} (1-\alpha\Delta)w_4 = 0, & \text{in } \Pi_\varepsilon \\ \operatorname{div} w_4 = 0, & \text{in } \Pi_\varepsilon \\ w_4 = -K^\alpha, & \text{on } \partial\Pi_\varepsilon. \end{cases} \tag{23}$$
Set
$$F \equiv \operatorname{curl} w_4. \tag{24}$$
Taking the curl of the first equation in (23) we obtain
$$F - \alpha\Delta F = 0 \quad \text{in } \Pi_\varepsilon. \tag{25}$$

Note that, because $\operatorname{div} w_4 = 0$, we have

$$\nabla^\perp F = \Delta w_4. \tag{26}$$

Evaluating the first equation in (23) on $\partial\Pi_\varepsilon$, using (26) and taking the inner product of the result with the unit vector $x^\perp/\varepsilon$, which is tangent to $\partial\Pi_\varepsilon$, yields

$$w_4 \cdot \frac{x^\perp}{\varepsilon} - \alpha\nabla^\perp F \cdot \frac{x^\perp}{\varepsilon} = 0 \quad \text{ on } |x| = \varepsilon.$$

Set $\widehat{\mathbf{n}} \equiv x/\varepsilon$, the *interior* unit normal to $\partial\Pi_\varepsilon$. Then, since $\nabla^\perp F \cdot x^\perp/\varepsilon = \nabla F \cdot \widehat{\mathbf{n}} = \partial F/\partial\widehat{\mathbf{n}}$, we have derived a Neumann boundary condition for F:

$$\frac{\partial F}{\partial \widehat{\mathbf{n}}} = -\frac{1}{\alpha}(K^\alpha) \cdot \widehat{\mathbf{n}}^\perp \qquad \text{at } |x| = \varepsilon. \tag{27}$$

Next we use the circular symmetry of H together with the radial symmetry of $\mathcal{G}_\alpha$ to deduce that K^α is a vector of the form $x^\perp$ times a radial function, see (12). Hence the right-hand-side of (27) is a real number which depends only on α, which is fixed, and on ε. We denote it by A_ε:

$$A_\varepsilon \equiv -\frac{1}{\alpha} K^\alpha\big|_{|x|=\varepsilon} \cdot \frac{x^\perp}{\varepsilon}.$$

With this notation we note, in particular, that

$$w_4\big|_{\partial\Pi_\varepsilon} = \frac{\alpha}{\varepsilon} A_\varepsilon x^\perp. \tag{28}$$

We also observe at this point that, thanks to Lemma 3, we have that

$$A_\varepsilon = O(\varepsilon|\log\varepsilon|) \quad \text{as } \varepsilon \to 0. \tag{29}$$

We have, thus far, deduced a Neumann boundary-value problem for F:

$$\begin{cases} (1-\alpha\Delta)F = 0, & \text{in } \Pi_\varepsilon \\ \frac{\partial F}{\partial \widehat{\mathbf{n}}} = A_\varepsilon, & \text{on } \partial\Pi_\varepsilon. \end{cases}$$

We already observed that w_3 and K^α are of the form $x^\perp$ times a radial function, so $w_4 = w_3 - K^\alpha$ is also of this form. Therefore $F = \operatorname{curl} w_4$ is radially symmetric in Π_ε. Hence the restriction of F to the boundary $\partial\Pi_\varepsilon$ depends only on α, fixed, and on ε. We denote this constant by B_ε:

$$B_\varepsilon \equiv F\big|_{\partial\Pi_\varepsilon}. \tag{30}$$

Our next step is to extend F to all of $\mathbb{R}^2$, find an equation satisfied by this extension and solve it. To this end consider the continuous extension of F given by

$$\overline{F} \equiv \begin{cases} F, \text{ if } |x| > \varepsilon \\ B_\varepsilon, \text{ if } |x| \leq \varepsilon. \end{cases} \tag{31}$$

