On the asymptotic behaviour of 2D stationary Navier-Stokes solutions with symmetry conditions

Agathe Decaster & Dragoş Iftimie

Abstract

We consider the 2D stationary incompressible Navier-Stokes equations in $\mathbb{R}^2$. Under suitable symmetry, smallness and decay at infinity conditions on the forcing we determine the behaviour at infinity of the solutions. Moreover, when the forcing is small, satisfies suitable symmetry conditions and decays at infinity like a vector field homogeneous of degree –3, we show that there exists a unique small solution whose asymptotic behaviour at infinity is homogeneous of degree –1.

1 Introduction

We consider the incompressible stationary Navier-Stokes equations in $\mathbb{R}^2$:

$$-\Delta U + (U \cdot \nabla U) + \nabla p = f,\ \operatorname{div} U = 0 \quad \text{in } \mathbb{R}^2,\ \lim_{|x|\to\infty} U(x) = 0. \tag{1.1}$$

The forcing term f is given. The unknowns are the velocity field U and the scalar pressure p, but the pressure p is uniquely determined (up to a constant) by f and U so by solution we mean only the velocity field U. The aim of this paper is to determine the asymptotic behaviour of the solutions at infinity under suitable assumptions on the forcing term. The most physically relevant case is the case when the forcing f is compactly supported. For example, the asymptotic behaviour of solutions in the case of an exterior domain with vanishing forcing can be reduced to the case of a compactly supported forcing in the domain $\mathbb{R}^2$.

This problem has been studied in various cases. First, let us observe that the 3D case is now relatively well understood at least in the setting of small solutions. We mention just a few results, all of them valid for small solutions. In [3] an explicit asymptotic behaviour as $O(1/|x|^2)$ when $|x| \to \infty$ is found under the assumption that $\int_{\mathbb{R}^3} f = 0$. If $\int_{\mathbb{R}^3} f \neq 0$, the authors of [18] proved that the asymptotic behaviour of the solutions is given by a vector field homogeneous of degree -1 and [17] shows that this homogeneous vector field is a Landau solution. In [15], the authors extended the result of [17] to time-periodic solutions.

In dimension 2, the study of (1.1) is much more difficult. The literature on this case is very rich. Let us mention a few results, in our opinion the most prominent among those directly related to our work. Amick [2] and Gilbarg and Weinberger [10] considered the case of an exterior domain and proved the existence of solutions such that $\int |\nabla U|^2 < \infty$. They also proved that for such solutions there exists some U_∞ such that $\lim_{|x|\to\infty} U(x) = U_\infty$.

But they could not prescribe the value of U_∞. In particular, they could not prove the existence of solutions going to 0 at infinity.

Finn and Smith [8] tried to prove the existence and the uniqueness of solutions in the case of small data (a small forcing term and small boundary conditions in the case of an exterior domain), but they only succeeded in the case where the velocity at infinity is non zero ($U_\infty \neq 0$). The reason is that their method relies on the linearization of (1.1) and, when $U_\infty \neq 0$, this linearized system (Oseen system) produces solutions that, in the neighborhood of infinity, are more regular than those corresponding to the linearized system when $U_\infty = 0$. Amick proved in [1] some partial results on the asymptotic behaviour at infinity when $U_\infty \neq 0$. He showed that if the data vanishes and the solution satisfies some symmetry properties around the direction of U_∞, then U admits an asymptotic expansion which is the same as the fundamental solution of the Oseen system. We also refer to the more recent paper [16] on further results for the case of non zero velocity at infinity.

Based on the 3D result, one might think that the relevant asymptotic behaviour at infinity is homogeneous of degree -1. So it is interesting to find the solutions of (1.1) which are homogeneous of degree -1. This was studied in [13], see also [19], who found that all homogeneous solutions of degree -1 with $f = 0$ are either the trivial solutions $U(x) = \mu \frac{x^\perp}{|x|^2}$ or a discrete family of solutions. In [12], the authors found solutions that are scale-invariant up to a rotation, which means that for any $\lambda > 0$,

$$\lambda U(\lambda x) = R_\lambda^{-1} U(R_\lambda x).$$

They proved that these solutions are a family with two continuous parameters and one discrete parameter, generalizing the solutions of Hamel and Šverák. Unfortunately, it is not known how to perform a perturbation argument around these homogeneous solutions (like was done in [17] for the 3D Landau solutions) so we don't know if they describe the asymptotic behaviour of the solutions of (1.1) or not. Let us mention at this point the paper of Hillairet and Wittwer [14] where the authors are able to make a perturbation argument around the particular solution $\mu x^\perp/|x|^2$ for sufficiently large μ. Guillod [11] conjectures (via a formal asymptotic expansion) that in the case of nonzero net force there is formation of a wake and the optimal decay at infinity of the velocity may be like $|x|^{-1/3}$.

Even the existence of small solutions of (1.1) is not known. There is however a particular case when it is possible to find solutions of (1.1) by using a standard fixed point argument. It was proved in [20] that if the forcing satisfies the symmetry conditions

$$f_1(x_1, -x_2) = f_1(x_1, x_2), \quad f_2(x_1, -x_2) = -f_2(x_1, x_2), \quad f_1(x_2, x_1) = f_2(x_1, x_2) \tag{1.2}$$

and is sufficiently small and sufficiently decaying at infinity, then there exists a unique small solution of (1.1). Yamazaki also proved that the velocity is $O(1/|x|)$ and the vorticity is $O(1/|x|^2)$ at infinity.

Our first goal here is to find the exact asymptotic behaviour at infinity of the solutions of Yamazaki. We prove that, if the forcing term satisfies the symmetry properties of Yamazaki and decays at infinity like $O(1/|x|^{5+\delta})$, $0 < \delta < 2$, then the vorticity ω decays like $1/|x|^4$ at infinity and has the following asymptotic expansion:

$$\omega(x) = C\frac{x_1 x_2 (x_1^2 - x_2^2)}{|x|^8} + O\left(\frac{1}{|x|^{4+\delta}}\right) \qquad \text{as } |x| \to \infty$$

for some constant C, see Theorem 2.5 below. We also obtain the following expansion at infinity for the velocity field:

$$U(x) = \frac{C}{12|x|^8}\left(x_1(x_1^4 + 3x_2^4 - 8x_1^2x_2^2), x_2(3x_1^4 + x_2^4 - 8x_1^2x_2^2)\right) + O\left(\frac{1}{|x|^{3+\delta}}\right).$$

In addition, we show that the above result is optimal in the sense that there exists a forcing f such that the constant C does not vanish.

A natural question that arises is what happens when the forcing term decays slower that $1/|x|^5$ at infinity? For example, when the forcing is $O(1/|x|^3)$ one can construct a solution bounded by $1/|x|$ at infinity due to the result of Yamazaki [20]. How this solution behaves at infinity? We restrict our analysis to this case, *i.e.* solutions which decay like $1/|x|$ at infinity. There are several reasons for that. First, the decay $1/|x|$ at infinity is critical in the sense that it corresponds to the critical scaling of the Navier-Stokes equations. Second, we recalled above that in dimension three the relevant asymptotic behaviour of stationary solutions for compactly supported forcing is homogeneous of degree -1. Furthermore, in [6] we considered 3D stationary solutions corresponding to a forcing whose decay at infinity is homogeneous of degree -3. In the case of small data, we found a necessary and sufficient condition on the forcing in order to obtain solutions with decay at infinity homogeneous of degree -1. Motivated by these results, we consider here a small forcing verifying the symmetry conditions (1.2) and whose asymptotic behaviour at infinity is homogeneous of degree -3. We show that there exists a unique small stationary solution whose asymptotic behaviour at infinity is homogeneous of degree -1.

The plan of this paper is the following. The notation, the functional spaces and the statement of our results are given in Section 2. We prove our main result (Theorem 2.5) in Section 3. In Section 4 we will extend it to exterior domains, see Theorem 2.6. In the last section we deal with solutions decaying like $1/|x|$ at infinity, see Theorem 5.4.

2 Notations and main result

Throughout this paper a solution of (1.1) is a vector field U such that there exists some p such that (1.1) is satisfied. Let $\omega = \operatorname{curl} U$. Taking the curl of the first equation in (1.1) we obtain the vorticity form of the Navier-Stokes equations:

$$-\Delta\omega + (U \cdot \nabla)\omega = \operatorname{curl} f \text{ in } \mathbb{R}^2 \tag{2.1}$$

$$U(x) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{2\pi|x-y|^2}\omega(y)\,dy. \tag{2.2}$$

The second equation is the well-known Biot-Savart law.

Let us introduce the functional spaces we will use. We consider solutions with the same symmetry conditions as in [20]:

$$\omega(-x_1, x_2) = \omega(x_2, x_1) = -\omega(x_1, x_2) \qquad \forall (x_1, x_2) \in \mathbb{R}^2. \tag{2.3}$$

We remark that these conditions imply directly the following properties:

$$\begin{aligned} \omega(x_1, -x_2) &= -\omega(x_1, x_2) \\ \omega(-x_1, -x_2) &= \omega(x_1, x_2). \end{aligned}$$

Our functional spaces are made of bounded functions that decay sufficiently at infinity. In [20] the authors consider ω decaying like $1/|x|^2$ at infinity. Here we will see that if the forcing f is sufficiently decaying at infinity (for example if we consider $f \in C_c^\infty(\mathbb{R}^2)$), then ω decays like $1/|x|^4$ (and in general not better). This leads us to the following definition:

Definition 2.1. *For $\alpha > 0$, the space X_α is the space of all functions $\omega \in L^1_{loc}(\mathbb{R}^2)$ such that the conditions (2.3) are satisfied and*

$$\|\omega\|_{X_\alpha} = \sup_{\mathbb{R}^2} (1+|x|)^\alpha |\omega(x)| < \infty.$$

Now we introduce the associated spaces for the velocity field. From (2.2) we easily deduce the symmetry conditions for U:

$$\begin{aligned} U(x_1,-x_2) &= \int_{\mathbb{R}^2} \frac{(x_2+y_2, x_1-y_1)}{2\pi|(x_1-y_1,-x_2-y_2)|^2}\omega(y)\,dy \\ &= \int_{\mathbb{R}^2} \frac{(x_2-y_2, x_1-y_1)}{2\pi|(x_1-y_1,-x_2+y_2)|^2}\omega(y_1,-y_2)\,dy \\ &= -\int_{\mathbb{R}^2} \frac{(-(-x_2+y_2), x_1-y_1)}{2\pi|x-y|^2}\omega(y)\,dy \\ &= (U_1(x),-U_2(x)). \end{aligned}$$

So U_1 is even and U_2 is odd with respect to x_2. In the same way, one can show that U_1 is odd and U_2 is even with respect to x_1, and that $U_1(x_2,x_1)=U_2(x_1,x_2)$. Conversely, if $U:\mathbb{R}^2\to\mathbb{R}^2$ satisfies

$$\begin{aligned} U_1(x_1,-x_2) &= U_1(x_1,x_2) \\ U_2(x_1,-x_2) &= -U_2(x_1,x_2) \\ U_1(x_2,x_1) &= U_2(x_1,x_2) \end{aligned} \tag{2.4}$$

then we get that $\omega=\operatorname{curl} U$ verifies the conditions (2.3). This motivates the following definition of functional spaces for the velocity U:

Definition 2.2. *Let Y_β be the space of all vector fields $U:\mathbb{R}^2\to\mathbb{R}^2$ such that (2.4) hold true and*

$$\|U\|_{Y_\beta} = \sup_{\mathbb{R}^2}(1+|x|)^\beta|U(x)|<\infty.$$

Finally, we will need to consider vector fields such as ωU or $f^\perp$ that satisfy different symmetry properties. The corresponding functional spaces are almost the same as the spaces Y_β but with different symmetry conditions.

Definition 2.3. *Let Z_β be the space of all vector fields $V:\mathbb{R}^2\to\mathbb{R}^2$ such that*

$$\|V\|_{Z_\beta} = \sup_{\mathbb{R}^2}(1+|x|)^\beta|V(x)|<\infty$$

and that

$$\begin{aligned} V_1(x_1,-x_2) &= -V_1(x_1,x_2) \\ V_2(x_1,-x_2) &= V_2(x_1,x_2) \\ V_1(x_2,x_1) &= -V_2(x_1,x_2). \end{aligned}$$

We state now a classical fixed point lemma that will be used several times in this paper.