Let us compute $\Delta\overline{F}$ in the sense of distributions on $\mathbb{R}^2$. Fix $\varphi \in C_c^\infty(\mathbb{R}^2)$. Then

$$\begin{aligned}\langle\Delta\overline{F},\varphi\rangle &= \langle\overline{F},\Delta\varphi\rangle\\ &= \int_{\{|x|>\varepsilon\}} F\Delta\varphi + \int_{\{|x|\le\varepsilon\}} B_\varepsilon\Delta\varphi\\ &= -\int_{\{|x|>\varepsilon\}} \nabla F\nabla\varphi + \int_{\{|x|=\varepsilon\}} F\nabla\varphi\cdot(-\widehat{\mathbf{n}}) + B_\varepsilon\int_{\{|x|=\varepsilon\}}\nabla\varphi\cdot\widehat{n}\\ &= \int_{\{|x|>\varepsilon\}} \varphi\Delta F - \int_{\{|x|=\varepsilon\}}\varphi\nabla F\cdot(-\widehat{\mathbf{n}})\\ &= \int_{\{|x|>\varepsilon\}} \varphi\Delta F + A_\varepsilon\int_{\{|x|=\varepsilon\}}\varphi.\end{aligned}$$

Recalling (25) and (31) we infer that

$$(1-\alpha\Delta)\overline{F} = B_\varepsilon\chi_{\{|x|\le\varepsilon\}} - \alpha A_\varepsilon\delta_{\{|x|=\varepsilon\}} \quad \text{in } \mathscr{D}'(\mathbb{R}^2).$$

We can now invert the operator $1-\alpha\Delta_{\mathbb{R}^2}$ to find a formula for $\overline{F}$:

$$\overline{F} = \mathcal{G}_\alpha * \left(B_\varepsilon\chi_{\{|x|\le\varepsilon\}} - \alpha A_\varepsilon\delta_{\{|x|=\varepsilon\}}\right).$$

Evaluating the above expression at 0 we have, by definition of $\overline{F}$,

$$B_\varepsilon = \overline{F}(0) = B_\varepsilon\int_{\{|x|\le\varepsilon\}}\mathcal{G}_\alpha - \alpha A_\varepsilon\int_{\{|x|=\varepsilon\}}\mathcal{G}_\alpha\, ds.$$

It follows, see property (P2) for $\mathcal{G}_\alpha$, that

$$B_\varepsilon = -\alpha A_\varepsilon\frac{\int_{\{|x|=\varepsilon\}}\mathcal{G}_\alpha\, ds}{\int_{\{|x|>\varepsilon\}}\mathcal{G}_\alpha}.$$

Using the properties of the kernel $\mathcal{G}_\alpha$ and the estimate of the size of A_ε given in (29) we can estimate the size of B_ε. More precisely, relation (8) implies that

$$\int_{\{|x|>\varepsilon\}}\mathcal{G}_\alpha \to \int_{\mathbb{R}^2}\mathcal{G}_\alpha = 1 \quad \text{as } \varepsilon\to 0,$$

and, thanks to (9), we have that

$$\int_{\{|x|=\varepsilon\}}\mathcal{G}_\alpha\, ds = O(\varepsilon|\log\varepsilon|).$$

Combining these two bounds with (29) implies that

$$B_\varepsilon = O(\varepsilon^2|\log\varepsilon|^2) \quad \text{as } \varepsilon\to 0. \tag{32}$$

Knowing B_ε, we can compute the H^1 norm of w_4. We multiply the first equation of (23) by w_4 and integrate on Π_ε. We obtain

$$0 = \int_{\Pi_\varepsilon}(w_4-\alpha\Delta w_4)\cdot w_4 = \int_{\Pi_\varepsilon}|w_4|^2 + \alpha\int_{\Pi_\varepsilon}|\nabla w_4|^2 + \alpha\int_{\partial\Pi_\varepsilon}\frac{\partial w_4}{\partial\widehat{\mathbf{n}}}\cdot w_4.$$