Lemma 2.4 ([5]). *Let X be a Banach space and let $B: X\times X\to X$ be a bilinear map. Assume that for all $x_1,x_2\in X$ we have that*

$$\|B(x_1,x_2)\|_X \leqslant \eta\|x_1\|_X\|x_2\|_X.$$

Then for all $y\in X$ satisfying $4\eta\|y\|_X<1$, the equation

$$x = y+B(x,x),$$

has a unique solution $x\in X$ in the ball $B_X(0,\frac{1}{2\eta})$. Moreover, we have that

$$\|x\|_X\leqslant 2\|y\|_X$$

and x is the limit of the sequence x_n defined recursively by

$$x_0=y,\quad x_{n+1}=y+B(x_n,x_n).$$

The main result of this article reads as follows.

Theorem 2.5. *Let $0 < \delta < 2$. There exist $\varepsilon_1, \varepsilon_2 > 0$ such that, for any $f \in Y_{5+\delta}$ such that $\|f\|_{Y_{5+\delta}} \leqslant \varepsilon_1$ there exists a unique ω in the ball $B(0, \varepsilon_2)$ of X_4 which solves* (2.1)*. Moreover, ω has the following asymptotic behaviour when $|x| \to \infty$:*

$$\omega(x) = m\frac{x_1x_2(x_1^2 - x_2^2)}{|x|^8} + O\left(\frac{1}{|x|^{4+\delta}}\right),$$

where the constant m is given by:

$$m = \frac{4}{\pi}\int_{\mathbb{R}^2}(3y_1^2y_2 - y_2^3)(U_1\omega - f_2)(y)\,dy.$$

Finally, this constant is generally non zero.

We extend next this result to exterior domains. We need to impose the following symmetry conditions on the pressure p:

$$p(x_1, -x_2) = p(-x_1, x_2) = p(x_2, x_1) = p(x_1, x_2) \qquad \forall x_1, x_2. \tag{2.5}$$

Theorem 2.6. *Let $R > 0$, $0 < \delta < 2$ and let f, U and p be defined for $|x| > R$. We assume:*

- *U and $\omega = \operatorname{curl} U$ are bounded and vanishing at infinity.*
- *$|x|^{5+\delta} f$ and $|x|^{5+\delta}U\omega$ are bounded.*
- *The symmetry conditions:* (2.4) *for the velocity U,* (1.2) *for the forcing f and* (2.5) *for the pressure p are satisfied.*
- *The stationary Navier-Stokes equations are verified on $|x| > R$ without boundary conditions at $|x| = R$:*

$$-\Delta U + (U\cdot\nabla)U + \nabla p = f, \quad \operatorname{div} U = 0 \quad \textit{in} \quad \{|x| > R\};$$

- *The "no outflow to infinity" condition*

$$\int_{|x|=R_1} U\cdot x\,d\sigma(x) = 0 \tag{2.6}$$

  is satisfied for all $R_1 > R$

Then there exists a constant m such that

$$\omega(x) = m\frac{x_1x_2(x_1^2 - x_2^2)}{|x|^8} + O\left(\frac{1}{|x|^{4+\delta}}\right) \qquad \textit{as } |x| \to \infty \tag{2.7}$$

and

$$U(x) = \frac{m}{12|x|^8}\left(x_1(x_1^4 + 3x_2^4 - 8x_1^2x_2^2), x_2(3x_1^4 + x_2^4 - 8x_1^2x_2^2)\right) + O\left(\frac{1}{|x|^{3+\delta}}\right) \qquad \textit{as } |x| \to \infty.$$

Remark 2.7. Integrating the divergence of U on an annulus and using the Stokes formula implies that the integral in the equation (2.6) is proportional to R_1. So the condition (2.6) does not depend on the choice of R_1: if (2.6) is true for one R_1, it is true for all of them. Moreover, the divergence free condition also implies that (2.6) holds true if U satisfies the homogeneous Dirichlet conditions on some exterior domain.

In Theorems 2.5 and 2.6 we assumed that the forcing is decaying sufficiently fast at infinity and we found an explicit asymptotic behavior as $1/|x|^4$ for the solution. Since in dimension three the relevant asymptotic behavior is like $1/|x|$, it would be interesting to construct solutions in dimension two that decay exactly like $1/|x|$ at infinity. For such solutions, the forcing should decay exactly like $1/|x|^3$ at infinity. We will consider in Section 5 such forcing terms who verify in addition the symmetry conditions of Yamazaki. The precise statements will be given only in the last section; indeed this requires introducing a substantial amount of new notions which will be used only in the last section. Let us simply say at this point that we will show that for small forcing homogeneous of degree -3 there exists a unique small solution homogeneous of degree -1, see Theorem 5.3. Moreover, if the forcing is small and admits an asymptotic behavior at infinity homogeneous of degree -3, then there exists a unique small solution which admits an asymptotic behavior at infinity homogeneous of degree -1, see Theorem 5.4. The important observation that allows us to deal with $O(1/|x|)$ solutions in Section 5 is that, even though $U \otimes U$ is not well-defined for such solutions, the quantities $U_1^2 - U_2^2$ and $U_1 U_2$ are well-defined in the principal value sense. And defining these two quantities suffice to give a sense to the PDE since the Navier-Stokes equations can be written under the following form:

$$
\begin{aligned}
-\Delta U_1 + \partial_1\big(p + \frac{|U|^2}{2}\big) + \partial_1\big(\frac{U_1^2 - U_2^2}{2}\big) + \partial_2(U_1U_2) = f_1 \\
-\Delta U_2 + \partial_2\big(p + \frac{|U|^2}{2}\big) - \partial_2\big(\frac{U_1^2 - U_2^2}{2}\big) + \partial_1(U_1U_2) = f_2.
\end{aligned}
$$

3 Proof of Theorem 2.5

First, we study how the Biot-Savart law operates in the spaces X_α and Y_β.

Lemma 3.1. *Let $\omega \in X_\alpha$ with $1 < \alpha < 6$. The associated velocity field U defined by* (2.2) *belongs to $Y_{\alpha-1}$, and we have the following inequality:*

$$\|U\|_{Y_{\alpha-1}} \leqslant C\|\omega\|_{X_\alpha}$$

for some constant $C = C(\alpha)$.

Proof. Let $\omega \in X_\alpha$ for $1 < \alpha < 6$. Clearly U is bounded so we only need to estimate $|x|^{\alpha-1}|U(x)|$. We can write

$$
\begin{aligned}
U(x) &= \int_{|y|\leqslant|x|/2} \frac{(x-y)^\perp}{2\pi|x-y|^2}\omega(y)\,dy + \int_{|x|/2<|y|<2|x|} \frac{(x-y)^\perp}{2\pi|x-y|^2}\omega(y)\,dy + \int_{|y|\geqslant 2|x|} \frac{(x-y)^\perp}{2\pi|x-y|^2}\omega(y)\,dy \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}
$$

We estimate the easiest integral first:

$$
\begin{aligned}
|I_3| &\leqslant C\|\omega\|_{X_\alpha} \int_{|y|\geqslant 2|x|} \frac{1}{|x-y||y|^\alpha}\,dy \\
&\leqslant C\|\omega\|_{X_\alpha} \int_{|y|\geqslant 2|x|} \frac{1}{|y|^{\alpha+1}}\,dy \\
&\leqslant \frac{C\|\omega\|_{X_\alpha}}{|x|^{\alpha-1}}
\end{aligned}
$$

where we used $\alpha > 1$ to deduce that $1/|y|^{\alpha+1}$ is integrable at infinity.

Next we consider I_2:

$$\begin{aligned}
|I_2| &\leqslant C\|\omega\|_{X_\alpha} \int_{|x|/2<|y|<2|x|} \frac{1}{|x-y||y|^\alpha}\,dy \\
&\leqslant \frac{C\|\omega\|_{X_\alpha}}{|x|^\alpha} \int_{|x|/2<|y|<2|x|} \frac{1}{|x-y|}\,dy \\
&\leqslant \frac{C\|\omega\|_{X_\alpha}}{|x|^\alpha} \int_{|z|\leqslant 3|x|} \frac{1}{|z|}\,dz \\
&\leqslant \frac{C\|\omega\|_{X_\alpha}}{|x|^{\alpha-1}}.
\end{aligned}$$

It remains to estimate I_1. In order to deal with it, we need to use the symmetry properties of ω. Let us notice first that

$$|I_1| \leqslant C\|\omega\|_{X_\alpha} \int_{|y|\leqslant|x|/2} \frac{1}{|x-y||y|^\alpha}\,dy \leqslant \frac{C\|\omega\|_{X_\alpha}}{|x|} \int_{|y|\leqslant|x|/2} \frac{1}{|y|^\alpha}\,dy.$$

So in the case $\alpha<2$ we have the desired bound:

$$|I_1| \leqslant \frac{C\|\omega\|_{X_\alpha}}{|x|^{\alpha-1}}.$$

If $2\leqslant\alpha<3$, we use the symmetry properties of ω to deduce that $\int_B \omega(y)\,dy=0$ where B is any ball centered in 0. Therefore we have:

$$\begin{aligned}
I_1 &= \int_{|y|\leqslant|x|/2} \left(\frac{(x-y)^\perp}{2\pi|x-y|^2} - \frac{x^\perp}{2\pi|x|^2}\right)\omega(y)\,dy \\
&= \int_{|y|\leqslant|x|/2} \left(\frac{-y^\perp|x|^2+2(x\cdot y)x^\perp-|y|^2x^\perp}{2\pi|x-y|^2|x|^2}\right)\omega(y)\,dy.
\end{aligned}$$

So

$$|I_1| \leqslant C\|\omega\|_{X_\alpha} \int_{|y|\leqslant|x|/2} \left(\frac{|y|}{|x|^2}+\frac{|y|^2}{|x|^3}\right)\frac{1}{|y|^\alpha}\,dy \leqslant \frac{C\|\omega\|_{X_\alpha}}{|x|^{\alpha-1}}$$

where we used that, for $\alpha<3$, $1/|y|^{\alpha-1}$ is integrable near 0.

Next, if $3\leqslant\alpha<4$, we use the fact that, for $i\in\{1,2\}$, $\int_B y_i\omega(y)\,dy=0$ for any ball B with center 0. This is a consequence of the fact that ω is odd with respect to y_{3-i}. We infer that

$$I_1 = \int_{|y|\leqslant|x|/2} \left(\frac{-y^\perp|x|^2+2(x\cdot y)x^\perp-|y|^2x^\perp}{2\pi|x-y|^2|x|^2} - \frac{-y^\perp|x|^2+2(x\cdot y)x^\perp}{2\pi|x|^4}\right)\omega(y)\,dy \tag{3.1}$$
$$= \int_{|y|\leqslant|x|/2} \left(\frac{-2(x\cdot y)|x|^2y^\perp+|y|^2|x|^2y^\perp+4(x\cdot y)^2x^\perp-2|y|^2(x\cdot y)x^\perp-|y|^2|x|^2x^\perp}{2\pi|x-y|^2|x|^4}\right)\omega(y)\,dy$$

so

$$\begin{aligned}
|I_1| &\leqslant C\|\omega\|_{X_\alpha} \int_{|y|\leqslant|x|/2} \left(\frac{|y|^2}{|x|^3}+\frac{|y|^3}{|x|^4}\right)\frac{1}{|y|^\alpha}\,dy \\
&\leqslant \frac{C\|\omega\|_{X_\alpha}}{|x|^{\alpha-1}}.
\end{aligned}$$