Given that w_4 is of the form $x^\perp$ times by a radial function, we can show, through an easy calculation, the following identity:

$$x\cdot\nabla w_4 = x^\perp\operatorname{curl} w_4 - w_4.$$

Recalling (28), (30) and (24) we observe that

$$\begin{aligned}\int_{\partial\Pi_\varepsilon} \frac{\partial w_4}{\partial \widehat{\mathbf{n}}}\cdot w_4 &= \frac{1}{\varepsilon}\int_{\partial\Pi_\varepsilon} x\cdot\nabla w_4\cdot w_4\\ &= \frac{1}{\varepsilon}\int_{\partial\Pi_\varepsilon}(x^\perp \operatorname{curl} w_4 - w_4)\cdot w_4\\ &= \frac{1}{\varepsilon}\int_{\partial\Pi_\varepsilon}(B_\varepsilon - \frac{\alpha}{\varepsilon}A_\varepsilon)x^\perp\cdot x^\perp\frac{\alpha}{\varepsilon}A_\varepsilon\\ &= 2\pi\alpha\varepsilon A_\varepsilon(B_\varepsilon - \frac{\alpha}{\varepsilon}A_\varepsilon).\end{aligned}$$

Finally, we conclude that

$$\int_{\Pi_\varepsilon}|w_4|^2 + \alpha\int_{\Pi_\varepsilon}|\nabla w_4|^2 = -\alpha\int_{\partial\Pi_\varepsilon}\frac{\partial w_4}{\partial\widehat{\mathbf{n}}}\cdot w_4 = 2\pi\alpha^2\varepsilon A_\varepsilon(\frac{\alpha}{\varepsilon}A_\varepsilon - B_\varepsilon).$$

Recalling that $w_4 = (1+\alpha\mathbb{A}_\varepsilon)^{-1}H - K^\alpha$ and using (29) and (32) completes the proof. □

We summarize the results of this section in the result below.

Theorem 9. *We have that $u_\varepsilon - \gamma K^\alpha - mH_\infty$ is bounded in $L^\infty_{loc}([0,\infty);H^1(\Pi_\varepsilon))$ independently of ε.*

Proof. We use the decomposition (22) to write

$$\begin{aligned}u_\varepsilon - \gamma K^\alpha - mH_\infty &= (1+\alpha\mathbb{A}_\varepsilon)^{-1}(\mathbf{K}_\varepsilon(q_\varepsilon) + m(H-H_\infty))\\ &\quad + \gamma\big[(1+\alpha\mathbb{A}_\varepsilon)^{-1}H - K^\alpha\big] + m\big[(1+\alpha\mathbb{A}_\varepsilon)^{-1}H_\infty - H_\infty\big].\end{aligned}$$

The first term on the rhs is bounded in $H^1(\Pi_\varepsilon)$ as a consequence of Lemma 6, the second term is bounded in $H^1(\Pi_\varepsilon)$ thanks to Proposition 8 and the H^1 bound for the third term follows from Lemma 7. □

5. Temporal estimates and passing to the limit

We will now prove the convergence result, which is part b) of Theorem 1.

We define

$$w_\varepsilon = K_\varepsilon(q_\varepsilon) + mH. \tag{33}$$

Let $\widetilde{u}_\varepsilon$ be the extension of u_ε to the whole of $\mathbb{R}^2$ with zero values for $|x|\le\varepsilon$. We define in a similar manner $\widetilde{q}_\varepsilon$ and $\widetilde{w}_\varepsilon$. Since u_ε and w_ε are tangent to the boundary, we infer that $\widetilde{u}_\varepsilon$ and $\widetilde{w}_\varepsilon$ are divergence free in the whole of $\mathbb{R}^2$.

First we note that $K^\alpha \in H^1_{loc}(\mathbb{R}^2)$. Indeed, this follows from parts (a) and (b) of Lemma 3 together with properties (P3) and (P4) of $\mathcal{G}_\alpha$. Therefore, using Theorem 9 together with the vanishing of u_ε on the boundary of Π_ε, we obtain that $\widetilde{u}_\varepsilon - \gamma K^\alpha - mH_\infty$ is bounded in $L^\infty_{loc}([0,\infty);H^1(\mathbb{R}^2))$. We infer that there exists some divergence free limit vector field u such that

$$u - \gamma K^\alpha - mH_\infty \in L^\infty_{loc}([0,\infty);H^1(\mathbb{R}^2)), \tag{34}$$

and, passing to subsequences as needed,

$$\widetilde{u}_\varepsilon - u \rightharpoonup 0 \text{ in } L^\infty_{loc}([0,\infty);H^1(\mathbb{R}^2)) \text{ weak}* \text{ as } \varepsilon\to 0. \tag{35}$$

In view of the discussion above we have, in particular, that $\widetilde{u}_\varepsilon$ is bounded in $L^\infty_{loc}([0,\infty);H^1_{loc}(\mathbb{R}^2))$ and u belongs to $L^\infty_{loc}([0,\infty);H^1_{loc}(\mathbb{R}^2))$.

We also find, thanks to Lemma 5, that $\widetilde{w}_\varepsilon$ is bounded in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ independently of ε, so we can further assume that

$$\widetilde{w}_\varepsilon \rightharpoonup w \quad \text{in } L^\infty(\mathbb{R}_+ \times \mathbb{R}^2) \text{ weak}* \text{ as } \varepsilon \to 0 \tag{36}$$

and

$$\widetilde{q}_\varepsilon \rightharpoonup q \quad \text{in } L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)) \text{ weak}* \text{ as } \varepsilon \to 0. \tag{37}$$

Recall the equation for the potential vorticity

$$\partial_t q_\varepsilon + u_\varepsilon \cdot \nabla q_\varepsilon = 0 \quad \text{in } (0,\infty) \times \Pi_\varepsilon.$$

Since u_ε is tangent to the boundary of Π_ε (it even vanishes), the extensions $\widetilde{u}_\varepsilon$ and $\widetilde{q}_\varepsilon$ satisfy the same PDE in all of $\mathbb{R}^2$:

$$\partial_t \widetilde{q}_\varepsilon + \widetilde{u}_\varepsilon \cdot \nabla \widetilde{q}_\varepsilon = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^2. \tag{38}$$

We observed above that Theorem 9 implies that $\widetilde{u}_\varepsilon - \gamma K^\alpha - mH_\infty$ is bounded in $L^\infty_{loc}([0,\infty); H^1(\mathbb{R}^2)) \hookrightarrow L^\infty_{loc}([0,\infty); L^4(\mathbb{R}^2))$. From Lemma 3 and the definition of H_∞ we observe that $\gamma K^\alpha + mH_\infty \in L^4(\mathbb{R}^2)$ and, therefore, $\widetilde{u}_\varepsilon$ is bounded in $L^\infty_{loc}([0,\infty); L^4(\mathbb{R}^2))$. Since $\widetilde{q}_\varepsilon$ is bounded in $L^\infty(\mathbb{R}_+; L^4(\mathbb{R}^2))$ we infer that $\widetilde{u}_\varepsilon \widetilde{q}_\varepsilon$ is bounded in the space $L^\infty_{loc}([0,\infty); L^2(\mathbb{R}^2))$, so that $\widetilde{u}_\varepsilon \cdot \nabla \widetilde{q}_\varepsilon = \operatorname{div}(\widetilde{u}_\varepsilon \widetilde{q}_\varepsilon)$ is bounded in $L^\infty_{loc}([0,\infty); H^{-1}(\mathbb{R}^2))$. Then $\partial_t \widetilde{q}_\varepsilon = -\widetilde{u}_\varepsilon \cdot \nabla \widetilde{q}_\varepsilon$ is also bounded in $L^\infty_{loc}([0,\infty); H^{-1}(\mathbb{R}^2))$. We infer that the $\widetilde{q}_\varepsilon$ are equicontinuous in time with values in $H^{-1}(\mathbb{R}^2)$. Using the compactness of the embedding $H^{-1}(\mathbb{R}^2) \hookrightarrow H^{-2}_{loc}(\mathbb{R}^2)$ and the Ascoli theorem, we infer that, passing to subsequences if necessary, $\widetilde{q}_\varepsilon \to q$ in $C^0([0,\infty); H^{-2}_{loc}(\mathbb{R}^2)$. Recalling that $\widetilde{q}_\varepsilon$ is bounded in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^2))$ we finally deduce that

$$\widetilde{q}_\varepsilon \to q \quad \text{in } C^0([0,\infty); H^{-1}_{loc}(\mathbb{R}^2)) \text{ strongly as } \varepsilon \to 0. \tag{39}$$