Assume now that $4 \leqslant \alpha < 5$. We notice that all the moments of order 2 of ω are zero. Indeed, if $i \in \{1,2\}$, we see that $\int_B y_i^2 \omega(y)\,dy = 0$ because ω is odd with respect to y_i. And $\int_B y_1 y_2 \omega(y)\,dy = 0$ because ω changes sign when one exchanges y_1 and y_2. So we can remove from the integral I_1 all the terms which are polynomials of degree 2 in the y variable. More precisely, we notice that we wrote in (3.1) the term I_1 under the form

$$I_1 = \int_{|y|\leqslant|x|/2} \left(\frac{P_1(x,y) + Q_1(x,y)}{|x-y|^2|x|^4} \right) \omega(y)\,dy$$

where P_1 is a polynomial homogeneous of degree 3 in x and homogeneous of degree 2 in y while Q_1 is a polynomial homogeneous of degree 3 in y and homogeneous of degree 2 in x. Since all the moments of order 2 of ω vanish, we infer that $\int_B P_1(x,y)\omega(y)\,dy = 0$, so we have:

$$\begin{aligned}
I_1 &= \int_{|y|\leqslant|x|/2} \left(P_1(x,y) \left(\frac{1}{|x-y|^2|x|^4} - \frac{1}{|x|^6} \right) + \frac{Q_1(x,y)}{|x-y|^2|x|^4} \right) \omega(y)\,dy \\
&= \int_{|y|\leqslant|x|/2} \left(\frac{2(x\cdot y)P_1(x,y) + |x|^2 Q_1(x,y) - |y|^2 P_1(x,y)}{|x-y|^2|x|^6} \right) \omega(y)\,dy \\
&= \int_{|y|\leqslant|x|/2} \left(\frac{P_2(x,y) + Q_2(x,y)}{|x-y|^2|x|^6} \right) \omega(y)\,dy
\end{aligned}$$

where P_2 is homogeneous of degree 4 in x and homogeneous of degree 3 in y, and Q_2 is homogeneous of degree 3 in x and homogeneous of degree 4 in y. So we have

$$|I_1| \leqslant C\|\omega\|_{X_\alpha} \int_{|y|\leqslant|x|/2} \left(\frac{|y|^3}{|x|^4} + \frac{|y|^4}{|x|^5} \right) \frac{1}{|y|^\alpha}\,dy \leqslant \frac{C\|\omega\|_{X_\alpha}}{|x|^{\alpha-1}}$$

because, for $\alpha < 5$, $1/|y|^{\alpha-3}$ is integrable near 0. Then, to reach the optimal bound $\alpha < 6$, we need to iterate the same operation on the polynomials of order 3 in y, that is on P_2. We notice once again that the moments of order 3 of ω vanish. Indeed, we have $\int_B y_i^3 \omega(y)\,dy = 0$ because ω is odd with respect to y_{3-i} and $\int_B y_i^2 y_{3-i}\omega(y)\,dy = 0$ because ω is odd with respect to y_i. Hence

$$\begin{aligned}
I_1 &= \int_{|y|\leqslant|x|/2} \left(P_2(x,y) \left(\frac{1}{|x-y|^2|x|^6} - \frac{1}{|x|^8} \right) + \frac{Q_2(x,y)}{|x-y|^2|x|^6} \right) \omega(y)\,dy \\
&= \int_{|y|\leqslant|x|/2} \left(\frac{2(x\cdot y)P_2(x,y) + |x|^2 Q_2(x,y) - |y|^2 P_2(x,y)}{|x-y|^2|x|^8} \right) \omega(y)\,dy \\
&= \int_{|y|\leqslant|x|/2} \left(\frac{P_3(x,y) + Q_3(x,y)}{|x-y|^2|x|^8} \right) \omega(y)\,dy
\end{aligned}$$

where P_3 is homogeneous of degree 5 in x and homogeneous of degree 4 in y, and Q_3 is homogeneous of degree 4 in x and homogeneous of degree 5 in y. We deduce that

$$|I_1| \leqslant C\|\omega\|_{X_\alpha} \int_{|y|\leqslant|x|/2} \left(\frac{|y|^4}{|x|^5} + \frac{|y|^5}{|x|^6} \right) \frac{1}{|y|^\alpha}\,dy \leqslant \frac{C\|\omega\|_{X_\alpha}}{|x|^{\alpha-1}}$$

because, for $\alpha < 6$, $1/|y|^{\alpha-4}$ is integrable near 0. This concludes the proof of the lemma. □

Remark 3.2. This argument cannot be continued because, to do so, we would need that $\int_B P_3(x,y)\omega(y)\,dy = 0$. But in this polynomial, we have terms like $y_1^3 y_2$, and the integral $\int_B y_1^3 y_2 \omega(y)\,dy$ does not necessarily vanish. The mapping $\omega \mapsto U$ is not continuous from X_6 to Y_5.

We can write the equation (2.1) under the following form:

$$\Delta\omega = U\cdot\nabla\omega - \operatorname{curl} f = \operatorname{div}(U\omega + f^{\perp}) \equiv \operatorname{div} V$$

where $V = U\omega + f^{\perp}$. We know that the inverse of the laplacian is given by the convolution with $\frac{1}{2\pi}\ln|x|$, so we can can write the solution of this equation as

$$\omega = \Delta^{-1}\operatorname{div} V = \frac{1}{2\pi}\ln|x| * \operatorname{div} V = \sum_{i=1}^{2}\frac{x_i}{2\pi|x|^2} * V_i.$$

From the result in [20], we know that there exists a unique small solution $\omega\in X_2$. In order to find its asymptotic behaviour, we assume that V belongs to Z_β for β large enough, and we determine the asymptotic expansion of $\Delta^{-1}\operatorname{div} V = \frac{x}{2\pi|x|^2} * V$ in the following lemma.

Lemma 3.3. *Let $V\in Z_\beta$ with $\beta > 1$, $\beta\neq 5$. Then $\Delta^{-1}\operatorname{div} V \in X_{\min(\beta-1,4)}$ and the mapping $V\mapsto \Delta^{-1}\operatorname{div} V$ is continuous from Z_β to $X_{\min(\beta-1,4)}$. Moreover, if $5<\beta<7$ we have the asymptotic expansion*

$$\Delta^{-1}\operatorname{div} V(x_1,x_2) = m\frac{x_1x_2(x_1^2-x_2^2)}{|x|^8} + O\left(\frac{1}{|x|^{\beta-1}}\right) \tag{3.2}$$

when $|x|\to\infty$ and

$$m = \frac{4}{\pi}\int_{\mathbb{R}^2}(3y_1^2y_2 - y_2^3)V_1(y)\,dy.$$

Proof. Let us observe first that $\Delta^{-1}\operatorname{div}$ is a convolution operator with kernel bounded by $\frac{C}{|x|}$, so $\Delta^{-1}\operatorname{div} V$ is bounded by classical results. Next, since $Z_{\beta'}\subseteq Z_\beta$ for all $\beta'\geqslant\beta$ we can assume without loss of generality that $\beta<7$.

We make the decomposition

$$\Delta^{-1}\operatorname{div} V(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}\cdot V(y)\,dy = I_1+I_2+I_3$$

where

$$\begin{aligned}
I_1 &= \frac{1}{2\pi}\int_{|y|\leqslant|x|/2}\frac{x-y}{|x-y|^2}\cdot V(y)\,dy\\
I_2 &= \frac{1}{2\pi}\int_{|x|/2<|y|<2|x|}\frac{x-y}{|x-y|^2}\cdot V(y)\,dy\\
I_3 &= \frac{1}{2\pi}\int_{|y|\geqslant 2|x|}\frac{x-y}{|x-y|^2}\cdot V(y)\,dy.
\end{aligned}$$

As in the proof of Lemma 3.1, we first deal with I_2 and I_3:

$$\begin{aligned}
|I_3| &\leqslant C\|V\|_{Z_\beta}\int_{|y|\geqslant 2|x|}\frac{1}{|x-y||y|^\beta}\,dy\\
&\leqslant C\|V\|_{Z_\beta}\int_{|y|\geqslant 2|x|}\frac{1}{|y|^{\beta+1}}\,dy\\
&\leqslant \frac{C\|V\|_{X_\beta}}{|x|^{\beta-1}}
\end{aligned}$$

and

$$
\begin{aligned}
|I_2| &\leqslant C\|V\|_{Z_\beta} \int_{|x|/2<|y|<2|x|} \frac{1}{|x-y||y|^\beta}\, dy \\
&\leqslant \frac{C\|V\|_{Z_\beta}}{|x|^\beta} \int_{|z|\leqslant 3|x|} \frac{1}{|z|} dz \\
&\leqslant \frac{C\|V\|_{Z_\beta}}{|x|^{\beta-1}}.
\end{aligned}
$$

To bound the integral I_1, we need the cancellation properties of the moments of V. Let B be a ball with center 0 and $i \in \{1,2\}$. We will study the moments of V_i until order 4.

Moments of order 0. Since V_i is odd with respect to x_{3-i}, we have $\int_B V_i = 0$.

Moments of order 1. Since V_i is odd with respect to x_{3-i}, we have $\int_B x_1 V_1 = \int_B x_2 V_2 = 0$. Since $V_1(x_2, x_1) = -V_2(x_1, x_2)$, we also have $\int_B x_1 V_2 = -\int_B x_2 V_1$.

Moments of order 2. All these moments vanish. Indeed, since V_i is odd with respect to x_{3-i}, we have $\int_B x_1^2 V_i = \int_B x_2^2 V_i = 0$ and since V_i is even with respect to x_i, we know that $\int_B x_1 x_2 V_i = 0$.

Moments of order 3. Using the odd parity of V_i with respect to x_{3-i}, we get that $\int_B x_1^3 V_1 = \int_B x_1 x_2^2 V_1 = \int_B x_2^3 V_2 = \int_B x_1^2 x_2 V_2 = 0$. The fact that $V_1(x_2, x_1) = -V_2(x_1, x_2)$ implies that $\int_B x_1^3 V_2 = -\int_B x_2^3 V_1$ and $\int_B x_1 x_2^2 V_2 = -\int_B x_1^2 x_2 V_1$.

Moments of order 4. Since V_i is odd with respect to x_{3-i}, we have $\int_B x_1^4 V_i = \int_B x_2^4 V_i = \int_B x_1^2 x_2^2 V_i = 0$. Since V_i is even with respect to x_i, we get $\int_B x_1^3 x_2 V_i = \int_B x_1 x_2^3 V_i = 0$. Finally, all the moments of order 4 of V vanish.

To find the asymptotic expansion of I_1, we apply the Taylor formula to the function $H(z) = \frac{z}{|z|^2}$. For $x \neq 0$ and $|y| \leqslant |x|/2$, we have:

$$
\begin{aligned}
H(x-y) = H(x) &- \nabla H(x)\cdot y + \frac{1}{2}\nabla^2 H(x)(y,y) - \frac{1}{6}\nabla^3 H(x)(y,y,y) + \\
&\frac{1}{24}\nabla^4 H(x)(y,y,y,y) - \int_0^1 \frac{(1-t)^4}{24}\nabla^5 H(x-ty)(y,y,y,y,y)dt.
\end{aligned}
\tag{3.3}
$$

Since $\nabla^5 H$ is homogeneous of degree -6 and $|y| \leqslant |x|/2$, we know that $|\nabla^5 H(x-ty)| \leqslant \frac{C}{|x|^6}$ for all $t \in [0;1]$ and we can conclude that the integral term above is bounded by $\frac{C|y|^5}{|x|^6}$.