The weak convergence of $\widetilde{u}_\varepsilon$ in H^1 obtained in (35) then implies that $\widetilde{u}_\varepsilon \widetilde{q}_\varepsilon \to uq$ in the sense of distributions. Thus we also have that $\operatorname{div}(\widetilde{u}_\varepsilon \widetilde{q}_\varepsilon) \to \operatorname{div}(uq)$ in the sense of the distributions in $\mathbb{R}^2$. Hence, we can pass to the limit $\varepsilon \to 0$ in (38) to obtain that

$$\partial_t q + u \cdot \nabla q = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^2.$$

In addition, the strong convergence found in (39) implies that we also have convergence for the initial data: $\widetilde{q}_\varepsilon(0,\cdot) \to q(0,\cdot)$ as $\varepsilon \to 0$ in $H^{-1}_{loc}(\mathbb{R}^2)$. We conclude that we have an initial condition for the equation of q:

$$q(0,x) = q_0(x).$$

Next we will obtain the relationship between u and q expressed in the system of PDE satisfied by q. We already know that u is divergence-free, as limit of u_ε, which are divergence-free. We will proceed to show that $\operatorname{curl}(1-\alpha\Delta)u = q + \gamma\delta$. This will be done in two steps: first we determine the equation for the limit of $\widetilde{w}_\varepsilon$ and then we determine the equation for the limit of $\widetilde{u}_\varepsilon$. The following lemma deals with the first step.

Lemma 10. *We have that* $w \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$, *that* $\operatorname{div} w = 0$ *and that* $\operatorname{curl} w = q$.

Proof. We already know that $w \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ and that $\operatorname{div} w = 0$. Let us compute $\operatorname{curl} w$. Let $\varphi \in C^\infty_c((0,\infty) \times \mathbb{R}^2)$ be a test function and choose some $\mu > 0$. Let $\eta_\mu(x) \equiv \eta(x/\mu)$ and $\varphi_\mu = \eta_\mu \varphi$, where η was introduced on page 3. We assume that

$\varepsilon < \mu$. We multiply $\widetilde{q}_\varepsilon$ by φ_μ and we integrate by parts, using that φ_μ is compactly supported in Π_ε and that $q_\varepsilon = \operatorname{curl} w_\varepsilon$:

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^2} \widetilde{q}_\varepsilon \varphi_\mu &= \int_0^\infty \int_{\Pi_\varepsilon} q_\varepsilon \varphi_\mu \\ &= \int_0^\infty \int_{\Pi_\varepsilon} \operatorname{curl} w_\varepsilon \varphi_\mu \\ &= -\int_0^\infty \int_{\Pi_\varepsilon} w_\varepsilon \cdot \nabla^\perp \varphi_\mu \\ &= -\int_0^\infty \int_{\mathbb{R}^2} \widetilde{w}_\varepsilon \cdot \nabla^\perp \varphi_\mu .\end{aligned}$$

We send $\varepsilon \to 0$ and use the weak convergences found in (36) and in (37) to obtain that

$$\int_0^\infty \int_{\mathbb{R}^2} q \varphi_\mu = -\int_0^\infty \int_{\mathbb{R}^2} w \cdot \nabla^\perp \varphi_\mu .$$

One can easily check that $\varphi_\mu \to \varphi$ weakly in H^1 as $\mu \to 0$. Recalling that w is bounded in space and time and, hence, it belongs to L^2_{loc}, and observing that φ_μ is supported in a compact set independent of μ, we can pass to the limit $\mu \to 0$ above to obtain that

$$\int_0^\infty \int_{\mathbb{R}^2} q \varphi = -\int_0^\infty \int_{\mathbb{R}^2} w \cdot \nabla^\perp \varphi .$$

This means that $\operatorname{curl} w = q$ in the sense of the distributions. This concludes the proof of the lemma. □

The second step consists in computing $u - \alpha \Delta u$ in terms of w.

Lemma 11. *We have that $u - \alpha \Delta u = w + \gamma H$.*

Proof. Let $\Psi \in C^\infty_{c,\sigma}((0,\infty) \times \mathbb{R}^2; \mathbb{R}^2)$ be a divergence-free test vector field and choose some $\mu > 0$. We assume $\varepsilon < \mu$.

Since Ψ is divergence-free, there exists $\Phi \in C^\infty((0,\infty) \times \mathbb{R}^2; \mathbb{R})$, compactly supported in time, such that $\Psi = \nabla^\perp \Phi$. We can assume, without loss of generality, that $\Phi(t, 0) = 0$ for all t.

As before, let $\eta_\mu(x) = \eta(x/\mu)$ and $\Psi_\mu = \nabla^\perp(\eta_\mu \Phi)$. Then Ψ_μ is divergence free and compactly supported in Π_ε so $(1 + \alpha \mathbb{A}_\varepsilon)\Psi_\mu = (1 - \alpha\Delta)\Psi_\mu$. Recall that $(1 + \alpha \mathbb{A}_\varepsilon) u_\varepsilon = w_\varepsilon + \gamma H$, see (13) and the definition of w_ε given in relation (33). We write

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^2} \widetilde{u}_\varepsilon \cdot (\Psi_\mu - \alpha \Delta \Psi_\mu) &= \int_0^\infty \int_{\Pi_\varepsilon} u_\varepsilon \cdot (1 + \alpha \mathbb{A}_\varepsilon) \Psi_\mu \\ &= \int_0^\infty \int_{\Pi_\varepsilon} (1 + \alpha \mathbb{A}_\varepsilon) u_\varepsilon \cdot \Psi_\mu \\ &= \int_0^\infty \int_{\Pi_\varepsilon} w_\varepsilon \cdot \Psi_\mu + \gamma \int_0^\infty \int_{\Pi_\varepsilon} H \cdot \Psi_\mu \\ &= \int_0^\infty \int_{\mathbb{R}^2} \widetilde{w}_\varepsilon \cdot \Psi_\mu + \gamma \int_0^\infty \int_{\mathbb{R}^2} H \cdot \Psi_\mu .\end{aligned}$$

We now let $\varepsilon \to 0$ and use (35) and (36) to pass to the limit. We obtain

$$\int_0^\infty \int_{\mathbb{R}^2} u \cdot (\Psi_\mu - \alpha \Delta \Psi_\mu) = \int_0^\infty \int_{\mathbb{R}^2} w \cdot \Psi_\mu + \gamma \int_0^\infty \int_{\mathbb{R}^2} H \cdot \Psi_\mu .$$

We rewrite the last term above in the following form:

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^2} H \cdot \Psi_\mu &= \int_0^\infty \int_{\mathbb{R}^2} H \cdot \nabla^\perp(\eta_\mu \Phi) \\ &= \int_0^\infty \int_{\mathbb{R}^2} H \cdot \nabla^\perp((\eta_\mu - 1)\Phi) + \int_0^\infty \int_{\mathbb{R}^2} H \cdot \nabla^\perp \Phi \\ &= \int_0^\infty \int_{\mathbb{R}^2} H \cdot \nabla^\perp((\eta_\mu - 1)\Phi) + \int_0^\infty \int_{\mathbb{R}^2} H \cdot \Psi.\end{aligned}$$

We now use that $\operatorname{curl} H = \delta$, we recall that $\eta_\mu - 1$ is C^∞ and compactly supported, and we write the following sequence of equalities in the sense of the distributions $\mathscr{D}'(\mathbb{R}^2)$:

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^2} H \cdot \nabla^\perp((\eta_\mu - 1)\Phi) = \int_0^\infty \langle H, \nabla^\perp((\eta_\mu - 1)\Phi)\rangle = -\int_0^\infty \langle \operatorname{curl} H, (\eta_\mu - 1)\Phi\rangle \\ = -\int_0^\infty \langle \delta, (\eta_\mu - 1)\Phi\rangle = \int_0^\infty \Phi(t,0)\, dt = 0,\end{aligned}$$

where we also used that $\eta_\mu(0) = 0$.