Now let us recall that

$$
I_1 = \frac{1}{2\pi} \int_{|y|\leqslant |x|/2} H(x-y)\cdot V(y)\, dy.
$$

We replace $H(x-y)$ in the expression above by the formula (3.3). Since all the moments of order 0, 2 or 4 of V vanish, we infer that

$$
\begin{aligned}
I_1 &= -\frac{1}{2\pi} \int_{|y|\leqslant |x|/2} \left(\nabla H(x)\cdot y + \frac{1}{6}\nabla^3 H(x)(y,y,y) + O\left(\frac{|y|^5}{|x|^6}\right)\right)\cdot V(y)\, dy \\
&\equiv I_4 + I_5 + I_6.
\end{aligned}
$$

We compute now successively the derivatives of H. For $x \neq 0$ and $i,j,k,l \in \{1,2\}$ we have

$$\begin{aligned}
(\nabla H(x))_{i,j} &= \partial_j H_i(x) = \frac{\delta_{i,j}}{|x|^2} - 2\frac{x_i x_j}{|x|^4} \\
(\nabla^2 H(x))_{i,j,k} &= \partial_k \partial_j H_i(x) = -2\frac{\delta_{i,j}x_k + \delta_{i,k}x_j + \delta_{j,k}x_i}{|x|^4} + 8\frac{x_i x_j x_k}{|x|^6} \\
(\nabla^3 H(x))_{i,j,k,l} &= \partial_l \partial_k \partial_j H_i(x) = -2\frac{\delta_{i,j}\delta_{k,l} + \delta_{i,k}\delta_{j,l} + \delta_{j,k}\delta_{i,l}}{|x|^4} \\
&\qquad + 8\frac{\delta_{i,j}x_k x_l + \delta_{i,k}x_j x_l + \delta_{j,k}x_i x_l + \delta_{i,l}x_j x_k + \delta_{j,l}x_i x_k + \delta_{k,l}x_i x_j}{|x|^6} - 48\frac{x_i x_j x_k x_l}{|x|^8}.
\end{aligned}$$

Hence

$$\nabla H(x) \cdot y = \frac{y}{|x|^2} - 2\frac{(x\cdot y)x}{|x|^4}$$

and

$$\nabla^3 H(x)(y,y,y) = -6\frac{|y|^2 y}{|x|^4} + 24\frac{(x\cdot y)^2 y + |y|^2 (x\cdot y)x}{|x|^6} - 48\frac{(x\cdot y)^3 x}{|x|^8}.$$

We can prove now that $I_4 = 0$:

$$\begin{aligned}
I_4 &= -\frac{1}{2\pi}\int_{|y|\leqslant |x|/2} (\nabla H(x)\cdot y)\cdot V(y)\,dy \\
&= -\frac{1}{2\pi}\int_{|y|\leqslant |x|/2} \left(\frac{y\cdot V(y)}{|x|^2} - 2\frac{(x\cdot y)(x\cdot V(y))}{|x|^4}\right) dy \\
&= -\frac{1}{2\pi|x|^2}\int_{|y|\leqslant |x|/2} (y_1V_1 + y_2V_2)\,dy + \frac{1}{\pi|x|^4}\int_{|y|\leqslant |x|/2} (x_1y_1 + x_2y_2)(x_1V_1 + x_2V_2)\,dy \\
&= \frac{1}{\pi|x|^4}\int_{|y|\leqslant |x|/2} (x_1x_2y_1V_2 + x_1x_2y_2V_1)\,dy \\
&= \frac{x_1x_2}{\pi|x|^4}\int_{|y|\leqslant |x|/2} (y_1V_2 + y_2V_1)\,dy \\
&= 0
\end{aligned}$$

where we used the cancellation properties of the moments of order 1 of V.

Next we estimate I_6:

$$|I_6| = \left|-\frac{1}{2\pi}\int_{|y|\leqslant |x|/2} O\left(\frac{|y|^5}{|x|^6}\right)\cdot V(y)\,dy\right| \leqslant \frac{C\|V\|_{Z_\beta}}{|x|^6}\int_{|y|\leqslant |x|/2}\frac{1}{|y|^{\beta-5}}\,dy \leqslant \frac{C\|V\|_{Z_\beta}}{|x|^{\beta-1}}$$

where we used that $\beta < 7$. Finally, we compute I_5:

$$\begin{aligned}
I_5 &= -\frac{1}{12\pi}\int_{|y|\leqslant |x|/2} \nabla^3 H(x)(y,y,y)\cdot V(y)\,dy \\
&= -\frac{1}{12\pi}\int_{|y|\leqslant |x|/2}\Big(-6\frac{|y|^2(y\cdot V)}{|x|^4} + 24\frac{(x\cdot y)^2(y\cdot V) + |y|^2(x\cdot y)(x\cdot V)}{|x|^6} \\
&\qquad - 48\frac{(x\cdot y)^3(x\cdot V)}{|x|^8}\Big)\,dy
\end{aligned}$$

Like in the computation of I_4 and thanks to the cancellation properties of the moments of order 3 of V, we see that the term containing $|y|^2(y\cdot V)$ vanishes. We also have that

$$\begin{aligned}\int_B |y|^2(x\cdot y)(x\cdot V) = x_1^2\int_B (y_1^3+y_1y_2^2)V_1(y)\,dy + x_2^2\int_B (y_1^2y_2+y_2^3)V_2(y)\,dy \\ + x_1x_2\int_B [y_1^3V_2(y)+y_2^3V_1(y)+y_1y_2^2V_2(y)+y_1^2y_2V_1(y)]\,dy \\ = 0.\end{aligned}$$

We deduce that

$$I_5 = \frac{2}{\pi}\int_{|y|\leqslant|x|/2} \frac{2(x\cdot y)^3(x\cdot V)-|x|^2(x\cdot y)^2(y\cdot V)}{|x|^8}\,dy.$$

Using again the cancellation properties of the moments of order 3 of V, we get that

$$\begin{aligned}\int_B (x\cdot y)^3(x\cdot V)\,dy &= \int_B (x_1^3y_1^3+3x_1x_2^2y_1y_2^2+3x_1^2x_2y_1^2y_2+x_2^3y_2^3)(x_1V_1+x_2V_2)\,dy\\ &= \int_B (3x_1^3x_2y_1^2y_2V_1+x_1x_2^3y_2^3V_1+x_1^3x_2y_1^3V_2+3x_1x_2^3y_1y_2^2V_2)\,dy\\ &= \int_B x_1x_2(3(x_1^2-x_2^2)y_1^2y_2V_1+(x_2^2-x_1^2)y_2^3V_1)\,dy\\ &= \int_B x_1x_2(x_1^2-x_2^2)(3y_1^2y_2V_1-y_2^3V_1)\,dy\end{aligned}$$

and

$$\begin{aligned}\int_B (x\cdot y)^2(y\cdot V)\,dy &= \int_B (x_1^2y_1^2+2x_1x_2y_1y_2+x_2^2y_2^2)(y_1V_1+y_2V_2)\,dy\\ &= \int_B 2x_1x_2(y_1^2y_2V_1+y_1y_2^2V_2)\,dy\\ &= 0.\end{aligned}$$

We conclude that

$$I_5 = \frac{4}{\pi|x|^8}\int_{|y|\leqslant|x|/2} x_1x_2(x_1^2-x_2^2)(3y_1^2y_2-y_2^3)V_1(y)\,dy.$$

We proved that

$$\Delta^{-1}\operatorname{div} V(x) = \frac{4x_1x_2(x_1^2-x_2^2)}{\pi|x|^8}\int_{|y|\leqslant|x|/2}(3y_1^2y_2-y_2^3)V_1(y)\,dy + O\left(\frac{\|V\|_{Z_\beta}}{|x|^{\beta-1}}\right) \tag{3.4}$$

when $|x|\to\infty$.

Assume now that $1<\beta<5$. Then

$$\left|\int_{|y|\leqslant|x|/2}(3y_1^2y_2-y_2^3)V_1(y)\,dy\right| \leqslant C\|V\|_{Z_\beta}\int_{|y|\leqslant|x|/2}|y|^{3-\beta}\,dy \leqslant C\|V\|_{Z_\beta}|x|^{5-\beta}$$

so

$$|\Delta^{-1}\operatorname{div} V(x)| \leqslant C\frac{\|V\|_{Z_\beta}}{|x|^{\beta-1}}$$

which completes the proof in the case $1<\beta<5$.

Assume next that $5 < \beta < 7$. Then

$$\Big|\int_{|y|\leqslant|x|/2}(3y_1^2y_2 - y_2^3)V_1(y)\,dy\Big| \leqslant C\|V\|_{Z_\beta}\int_{|y|\leqslant|x|/2}\frac{|y|^3}{(1+|y|)^\beta}\,dy \leqslant C\|V\|_{Z_\beta}$$

which in view of (3.4) implies that the operator $V \mapsto \Delta^{-1}\operatorname{div} V$ is continuous from Z_β to X_4.

To show the asymptotic behaviour stated in (3.2) it remains to prove that we can replace by $\mathbb{R}^2$ the domain of integration in the integral from (3.4). We can estimate

$$\Big|\frac{4x_1x_2(x_1^2-x_2^2)}{\pi|x|^8}\int_{|y|>|x|/2}(3y_1^2y_2V_1 - y_2^3V_1)(y)\,dy\Big| \leqslant \frac{C\|V\|_{Z_\beta}}{|x|^4}\int_{|y|>|x|/2}\frac{dy}{|y|^{\beta-3}} \leqslant \frac{C\|V\|_{Z_\beta}}{|x|^{\beta-1}}.$$

We deduce from (3.4) that

$$\Delta^{-1}\operatorname{div} V(x) = \frac{mx_1x_2(x_1^2-x_2^2)}{|x|^8} + O\left(\frac{\|V\|_{Z_\beta}}{|x|^{\beta-1}}\right) \qquad \text{as } |x|\to\infty$$

which concludes the proof of the lemma. □

We go back to the proof of Theorem 2.5. We write the equation (2.1) under the following form:

$$\omega = \Delta^{-1}\operatorname{div}(U\omega + f^\perp) = B(\omega,\omega) + F \tag{3.5}$$

where

$$B(\omega,\omega) = \Delta^{-1}\operatorname{div}(U\omega) \quad \text{and} \quad F = \Delta^{-1}\operatorname{div} f^\perp.$$

Since $f \in Y_{5+\delta}$, we know that $f^\perp$ is in $Z_{5+\delta}$ so according to Lemma 3.3 we have that $F \in X_4$ and

$$\|F\|_{X_4} \leqslant C_1\|f\|_{Y_{5+\delta}}$$

for some constant C_1.

We apply Lemma 2.4 to the space X_4 and the bilinear map $B(\omega_1,\omega_2) = \Delta^{-1}\operatorname{div}(U^1\omega_2)$ where U^1 denotes the velocity field associated to the vorticity ω_1. We notice that B is continuous on X_4. Indeed, if $\omega_1,\omega_2 \in X_4$ then according to Lemma 3.1 we know that $U^1 \in Y_3$ so $U^1\omega_2 \in Z_7$. According to Lemma 3.3, it follows that $B(\omega_1,\omega_2) \in X_4$. Lemma 2.4 implies that there exists ε_0 such that, if $\|F\|_{X_4} \leqslant C_1\|f\|_{Y_{5+\delta}} \leqslant \varepsilon_0$ then there exists a unique ω in the ball $B(0, 2\varepsilon_0)$ of X_4 that solves (3.5). Moreover,

$$\|\omega\|_{X_4} \leqslant 2\|F\|_{X_4} \leqslant 2C_1\|f\|_{Y_{5+\delta}}.$$

To get the desired asymptotic expansion for ω, we notice that $U\omega$ belongs to Z_7. Applying Lemma 3.3 to $V = U\omega + f^\perp \in Z_{5+\delta}$ we deduce that

$$\omega(x) = m\frac{x_1x_2(x_1^2-x_2^2)}{|x|^8} + O\left(\frac{1}{|x|^{4+\delta}}\right)$$

where

$$m = \frac{4}{\pi}\int_{\mathbb{R}^2}(3y_1^2y_2 - y_2^3)V_1(y)\,dy = \frac{4}{\pi}\int_{\mathbb{R}^2}(3y_1^2y_2 - y_2^3)(U_1\omega - f_2)(y)\,dy.$$

Now we construct an example of forcing term f as small as we want such that $m \neq 0$. Let us introduce the function

$$\Psi(V) = \frac{4}{\pi} \int_{\mathbb{R}^2} (3y_1^2 y_2 - y_2^3) V_1(y)\, dy.$$

We want to find an f such that $m = \Psi(U\omega + f^\perp) \neq 0$. Let $f \in Y_{5+\delta}$ and $\varepsilon > 0$ be small enough such that there exists a solution ω_ε associated to the forcing term εf. We have that

$$\|\omega_\varepsilon\|_{X_4} \leqslant 2C_1 \|\varepsilon f\|_{Y_{5+\delta}} \leqslant C\varepsilon$$

so

$$|\Psi(U_\varepsilon \omega_\varepsilon)| \leqslant C\|U_\varepsilon \omega_\varepsilon\|_{Z_7} \leqslant C\|U_\varepsilon\|_{Y_3} \|\omega_\varepsilon\|_{X_4} \leqslant C\varepsilon^2.$$

Since Ψ is linear, we infer that

$$|\Psi(\varepsilon f^\perp + U_\varepsilon \omega_\varepsilon)| = |\varepsilon \Psi(f^\perp) + O(\varepsilon^2)| \geqslant \frac{\varepsilon}{2} |\Psi(f^\perp)|$$

if ε is small enough and if $\Psi(f^\perp) \neq 0$. So we just need to find a function $f \in Y_{5+\delta}$ such that $\Psi(f^\perp) \neq 0$. This is quite easy to obtain. Take a function $h \in C_0^\infty(\mathbb{R}^+, [0,1])$ such that $\int_{\mathbb{R}^+} r^7 h(r)\, dr = 1$ and $h = 1$ near 0. Let $f(x_1, x_2) = (x_1^3, x_2^3) h(|x|)$. Then $f \in C_0^\infty(\mathbb{R}^2)$ and it satisfies the required symmetry conditions, so $f \in Y_{5+\delta}$. We have that

$$\begin{aligned}
\Psi(f^\perp) &= -\frac{4}{\pi} \int_{\mathbb{R}^2} (3x_1^2 x_2 - x_2^3) f_2(x)\, dx \\
&= -\frac{4}{\pi} \int_{\mathbb{R}^2} (3x_1^2 x_2^4 - x_2^6) h(|x|)\, dx \\
&= -\frac{4}{\pi} \int_0^{+\infty} r^7 h(r)\, dr \int_0^{2\pi} (3\cos^2\theta \sin^4\theta - \sin^6\theta)\, d\theta \\
&= -\frac{4}{\pi} \int_0^{2\pi} \frac{1}{8} (-1 + 3\cos(2\theta) - 3\cos(4\theta) + \cos(6\theta))\, d\theta \\
&= 1.
\end{aligned}$$

This concludes the proof of Theorem 2.5.

Remark 3.4. Once we know the asymptotic behaviour of the vorticity, it is easy to determine the asymptotic behaviour of the velocity. If ω satisfies the symmetry conditions (2.3) and

$$\omega(x) = m \frac{x_1 x_2 (x_1^2 - x_2^2)}{|x|^8} + O\left(\frac{1}{|x|^{4+\delta}}\right)$$

when $|x| \to \infty$, then the velocity U has the following asymptotic behaviour:

$$U(x) = \frac{m}{12|x|^8} \left(x_1(x_1^4 + 3x_2^4 - 8x_1^2 x_2^2), x_2(3x_1^4 + x_2^4 - 8x_1^2 x_2^2) \right) + O\left(\frac{1}{|x|^{3+\delta}}\right). \tag{3.6}$$

Indeed, one can check that

$$\operatorname{curl}\left[\frac{1}{12|x|^8} \left(x_1(x_1^4 + 3x_2^4 - 8x_1^2 x_2^2), x_2(3x_1^4 + x_2^4 - 8x_1^2 x_2^2) \right)\right] = \frac{x_1 x_2 (x_1^2 - x_2^2)}{|x|^8}.$$

In addition, we observe that

$$\Delta\left(\frac{x_1x_2(x_1^2-x_2^2)}{|x|^6}\right)=-12\frac{x_1x_2(x_1^2-x_2^2)}{|x|^8}$$

and

$$\frac{1}{12|x|^8}\left(x_1(x_1^4+3x_2^4-8x_1^2x_2^2),x_2(3x_1^4+x_2^4-8x_1^2x_2^2)\right)=-\nabla^\perp\left(\frac{x_1x_2(x_1^2-x_2^2)}{12|x|^6}\right).$$

Let us introduce a function $\varphi\in C^\infty(\mathbb{R}^2;[0,1])$ such that $0\leqslant\varphi(x)\leqslant 1$, $\varphi(x)=0$ for $|x|\leqslant 1/2$ and $\varphi(x)=1$ for $|x|\geqslant 1$. If we define

$$W=-m\nabla^\perp\left(\varphi(x)\frac{x_1x_2(x_1^2-x_2^2)}{12|x|^6}\right)$$

then W is divergence free and

$$\begin{aligned}\operatorname{curl}W&=-m\Delta\left(\varphi(x)\frac{x_1x_2(x_1^2-x_2^2)}{12|x|^6}\right)\\&=m\varphi\frac{x_1x_2(x_1^2-x_2^2)}{|x|^8}-m\Delta\varphi(x)\frac{x_1x_2(x_1^2-x_2^2)}{12|x|^6}-2m\nabla\varphi\cdot\nabla\left(\frac{x_1x_2(x_1^2-x_2^2)}{12|x|^6}\right).\end{aligned}$$

We infer that $\operatorname{curl}(W-U)=O(1/|x|^{4+\delta})$ so according to Lemma 3.1 we get that $W-U=O(1/|x|^{3+\delta})$. But we also have

$$\begin{aligned}W(x)&=-m\varphi(x)\nabla^\perp\left(\frac{x_1x_2(x_1^2-x_2^2)}{12|x|^6}\right)-m\nabla^\perp\varphi(x)\frac{x_1x_2(x_1^2-x_2^2)}{12|x|^6}\\&=\frac{m}{12|x|^8}\left(x_1(x_1^4+3x_2^4-8x_1^2x_2^2),x_2(3x_1^4+x_2^4-8x_1^2x_2^2)\right)+O\left(\frac{1}{|x|^{3+\delta}}\right)\end{aligned}$$

and relation (3.6) follows.

4 The exterior domain case

In this section, we prove Theorem 2.6. We proceed as in [17]. We extend U to the whole plane, we study the additional forcing term that appears and we apply Lemma 3.3. We will need to check that the extended solution we obtain still verifies the symmetry conditions.

Let $\Omega=\{|x|>R\}$. First, let us observe that U and p are more regular than stated:

Lemma 4.1. *We have that $(U,p)\in W^{2,q}_{loc}(\Omega)\times W^{1,q}_{loc}(\Omega)$ for any $q>1$.*

This lemma was already proved in the 3-dimensional case in [6]. Its proof goes through to the bidimensional case without difficulty.

Let $R<R_0<R_1$ and consider a radial cut-off function $\eta\in C^\infty(\mathbb{R}^2,[0,1])$ such that $\eta=0$ on $B(0,R_0)$ and $\eta=1$ on $B(0,R_1)^c$. We define the following extension of the solution (U,p):

$$\begin{aligned}&\widetilde{U}=U, &&\widetilde{p}=p &&\text{on } B(0,R_1)^c\\&\widetilde{U}=\eta U+v, &&\widetilde{p}=\eta p &&\text{on } B(0,R_1)\end{aligned}$$

where we extended ηU and ηp with zero values for $|x| \leqslant R$. The vector field v is constructed in such a manner as to ensure that $\widetilde{U} \in L^\infty$, $\operatorname{div} \widetilde{U} = 0$ everywhere and that (2.4) hold for $\widetilde{U}$. We assume that v is a solution of the following problem

$$\begin{aligned} \operatorname{div} v &= -U \cdot \nabla \eta \text{ in } B(0, R_1) \\ v &= 0 \text{ on } \partial B(0, R_1). \end{aligned} \tag{4.1}$$

This problem has many solutions, and a way to find one with good estimates is given by the Bogovskii operators, see [4]. In particular, we have the following result:

Theorem 4.2 ([4])**.** *Let $g \in W_0^{k,q}(B)$ where B is a ball and $k \in \mathbb{N}$, $1 < q < \infty$. Assume that $\int_B g = 0$. Then there exists a solution $V \in W_0^{k+1,q}(B)$ of the equation $\operatorname{div} V = g$, with the following estimate:*

$$\|V\|_{W^{k+1,q}(B)} \leqslant C(q, k, B)\|g\|_{W^{k,q}(B)}.$$

We have that

$$\int_{B(0,R_1)} U \cdot \nabla \eta = \int_{C(0,R_1)} U \cdot \nu\, \eta - \int_{B(0,R_1)} \eta \operatorname{div} U = \int_{C(0,R_1)} U \cdot \nu = 0$$

where we used (2.6). Moreover, because $U \in W^{2,q}_{loc}(\Omega)$ for any $1 < q < \infty$ we have that $U \cdot \nabla \eta \in W_0^{2,q}(B(0, R_1))$ for any $1 < q < \infty$. Using Theorem 4.2 we infer that there exists some $v \in W_0^{3,q}(B(0, R_1))$ for all $1 < q < \infty$ which solves (4.1). We extend v to the whole space $\mathbb{R}^3$ by setting $v = 0$ for $|x| > R_1$ so that $v \in W^{3,q}(\mathbb{R}^3)$. Then we have to check that v satisfies (2.4) in order to deduce that $\widetilde{U}$ satisfies them too. We recall the formula for v introduced in Bogovskii's paper:

$$v(x) = -\int_{B(0,R_1)} \int_0^1 \frac{(x-y)}{t} \chi\left(y + \frac{x-y}{t}\right) \frac{dt}{t^n} U(y) \cdot \nabla \eta(y)\, dy$$

where χ is any function in $C_0^\infty(B(0, R_0))$ such that $\int \chi = 1$. Here we make the additional assumption that χ is radial. With this assumption, we can prove that v satisfies the conditions (2.4). Indeed, we can reformulate these conditions as:

$$U(x^\perp) = (U(x))^\perp \qquad \text{and} \qquad U(\overline{x}) = \overline{U(x)}$$

where $\overline{z}$ denotes the complex conjugate of z (here we identify $\mathbb{R}^2$ and $\mathbb{C}$). We have

$$\begin{aligned} v(x^\perp) &= -\int_{B(0,R_1)} \int_0^1 \frac{(x^\perp - y)}{t} \chi\left(y + \frac{x^\perp - y}{t}\right) \frac{dt}{t^n} U(y) \cdot \nabla \eta(y)\, dy \\ &= -\int_{B(0,R_1)} \int_0^1 \frac{(x-y)^\perp}{t} \chi\left(\left(y + \frac{x-y}{t}\right)^\perp\right) \frac{dt}{t^n} U(y^\perp) \cdot \nabla \eta(y^\perp)\, dy. \end{aligned}$$

The fact that η is radial implies $\nabla \eta(y^\perp) = (\nabla \eta(y))^\perp$. Then, using the fact that χ is radial and the symmetry properties of U we infer that

$$\begin{aligned} v(x^\perp) &= -\int_{B(0,R_1)} \int_0^1 \frac{(x-y)^\perp}{t} \chi\left(y + \frac{x-y}{t}\right) \frac{dt}{t^n} (U(y))^\perp \cdot (\nabla \eta(y))^\perp\, dy \\ &= -\int_{B(0,R_1)} \int_0^1 \frac{(x-y)^\perp}{t} \chi\left(y + \frac{x-y}{t}\right) \frac{dt}{t^n} U(y) \cdot \nabla \eta(y)\, dy \\ &= (v(x))^\perp. \end{aligned}$$

Next, one can check that $\nabla\eta(\overline{y}) = \overline{\nabla\eta(y)}$ so

$$\begin{aligned}
v(\overline{x}) &= -\int_{B(0,R_1)}\int_0^1 \frac{(\overline{x}-y)}{t}\chi\left(y+\frac{\overline{x}-y}{t}\right)\frac{dt}{t^n}U(y)\cdot\nabla\eta(y)\,dy \\
&= -\int_{B(0,R_1)}\int_0^1 \frac{\overline{(x-y)}}{t}\chi\left(\overline{y+\frac{x-y}{t}}\right)\frac{dt}{t^n}U(\overline{y})\cdot\nabla\eta(\overline{y})\,dy \\
&= \int_{B(0,R_1)}\int_0^1 \frac{\overline{(x-y)}}{t}\chi\left(y+\frac{x-y}{t}\right)\frac{dt}{t^n}\overline{U(y)}\cdot\overline{\nabla\eta(y)}\,dy \\
&= \int_{B(0,R_1)}\int_0^1 \frac{\overline{(x-y)}}{t}\chi\left(y+\frac{x-y}{t}\right)\frac{dt}{t^n}U(y)\cdot\nabla\eta(y)\,dy \\
&= \overline{v(x)}.
\end{aligned}$$

We deduce that v satisfies the conditions (2.4).