We infer that

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^2} (u \cdot \Psi_\mu + \alpha \nabla u \cdot \nabla \Psi_\mu) &= \int_0^\infty \int_{\mathbb{R}^2} u \cdot (\Psi_\mu - \alpha \Delta \Psi_\mu) \\ &= \int_0^\infty \int_{\mathbb{R}^2} w \cdot \Psi_\mu + \gamma \int_0^\infty \int_{\mathbb{R}^2} H \cdot \Psi.\end{aligned}$$

Since $\Phi(t,0) = 0$ one can easily check that $\Psi_\mu \to \Psi$ in $L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^2))$ weak$*$ and, moreover, the support of Ψ_μ is included in a compact set uniformly with respect to μ. Since $u \in L^\infty_{loc}([0,\infty; H^1_{loc}(\mathbb{R}^2))$ one can pass to the limit $\mu \to 0$ above to obtain

$$\int_0^\infty \int_{\mathbb{R}^2} (u \cdot \Psi + \alpha \nabla u \cdot \nabla \Psi) = \int_0^\infty \int_{\mathbb{R}^2} w \cdot \Psi + \gamma \int_0^\infty \int_{\mathbb{R}^2} H \cdot \Psi.$$

This can be written in the following form in the sense of the distributions:

$$\langle u - \alpha \Delta u - w - \gamma H, \Psi\rangle = 0$$

for all divergence-free test vector fields $\Psi \in C^\infty_{c,\sigma}((0,\infty) \times \mathbb{R}^2; \mathbb{R}^2)$. Since the vector field $u - \alpha \Delta u - w - \gamma H$ is divergence free, we deduce from the relation above that it must vanish. This completes the proof of the proposition. □

Recall that $\operatorname{curl} H = \delta$ in $\mathbb{R}^2$. Then, by virtue of Lemmas 10 and 11, it follows that

$$\operatorname{curl}(u - \alpha \Delta u) = \operatorname{curl}(w + \gamma H) = \operatorname{curl} w + \gamma \delta = q + \gamma \delta. \tag{40}$$

We have shown the convergence of a subsequence of q_ε towards a solution of (5). In the next section we will show that the solutions of (5) are unique, which, in turn, implies that the full sequence q_ε converges to q, without the need to pass to a subsequence.

To conclude the proof of part b) of Theorem 1 it remains to show that $u \in L^\infty_{loc}(\mathbb{R}_+; L^p(\mathbb{R}^2))$ for any $p > 2$. This follows immediately from (34) since it is easy to see, using Lemma 3 part b) and the definition of H_∞, that both K^α and $H_\infty \in L^p(\mathbb{R}^2)$, for all $p > 2$, and since $H^1(\mathbb{R}^2) \subset L^p(\mathbb{R}^2)$ for all $p \geq 1$.

Remark 12. *Recall that the kernel of the solution operator in the full plane for* $\operatorname{curl}(1-\alpha\Delta)$*, on divergence-free vector fields vanishing at infinity, is* $K^\alpha = \mathcal{G}_\alpha * H$*. This solution operator can be easily extended to solenoidal vector fields in* $L^p(\mathbb{R}^2)$*. Therefore, using* (40) *and that* $u \in L^\infty_{loc}(\mathbb{R}_+; L^p(\mathbb{R}^2))$, $p > 2$*, we infer that* u *can be expressed as:*

$$u = K^\alpha * (q + \gamma\delta) = K^\alpha * q + \gamma K^\alpha.$$

Moreover, if we denote by $\check{v} = H * q$ *the velocity field associated to* q *in* $\mathbb{R}^2$ *then one can check that*

$$\operatorname{curl}(\partial_t \check{v} + u\cdot\nabla\check{v} + \sum_j \check{v}_j \nabla u_j) = \partial_t \operatorname{curl}\check{v} + u\cdot\nabla\operatorname{curl}\check{v} = \partial_t q + u\cdot\nabla q = 0.$$

So the velocity formulation of (5) *can be written in the form*

$$\begin{aligned}\partial_t \check{v} + u\cdot\nabla\check{v} + \sum_j \check{v}_j \nabla u_j + \nabla p &= 0\\ \operatorname{div} u &= 0\\ u - \alpha\Delta u &= \check{v} + \gamma H.\end{aligned}$$

6. Uniqueness for the limit system

The global existence of solutions of (5) follows from the convergence result established in the previous section. Here we will prove uniqueness of solutions of (5), thereby completing the proof of Theorem 1.

Let us observe that the α–Euler equations in $\mathbb{R}^2$, with an initial vorticity given by a bounded measure such as $q_0 + \gamma\delta$, have a global unique solution, see [11]. But, as noted in the introduction, even though the limit system (5) is very similar to the α–Euler system, it is not the same. In addition, [11] proves uniqueness of Lagrangian solutions by working on the trajectories of the velocity field. Even if we could adapt the proof of uniqueness for α–Euler to (5), we would still have to make the connection between Lagrangian solutions and the Eulerian solutions considered here. Instead, we will give below a more classical proof of uniqueness, based on energy estimates.