We observe now that the extension $(\widetilde{U},\widetilde{p})$ verifies the following stationary Navier-Stokes equation in the whole plane:

$$-\Delta\widetilde{U} + (\widetilde{U}\cdot\nabla)\widetilde{U} + \nabla\widetilde{p} = \eta f + F \equiv \widetilde{f}, \qquad \operatorname{div}\widetilde{U} = 0 \tag{4.2}$$

where

$$F = -\Delta v - \Delta\eta U - 2\nabla U\cdot\nabla\eta + \operatorname{div}(\eta U\otimes v + \eta v\otimes U + v\otimes v + \eta^2 U\otimes U) - \eta\operatorname{div}(U\otimes U) + p\nabla\eta.$$

We observe that F is compactly supported in $B(0,R_1)$. Moreover, since $U \in W^{2,q}_{loc}(\Omega)$ and $v \in W^{3,q}$ for any $1 < q < \infty$ we deduce that $F \in W^{1,q}$ for any $1 < q < \infty$. By Sobolev embeddings we infer that F is bounded so $\widetilde{f} \in Y_{5+\delta}$.

Taking the curl of (4.2) yields the following PDE for $\widetilde{\omega} = \operatorname{curl}\widetilde{U}$:

$$\Delta\widetilde{\omega} = \operatorname{div}(\widetilde{U}\widetilde{\omega} + \widetilde{f}^\perp).$$

Since $\widetilde{\omega} = \omega$ for $|x| > R_1$ we have that $\widetilde{\omega}$ vanishes at infinity. So one can invert the laplacian in the relation above to obtain

$$\widetilde{\omega} = \Delta^{-1}\operatorname{div}(\widetilde{U}\widetilde{\omega} + \widetilde{f}^\perp).$$

We have that $\widetilde{U} \in W^{2,q}_{loc}$ for any $1 < q < \infty$ so $\widetilde{U}\widetilde{\omega}$ is locally bounded. Since $\widetilde{U}\widetilde{\omega} = U\omega$ for $|x| > R_1$ and by hypothesis $|U\omega| \leqslant C/|x|^{5+\delta}$ we infer that $\widetilde{U}\widetilde{\omega} \in Z_{5+\delta}$. We conclude that $\widetilde{U}\widetilde{\omega} + \widetilde{f}^\perp \in Z_{5+\delta}$. The asymptotic behaviour for the vorticity ω stated in relation (2.7) now follows from Lemma 3.3 applied to $V = \widetilde{U}\widetilde{\omega} + \widetilde{f}^\perp$ (recall that $\widetilde{\omega} = \omega$ for $|x| > R_1$). Once (2.7) is proved, the asymptotic behaviour for the velocity follows from the argument given in Remark 3.4. This completes the proof of Theorem 2.6.

Remark 4.3. Let us remark that in Theorem 2.6 there is no smallness hypothesis, in contrast to similar results in dimension three, see [17] and [6]. We assume however faster decay at infinity. It is possible to replace the hypothesis that $U\omega = O(1/|x|^{5+\delta})$ by less decay plus smallness. More precisely, if instead of assuming $U\omega = O(1/|x|^{5+\delta})$ we assume that $|x|^{5+\delta}f$, $|x|U$ and $|x|^2\omega$ are bounded by a small constant then the conclusion of Theorem 2.6 still holds true. Indeed, the same proof shows that $\widetilde{U}$ is small in Y_1, $\widetilde{\omega}$ is small in X^2 and $\widetilde{f}$ is small in $Y_{5+\delta}$. By Theorem 2.5 there exists a small solution W associated to the forcing term $\widetilde{f}$. From the uniqueness part of the result of Yamazaki [20] we deduce that $\widetilde{U} = W$, so the desired asymptotic behaviour follows.

5 Asymptotic behaviour like $1/|x|$

The aim of this last section is to construct solutions that decay like $1/|x|$ at infinity. Under our symmetry conditions and assuming that the forcing decays like $1/|x|^3$ at infinity we seek to prove that a unique solution exists and decays like $1/|x|$ at infinity. In [6] we considered exactly the same problem in dimension three. For many reasons, that result cannot be adapted to the dimension two in a straightforward manner. Let us mention just one of them. In dimension three, if U is homogeneous of degree -1, the product $U \otimes U$ is locally integrable. So the term $U \cdot \nabla U = \operatorname{div}(U \otimes U)$ is well defined in the sense of the distributions. In dimension two, this is no longer the case. It is not at all obvious what means a solution of the Navier-Stokes equations on $\mathbb{R}^2$ homogeneous of degree -1 (solution up to 0 and not only on $\mathbb{R}^2 \setminus \{0\}$). The product $U \otimes U$ is not even well-defined in the principal value sense. Indeed, one of the components of $U \otimes U$ is U_1^2 and U_1^2 cannot be defined in the principal value sense since it is homogeneous and non-negative so its integral on the unit circle does not vanish. There are other ways to extend an homogeneous function to a distribution in $\mathbb{R}^2$ but the result will not be an homogeneous distribution (fact which is crucial in the proof). The approach we take in dimension two is the following. We know since the work of Delort [7] on vortex sheets that to give a sense to the term $U \cdot \nabla U$ up to a gradient, it suffices to define $U_1^2 - U_2^2$ and $U_1 U_2$ in the sense of distributions. And it happens that the symmetry conditions of Yamazaki imply that these two expressions have vanishing mean on the unit circle, so their principal value is well-defined.

We will now rewrite equation (1.1) under the form of an integral equation for the velocity. To do that, let us recall that the kernel of the Green function of the operator $\mathbb{P}\Delta^{-1}$ in $\mathbb{R}^2$ is given by the following formula (see [9, Section IV.2]):

$$G_2(x) = -\frac{1}{4\pi}\Big(-\log|x| I_2 + \frac{x \otimes x}{|x|^2}\Big).$$

Applying the Leray projector $\mathbb{P}$ to (1.1) and inverting the laplacian, we find the following equivalent relation:

$$U = \mathbb{P}\Delta^{-1}(U \cdot \nabla U) - \mathbb{P}\Delta^{-1} f = \mathbb{P}\Delta^{-1} \operatorname{div}(U \otimes U) - \mathbb{P}\Delta^{-1} f.$$

Since the operator $\mathbb{P}\Delta^{-1}$ is a convolution operator with kernel G_2, one can use the explicit formula for G_2 to obtain the following equivalent formulation for (1.1):

$$\begin{aligned} U(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^2}\Big(\frac{x-y}{|x-y|^2}|U(y)|^2 - 2\frac{x-y}{|x-y|^4}\big[(x-y)\cdot U(y)\big]^2\Big)\,dy \\ - \frac{1}{4\pi}\int_{\mathbb{R}^2}\Big(\log|x-y| f(y) - \frac{x-y}{|x-y|^2}(x-y)\cdot f(y)\Big)\,dy. \end{aligned} \tag{5.1}$$

Let us first observe that if U is homogeneous of degree -1 and verifies the symmetry conditions (2.4), then the first integral on the right-hand side is well-defined in the sense of the principal value. Indeed, when $y \to 0$ the integrand is equivalent to

$$\frac{x}{|x|^4}\big[(x_2^2 - x_1^2)(U_1^2(y) - U_2^2(y)) - 4x_1x_2U_1(y)U_2(y)\big].$$

We immediately see from (2.4) that $U_1^2 - U_2^2$ and U_1U_2 have vanishing mean on the unit circle, so the integral is well-defined in the principal value sense. One can check in a similar way that if f is homogeneous of degree -3 and verifies the symmetry relations (2.4), then the last integral in (5.1) is also well-defined in the principal value sense.

Before going further, let us introduce some notation. If g is a function integrable at infinity but with a singularity in the origin, we define the principal value of the integral of g by

$$\mathrm{pv} \int_{\mathbb{R}^2} g(x)\, dx = \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} g(x)\, dx$$

provided that the limit exists.

We introduce the weighted homogeneous spaces $\dot{Y}_\beta$ like in Definition 2.2 by replacing $1+|x|$ by $|x|$:

$$\dot{Y}_\beta = \{U : \mathbb{R}^2 \to \mathbb{R}^2 \text{ verifying (2.4) and such that } \|U\|_{\dot{Y}_\beta} = \sup_{\mathbb{R}^2} |x|^\beta |U(x)| < \infty\}.$$

We define the bilinear form

(5.2)

$$\widetilde{B}(U,V)(x) = -\frac{1}{4\pi}\,\mathrm{pv} \int_{\mathbb{R}^2} \Big(\frac{x-y}{|x-y|^2} U(y)\cdot V(y) - 2\frac{x-y}{|x-y|^4} \big[(x-y)\cdot U(y)\big]\big[(x-y)\cdot V(y)\big] \Big)\, dy$$

and the vector field

$$\widetilde{F}(x) = -\frac{1}{4\pi}\,\mathrm{pv} \int_{\mathbb{R}^2} \Big(\log|x-y| f(y) - \frac{x-y}{|x-y|^2}(x-y)\cdot f(y) \Big)\, dy. \tag{5.3}$$

We defined these integrals in the principal value sense because we will use these quantities for U, V and f with singularities. Obviously, if U, V and f have no singularities then the integrals converge in the usual sense.

According to relation (5.1), the stationary Navier-Stokes equations can be written under the following equivalent form

$$U = \widetilde{B}(U,U) + \widetilde{F}. \tag{5.4}$$

Let us show an estimate on $\widetilde{F}$.

Lemma 5.1. *If $f \in \dot{Y}_\alpha$ where $2 < \alpha < 4$ then $\widetilde{F}$ is well-defined and the mapping $f \mapsto \widetilde{F}$ is continuous from $\dot{Y}_\alpha$ into $\dot{Y}_{\alpha-2}$. Moreover, if f is homogeneous of degree $-\alpha$ then $\widetilde{F}$ is homogeneous of degree $2-\alpha$.*

Proof. Let $f \in \dot{Y}_\alpha$. Let us show first that $\widetilde{F}$ is well-defined, that is to say the integral defining $\widetilde{F}$ in (5.3) converges in the principal value sense. To do that, we must expand the integrand in 0 until we find an integrable remainder. The Taylor formula applied to the functions $\log|x|$ and $\frac{x\otimes x}{|x|^2}$ gives

$$\log|x-y| = \log|x| - \frac{x\cdot y}{|x|^2} + O(|y|^2)$$

and

$$\frac{(x-y)\otimes(x-y)}{|x-y|^2} = \frac{x\otimes x}{|x|^2} - \frac{x\otimes y}{|x|^2} - \frac{y\otimes x}{|x|^2} + 2x\cdot y\frac{x\otimes x}{|x|^4} + O(|y|^2)$$

when $y \to 0$. So

(5.5)

$$\begin{aligned}
\log|x-y| f(y) - &\frac{x-y}{|x-y|^2}(x-y)\cdot f(y) \\
&= \log|x| f(y) - \frac{x\cdot y}{|x|^2} f(y) - \frac{x}{|x|^2} x\cdot f(y) + \frac{x}{|x|^2} y\cdot f(y) + \frac{y}{|x|^2} x\cdot f(y) \\
&\qquad\qquad - 2x\cdot y \frac{x}{|x|^4} x\cdot f(y) + O(|y|^2) f(y) \\
&\equiv H(x,y) + O(|y|^2) f(y).
\end{aligned}$$

Let $a \in (0, |x|)$. Let us show that the integral

$$\mathrm{pv} \int_{|y|<a} \Big(\log|x-y| f(y) - \frac{x-y}{|x-y|^2}(x-y)\cdot f(y) \Big)\, dy$$

converges in the principal value sense. We write

$$\int_{\varepsilon<|y|<a} \Big(\log|x-y| f(y) - \frac{x-y}{|x-y|^2}(x-y)\cdot f(y) \Big)\, dy = \int_{\varepsilon<|y|<a} H(x,y)\, dy + \int_{\varepsilon<|y|<a} O(|y|^2) f(y)\, dy. \tag{5.6}$$

Since $f \in Y_\alpha$ we have that $O(|y|^2)f(y) = O(|y|^{2-\alpha})$ is integrable in 0. Therefore the limit

$$\lim_{\varepsilon\to 0} \int_{\varepsilon<|y|<a} O(|y|^2) f(y)\, dy$$

exists. Recall now that f verifies the symmetry relations (2.4). Since f_1 is odd with respect to x_1 and f_2 is odd with respect to x_2 we have that $\int_{\varepsilon<|y|<a} f(y)\, dy = 0$ so

$$\int_{\varepsilon<|y|<a} H(x,y)\, dy = \int_{\varepsilon<|y|<a} \Big(-\frac{x\cdot y}{|x|^2} f(y) + \frac{x}{|x|^2} y\cdot f(y) + \frac{y}{|x|^2} x\cdot f(y) - 2x\cdot y \frac{x}{|x|^4} x\cdot f(y) \Big)\, dy.$$

An easy calculation shows that the first component of the integrand on the right-hand side is

$$\frac{x_2(x_2^2-x_1^2)}{|x|^4} y_1 f_2(y) - \frac{x_2(3x_1^2+x_2^2)}{|x|^4} y_2 f_1(y) + \frac{x_1(x_1^2-x_2^2)}{|x|^4}(y_2 f_2(y) - y_1 f_1(y)).$$

We also have that $y_1 f_2(y)$ and $y_2 f_1(y)$ are odd with respect to y_1 (and y_2) and $y_2 f_2(y) - y_1 f_1(y)$ is odd when exchanging y_1 and y_2, so

$$\int_{\varepsilon<|y|<a} H_1(x,y)\, dy = 0.$$

One can show in a similar manner that

$$\int_{\varepsilon<|y|<a} H_2(x,y)\, dy = 0$$

so the integral from (5.6) has a limit when $\varepsilon \to 0$. We showed that the integral defining $\widetilde{F}$ converges in the principal value sense.