Let $q, q' \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2))$ be two solutions of the limit system (5) with the same initial data $q(0,x) = q'(0,x) = q_0(x)$. Then $u = K^\alpha * q + \gamma K^\alpha$ and $u' = K^\alpha * q' + \gamma K^\alpha$, see Remark 12.

Let $\overline{q} = q - q'$, $\overline{u} = u - u'$ and $\overline{v} = H * \overline{q} = \nabla^\perp \Delta^{-1}\overline{q}$. Clearly $\int_{\mathbb{R}^2} q\,dx = \int_{\mathbb{R}^2} q_0\,dx = \int_{\mathbb{R}^2} q'\,dx$ so $\int_{\mathbb{R}^2} \overline{q}\,dx = 0$. In addition, q and q' are obviously compactly supported in space. We infer that $\overline{v}$ decays like $O(1/|x|^2)$ at infinity, so that it belongs to L^2.

We have the following PDE for $\overline{q}$:

$$\partial_t \overline{q} + u\cdot\nabla q - u'\cdot\nabla q' = 0.$$

We multiply by $\Delta^{-1}\overline{q}$ and integrate in $[0,T]\times\mathbb{R}^2$. We follow the same argument as in Section 3, when we estimated $\frac{\mathrm{d}}{\mathrm{d}t}\|\overline{v}^{n+1} - \overline{v}^n\|^2_{L^2(\Pi_\varepsilon)}$. Redoing the same estimates as those found on pages 9–10 we find

$$\begin{aligned}\frac{1}{2}\|\overline{v}(T)\|_{L^2}^2 &= -\int_0^T\int_{\mathbb{R}^2}(\partial_2 u_1+\partial_1 u_2)\overline{v}_1\overline{v}_2 - \frac{1}{2}\int_0^T\int_{\mathbb{R}^2}(-\partial_1 u_1+\partial_2 u_2)(\overline{v}_2^2-\overline{v}_1^2)\\ &\qquad + \int_0^T\int_{\mathbb{R}^2}\overline{u}\cdot\overline{v}^\perp q'\\ &\leq C\int_0^T(\|\partial_2 u_1+\partial_1 u_2\|_{L^\infty}+\|\partial_1 u_1-\partial_2 u_2\|_{L^\infty})\|\overline{v}\|_{L^2}^2\\ &\qquad + C\int_0^T\|\overline{u}\|_{L^2}\|\overline{v}\|_{L^2}\|q'\|_{L^\infty}\\ &\leq C\int_0^T(1+\|\partial_2 u_1+\partial_1 u_2\|_{L^\infty}+\|\partial_1 u_1-\partial_2 u_2\|_{L^\infty})\|\overline{v}\|_{L^2}^2,\end{aligned}$$

where we used the relation $\overline{u}-\alpha\Delta\overline{u}=\overline{v}$ to bound $\|\overline{u}\|_{L^2}\leq\|\overline{v}\|_{L^2}$ and, also, that q' is bounded in L^∞.

We now use (11) to obtain

$$\begin{aligned}\|\partial_1 u_1-\partial_2 u_2\|_{L^\infty} &\leq \|(\partial_1 K_1^\alpha-\partial_2 K_2^\alpha)*q\|_{L^\infty}+|\gamma|\|\partial_1 K_1^\alpha-\partial_2 K_2^\alpha\|_{L^\infty}\\ &\leq \|(\partial_1 K_1^\alpha-\partial_2 K_2^\alpha)\|_{L^\infty}(\|q\|_{L^1}+|\gamma|)\\ &\leq C,\end{aligned}$$

where C is uniform in time. A similar estimate holds true for $\|\partial_2 u_1+\partial_1 u_2\|_{L^\infty}$. We deduce that

$$\|\overline{v}(T)\|_{L^2}^2\leq C\int_0^T\|\overline{v}\|_{L^2}^2.$$

The Gronwall inequality then implies that $\overline{v}=0$, so that $q=q'$. This completes the proof of Theorem 1.

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