To estimate $\widetilde{F}$, we shall first make the integrand from (5.3) homogeneous. Observing that

$$\mathrm{pv} \int_{\mathbb{R}^2} f(y)\, dy = 0$$

we can write $\widetilde{F}$ under the form

$$\begin{aligned} \widetilde{F}(x) &= -\frac{1}{4\pi}\, \mathrm{pv} \int_{\mathbb{R}^2} \Big(\log\Big(\frac{|x-y|}{|x|}\Big) f(y) - \frac{x-y}{|x-y|^2}(x-y)\cdot f(y) \Big)\, dy \\ &= -\frac{1}{4\pi}\, \mathrm{pv} \int_{\mathbb{R}^2} L(x,y)\, dy. \end{aligned}$$

where

$$L(x,y) = \log\left(\frac{|x-y|}{|x|}\right) f(y) - \frac{x-y}{|x-y|^2}(x-y)\cdot f(y). \tag{5.7}$$

We decompose now the integral defining $\widetilde{F}$ in two regions:

$$\widetilde{F}(x) = -\frac{1}{4\pi}\int_{|y|\geqslant\frac{|x|}{2}} L(x,y)\,dy - \frac{1}{4\pi}\,\mathrm{pv}\int_{|y|\leqslant\frac{|x|}{2}} L(x,y)\,dy \equiv \widetilde{F}_1(x) + \widetilde{F}_2(x).$$

We bound $\widetilde{F}_1$ in the following way:

$$|\widetilde{F}_1(x)| \leqslant \frac{\|f\|_{\dot{Y}_\alpha}}{4\pi}\int_{|y|\geqslant\frac{|x|}{2}} \Big[1+\Big|\log\big(\frac{|x-y|}{|x|}\big)\Big|\Big]|y|^{-\alpha}\,dy. \tag{5.8}$$

The integral

$$I(x) = \int_{|y|\geqslant\frac{|x|}{2}} \Big[1+\Big|\log\big(\frac{|x-y|}{|x|}\big)\Big|\Big]|y|^{-\alpha}\,dy$$

defines a function of x which is radial and homogeneous of degree $2-\alpha$. Indeed, if A is an orthogonal matrix we have

$$\begin{aligned}
I(Ax) &= \int_{|y|\geqslant\frac{|Ax|}{2}} \Big[1+\Big|\log\big(\frac{|Ax-y|}{|Ax|}\big)\Big|\Big]|y|^{-\alpha}\,dy \\
&= \int_{|Az|\geqslant\frac{|Ax|}{2}} \Big[1+\Big|\log\big(\frac{|Ax-Az|}{|Ax|}\big)\Big|\Big]|Az|^{-\alpha}\,dz \\
&= \int_{|z|\geqslant\frac{|x|}{2}} \Big[1+\Big|\log\big(\frac{|x-z|}{|x|}\big)\Big|\Big]|z|^{-\alpha}\,dz \\
&= I(x)
\end{aligned}$$

where we made the change of variables $y = Az$. Next, if $\lambda > 0$ we have

$$\begin{aligned}
I(\lambda x) &= \int_{|y|\geqslant\frac{\lambda|x|}{2}} \Big[1+\Big|\log\big(\frac{|\lambda x-y|}{|\lambda x|}\big)\Big|\Big]|y|^{-\alpha}\,dy \\
&= \int_{|z|\geqslant\frac{|x|}{2}} \Big[1+\Big|\log\big(\frac{|x-z|}{|x|}\big)\Big|\Big]|\lambda z|^{-\alpha}\lambda^2\,dz \\
&= \lambda^{2-\alpha} I(x).
\end{aligned}$$

We deduce that there exists a constant C_0 such that $I(x) = C_0|x|^{2-\alpha}$, and relation (5.8) implies the bound

$$\|\widetilde{F}_1\|_{\dot{Y}_{\alpha-2}} \leqslant \frac{C_0}{4\pi}\|f\|_{\dot{Y}_\alpha}.$$

To estimate the term $\widetilde{F}_2$, we recall relation (5.5) to write

$$L(x,y) = -\frac{x\cdot y}{|x|^2}f(y) - \frac{x}{|x|^2}x\cdot f(y) + \frac{x}{|x|^2}y\cdot f(y) + \frac{y}{|x|^2}x\cdot f(y) - 2x\cdot y\frac{x}{|x|^4}x\cdot f(y) + O(|y|^2)f(y).$$

But the term $O(|y|^2)$ which appears above comes by applying the Taylor formula to the functions $\log|x|$ and $\frac{x\otimes x}{|x|^2}$. So it can be bounded by $|y|^2$ times the uniform norm on the segment $[x, x-y]$ of the second order derivatives of these two functions. These second order derivatives are homogeneous of degree -2 and if $|y| \leqslant |x|/2$ then every point $\xi \in [x, x-y]$

verifies $|x|/2 \leqslant |\xi| \leqslant 3|x|/2$. We can therefore bound these second order derivatives by $C/|x|^2$ and we conclude that the term $O(|y|^2)f(y)$ can be bounded by

$$|O(|y|^2)f(y)| \leqslant C\frac{|y|^2}{|x|^2}|f(y)| \leqslant C\frac{|y|^{2-\alpha}}{|x|^2}\|f\|_{\dot{Y}_\alpha}.$$

We also proved that

$$\begin{aligned}\operatorname{pv}\int_{|y|\leqslant\frac{|x|}{2}} \Big[-\frac{x\cdot y}{|x|^2}f(y) + \frac{x}{|x|^2}x\cdot f(y) - \frac{x}{|x|^2}y\cdot f(y) - \frac{y}{|x|^2}x\cdot f(y) + 2x\cdot y\frac{x}{|x|^4}x\cdot f(y)\Big]\,dy \\ = \operatorname{pv}\int_{|y|\leqslant\frac{|x|}{2}} (H(x,y) - \log|x|f(y))\,dy \\ = 0.\end{aligned}$$

Consequently

$$\begin{aligned}|\widetilde{F}_2(x)| &= \frac{1}{4\pi}\Big|\operatorname{pv}\int_{|y|\leqslant\frac{|x|}{2}} L(x,y)\,dy\Big| \\ &= \frac{1}{4\pi}\Big|\int_{|y|\leqslant\frac{|x|}{2}} O(|y|^2)f(y)|\,dy\Big| \\ &\leqslant C\frac{\|f\|_{\dot{Y}_\alpha}}{|x|^2}\int_{|y|\leqslant\frac{|x|}{2}}|y|^{2-\alpha}\,dy \\ &\leqslant C\|f\|_{\dot{Y}_\alpha}|x|^{2-\alpha}.\end{aligned}$$

We infer that

$$\|\widetilde{F}_2\|_{\dot{Y}_{\alpha-2}} \leqslant C\|f\|_{\dot{Y}_\alpha}.$$

and this ends the proof of the continuity of the mapping $f \mapsto \widetilde{F}$ from $\dot{Y}_\alpha$ into $\dot{Y}_{\alpha-2}$.

Finally, if f is homogeneous of degree $-\alpha$ then we immediately see that the function $L(x,y)$ defined in (5.7) verifies $L(\lambda x, \lambda y) = \lambda^{-\alpha}L(x,y)$. Therefore

$$\begin{aligned}\widetilde{F}(\lambda x) &= -\frac{1}{4\pi}\operatorname{pv}\int_{\mathbb{R}^2} L(\lambda x, y)\,dy \\ &= -\frac{1}{4\pi}\operatorname{pv}\int_{\mathbb{R}^2} L(\lambda x, \lambda z)\lambda^2\,dz \\ &= -\lambda^{2-\alpha}\frac{1}{4\pi}\operatorname{pv}\int_{\mathbb{R}^2} L(x,z)\,dz \\ &= \lambda^{2-\alpha}\widetilde{F}(x)\end{aligned}$$

which means that $\widetilde{F}$ is homogeneous of degree $2-\alpha$. This ends the proof of the lemma. □

The next lemma gives an estimate for the bilinear form $\widetilde{B}$.

Lemma 5.2. *Let α_1, α_2 be two real numbers such that $1 < \alpha_1 + \alpha_2 < 3$. If $U \in \dot{Y}_{\alpha_1}$ and $V \in \dot{Y}_{\alpha_2}$ then $\widetilde{B}(U,V)$ is well-defined and belongs to $\dot{Y}_{\alpha_1+\alpha_2-1}$. Moreover, we have the inequality*

$$\|\widetilde{B}(U,V)\|_{\dot{Y}_{\alpha_1+\alpha_2-1}} \leqslant C\|U\|_{\dot{Y}_{\alpha_1}}\|V\|_{\dot{Y}_{\alpha_2}}.$$

Proof. We show first that $\widetilde{B}$ is well defined, that is to say the integral which defines $\widetilde{B}$ converges in the principal value sense. We observe first that the integrand can be bounded by $\dfrac{C}{|x-y||y|^{\alpha_1+\alpha_2}}$ which is a function integrable (with respect to the y variable) in x and at infinity. Only 0 is a possible non-integrable singularity. When $y \to 0$, the integrand from (5.2) can be written as

$$\begin{aligned}
&\frac{x-y}{|x-y|^2}U(y)\cdot V(y) - 2\frac{x-y}{|x-y|^4}\big[(x-y)\cdot U(y)\big]\big[(x-y)\cdot V(y)\big] \\
&\qquad = \frac{x}{|x|^2}U(y)\cdot V(y) - 2\frac{x}{|x|^4}(x\cdot U(y))(x\cdot V(y)) + O(|y|^{1-\alpha_1-\alpha_2}) \\
&\qquad = \frac{x(x_2^2-x_1^2)}{|x|^4}[U_1(y)V_1(y) - U_2(y)V_2(y)] - 2\frac{xx_1x_2}{|x|^4}[U_1(y)V_2(y)+U_2(y)V_1(y)] \\
&\qquad\qquad + O(|y|^{1-\alpha_1-\alpha_2}).
\end{aligned}$$

Since $\alpha_1+\alpha_2<3$, the remainder $O(|y|^{1-\alpha_1-\alpha_2})$ is integrable in 0. The symmetry conditions (2.4) verified by U and V imply that $U_1(y)V_2(y)+U_2(y)V_1(y)$ is odd with respect to y_1 and y_2 and that $U_1(y)V_1(y)-U_2(y)V_2(y)$ is odd when exchanging y_1 and y_2. The integral in y of the term

$$\frac{x(x_2^2-x_1^2)}{|x|^4}[U_1(y)V_1(y) - U_2(y)V_2(y)] - 2\frac{xx_1x_2}{|x|^4}[U_1(y)V_2(y)+U_2(y)V_1(y)]$$

is therefore 0 on any annulus centered in 0 and this suffices to conclude that the integral defining $\widetilde{B}(U,V)$ exists in the principal value sense.

We bound now $\widetilde{B}(U,V)$ by decomposing

$$\begin{aligned}
\widetilde{B}(U,V) &= -\frac{1}{4\pi}\,\mathrm{pv}\int_{\mathbb{R}^2}\Big(\frac{x-y}{|x-y|^2}U(y)\cdot V(y) - 2\frac{x-y}{|x-y|^4}\big[(x-y)\cdot U(y)\big]\big[(x-y)\cdot V(y)\big]\Big)\,dy \\
&= -\frac{1}{4\pi}\,\mathrm{pv}\int_{|y|\leqslant\frac{|x|}{2}}\Big(\frac{x-y}{|x-y|^2}U(y)\cdot V(y) - 2\frac{x-y}{|x-y|^4}\big[(x-y)\cdot U(y)\big]\big[(x-y)\cdot V(y)\big]\Big)\,dy \\
&\quad - \frac{1}{4\pi}\int_{|y|\geqslant\frac{|x|}{2}}\Big(\frac{x-y}{|x-y|^2}U(y)\cdot V(y) - 2\frac{x-y}{|x-y|^4}\big[(x-y)\cdot U(y)\big]\big[(x-y)\cdot V(y)\big]\Big)\,dy \\
&\equiv \widetilde{B}_1(U,V) + \widetilde{B}_2(U,V).
\end{aligned}$$

The term $\widetilde{B}_2$ can be easily estimated

$$\begin{aligned}
|\widetilde{B}_2(U,V)| &\leqslant C\int_{|y|\geqslant\frac{|x|}{2}}\frac{|U(y)||V(y)|}{|x-y|}\,dy \\
&\leqslant C\|U\|_{\dot{Y}_{\alpha_1}}\|V\|_{\dot{Y}_{\alpha_2}}\int_{|y|\geqslant\frac{|x|}{2}}\frac{|y|^{-\alpha_1-\alpha_2}}{|x-y|}\,dy \\
&= C'\|U\|_{\dot{Y}_{\alpha_1}}\|V\|_{\dot{Y}_{\alpha_2}}|x|^{1-\alpha_1-\alpha_2}
\end{aligned}$$

where we used, as in the previous lemma, that the integral

$$\int_{|y|\geqslant\frac{|x|}{2}}\frac{|y|^{-\alpha_1-\alpha_2}}{|x-y|}\,dy$$

defines a radial function homogeneous of degree $1-\alpha_1-\alpha_2$. We infer that

$$\|\widetilde{B}_2(U,V)\|_{\dot{Y}_{\alpha_1+\alpha_2-1}} \leqslant C'\|U\|_{\dot{Y}_{\alpha_1}}\|V\|_{\dot{Y}_{\alpha_2}}.$$

To bound the term $\widetilde{B}_1$, we recall that

$$\mathrm{pv}\int_{|y|\leqslant\frac{|x|}{2}}\Big(\frac{x}{|x|^2}U(y)\cdot V(y)-2\frac{x}{|x|^4}(x\cdot U(y))(x\cdot V(y))\Big)=0$$

to deduce that

$$\begin{aligned}\widetilde{B}_1(U,V)=-\frac{1}{4\pi}\int_{|y|\leqslant\frac{|x|}{2}}\Big\{&\frac{x-y}{|x-y|^2}U(y)\cdot V(y)-\frac{x}{|x|^2}U(y)\cdot V(y)\\&-2\frac{x-y}{|x-y|^4}\big[(x-y)\cdot U(y)\big]\big[(x-y)\cdot V(y)\big]+2\frac{x}{|x|^4}(x\cdot U(y))(x\cdot V(y))\Big\}\,dy.\end{aligned}$$

When $|y|\leqslant\frac{|x|}{2}$ we have that $x-y$ is of the same size as x: $|x|/2\leqslant|x-y|\leqslant 3|x|/2$. It is easy to see that, for $|y|\leqslant\frac{|x|}{2}$, we have the following estimate for the above integrand:

$$\begin{aligned}\Big|\frac{x-y}{|x-y|^2}U(y)\cdot V(y)-\frac{x}{|x|^2}U(y)\cdot V(y)-2\frac{x-y}{|x-y|^4}\big[(x-y)\cdot U(y)\big]\big[(x-y)\cdot V(y)\big]&\\+2\frac{x}{|x|^4}(x\cdot U(y))(x\cdot V(y))\Big|&\\\leqslant C\frac{|y|}{|x|^2}|U(y)||V(y)|&\\\leqslant C\|U\|_{\dot{Y}_{\alpha_1}}\|V\|_{\dot{Y}_{\alpha_2}}\frac{|y|^{1-\alpha_1-\alpha_2}}{|x|^2}.&\end{aligned}$$

We can therefore bound $\widetilde{B}_1$ in the following way

$$|\widetilde{B}_1(U,V)|\leqslant C\frac{\|U\|_{\dot{Y}_{\alpha_1}}\|V\|_{\dot{Y}_{\alpha_2}}}{|x|^2}\int_{|y|\leqslant\frac{|x|}{2}}|y|^{1-\alpha_1-\alpha_2}\,dy\leqslant C\|U\|_{\dot{Y}_{\alpha_1}}\|V\|_{\dot{Y}_{\alpha_2}}|x|^{1-\alpha_1-\alpha_2}.$$

Consequently

$$\|\widetilde{B}_1(U,V)\|_{\dot{Y}_{\alpha_1+\alpha_2-1}}\leqslant C\|U\|_{\dot{Y}_{\alpha_1}}\|V\|_{\dot{Y}_{\alpha_2}}$$

and this completes the proof of the lemma. □

We can now find the asymptotic behaviour of the solutions of (5.4). We prove first the existence and uniqueness of homogeneous solutions.

Theorem 5.3. *Let $f_0\in\dot{Y}_3$ be homogeneous of degree -3 and let us define $\widetilde{F}_0$ by (5.3) where f is replaced by f_0. There exist two universal constants $\varepsilon_0,\varepsilon_0'>0$ such that if $\|f_0\|_{\dot{Y}_3}\leqslant\varepsilon_0$, then there exists exactly one solution $U_0\in\dot{Y}_1$ homogeneous of degree -1 of the equation $U_0=\widetilde{B}(U_0,U_0)+\widetilde{F}_0$ such that $\|U_0\|_{\dot{Y}_1}\leqslant\varepsilon_0'$.*

Proof. Lemma 5.2 shows that the operator $\widetilde{B}$ is bilinear and continuous from $\dot{Y}_1\times\dot{Y}_1$ into $\dot{Y}_1$. Lemma 5.1 implies that $\widetilde{F}_0$ is homogeneous of degree -1 and furthermore $\|\widetilde{F}_0\|_{\dot{Y}_1}\leqslant C\|f_0\|_{\dot{Y}_3}$. The existence and the uniqueness of a small solution $U_0\in\dot{Y}_1$ follows from the fixed point lemma 2.4. The homogeneity of U_0 is an immediate consequence of the homogeneity of $\widetilde{F}_0$, of the homogeneity of $\widetilde{B}$ and of the uniqueness of the solutions. □

The last result of this paper shows that if f is bounded by $C/(1+|x|)^3$, verifies the symmetry conditions (2.4) and admits an asymptotic behaviour at infinity homogeneous of degree -3, then the solution U also admits an asymptotic behaviour at infinity homogeneous of degree -1. Recall that we denoted by φ a smooth function such that $\varphi(x)=0$ for $|x|\leqslant 1/2$ and $\varphi(x)=1$ for $|x|\geqslant 1$.

Theorem 5.4. *Let $f \in Y_3$ be a vector field which of the form*

$$f = \varphi f_0 + f_1$$

where f_0 is homogeneous of degree -3 and $f_1 \in Y_\alpha$ for some $\alpha \in (3,4)$. There exist two constants $\varepsilon, \varepsilon' > 0$ depending solely on α such that if $\|f_0\|_{\dot{Y}_3} + \|f_1\|_{Y_\alpha} \leqslant \varepsilon$, then the equation (5.4) has a unique solution $U \in Y_1$ such that $\|U\|_{Y_1} \leqslant \varepsilon'$. Moreover, U has the following asymptotic behaviour:

$$U(x) = U_0(x) + O(|x|^{2-\alpha}) \tag{5.9}$$

when $|x| \to \infty$, where U_0 is the unique homogeneous solution associated to f_0 given by Theorem 5.3.

Proof. It can be easily checked that Lemmas 5.1 and 5.2 remain true in the setting of the inhomogeneous spaces Y_α. As f is small in Y_3, the existence and uniqueness of a small solution Y_1 can be shown as in Theorem 5.3.

Let us show now that the asymptotic behaviour stated in (5.9). We have that $\|U\|_{\dot{Y}_1} \leqslant \|U\|_{Y_1} \leqslant C\varepsilon$ and $\|U_0\|_{\dot{Y}_1} \leqslant C\varepsilon$. The vector field $V = U - U_0$ verifies the equation

$$V = \widetilde{B}(V, U) + \widetilde{B}(U_0, V) + \widetilde{F} - \widetilde{F}_0$$

and is the limit of the following sequence defined recursively

$$V_0 = \widetilde{F} - \widetilde{F}_0, \qquad V_{k+1} = \widetilde{B}(V_k, U) + \widetilde{B}(U_0, V_k) + \widetilde{F} - \widetilde{F}_0.$$

The vector field $\widetilde{F} - \widetilde{F}_0$ is associated to $f - f_0 = (\varphi - 1)f_0 + f_1$ which is small in $\dot{Y}_\alpha$. By Lemma 5.1, there exists a constant C_1 such that

$$\|\widetilde{F} - \widetilde{F}_0\|_{\dot{Y}_{\alpha-2}} \leqslant C_1\varepsilon. \tag{5.10}$$

We show by induction on k the following bound:

$$\|V_k\|_{\dot{Y}_{\alpha-2}} \leqslant 2C_1\varepsilon. \tag{5.11}$$

Relation (5.10) shows this bound for $k = 0$. Assume it is true for k and let us show it for $k+1$. Thanks to Lemma 5.2 we can bound

$$\begin{aligned}
\|V_{k+1}\|_{\dot{Y}_{\alpha-2}} &\leqslant \|\widetilde{B}(V_k, U)\|_{\dot{Y}_{\alpha-2}} + \|\widetilde{B}(U_0, V_k)\|_{\dot{Y}_{\alpha-2}} + \|\widetilde{F} - \widetilde{F}_0\|_{\dot{Y}_{\alpha-2}} \\
&\leqslant C\|V_k\|_{\dot{Y}_{\alpha-2}}(\|U\|_{\dot{Y}_1} + \|U_0\|_{\dot{Y}_1}) + \|\widetilde{F} - \widetilde{F}_0\|_{\dot{Y}_{\alpha-2}} \\
&\leqslant 4C^2C_1\varepsilon^2 + C_1\varepsilon \\
&\leqslant 2C_1\varepsilon
\end{aligned}$$

if $\varepsilon \leqslant 1/4C^2$.

Relation (5.11) is therefore verified for all k. Passing to the limit $k \to \infty$ implies that (5.11) is verified for V too. In particular $V \in \dot{Y}_{\alpha-2}$ which implies the asymptotic behaviour described in (5.9). Theorem 5.4 is completely proved. □

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Agathe Decaster: École Normale Supérieure de Lyon, Unité de Mathématiques Pures et Appliquées, UMR CNRS 5669, 46, allée d'Italie, 69364 Lyon Cedex 07, France.
Email: `agathe.decaster@ens-lyon.fr`

Dragoş Iftimie: Université de Lyon, CNRS, Université Lyon 1, Institut Camille Jordan, 43 bd. du 11 novembre, Villeurbanne Cedex F-69622, France.
Email: `iftimie@math.univ-lyon1.fr`