On the asymptotic behaviour of solutions of the stationary Navier-Stokes equations in dimension 3

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Abstract

In this paper, we address the problem of determining the asymptotic behaviour of the solutions of the incompressible stationary Navier-Stokes system in the full space, with a forcing term whose asymptotic behaviour at infinity is homogeneous of degree -3. We identify the asymptotic behaviour at infinity of the solution. We prove that it is homogeneous and that the leading term in the expansion at infinity uniquely solves the homogeneous Navier-Stokes equations with a forcing term which involves an additional Dirac mass. This also applies to the case of an exterior domain.

1 Introduction

We consider the incompressible stationary Navier-Stokes equations with a forcing term in $\mathbb{R}^3$:

\begin{equation}
-\Delta U + (U \cdot \nabla)U + \nabla p = f, \quad \text{div} U = 0 \quad \text{in} \mathbb{R}^3, \quad \lim_{|x| \to 0} U(x) = 0.
\end{equation}

The forcing $f$ is given and the unknowns are the velocity field $U$ and the scalar pressure $p$. Clearly $p$ is uniquely (up to a constant) determined by $f$ and $U$. For this reason, by solution we mean only the velocity field $U$. In other words, throughout this paper a solution of (1.1) is a vector field $U$ such that there exists some $p$ such that (1.1) is satisfied.

The aim of this paper is to determine the asymptotic behaviour of the solutions at infinity under reasonable assumptions on the forcing $f$. Several authors investigated this problem.

In [1] the authors studied the existence and uniqueness of solutions under a smallness assumption in the critical space $L^{3,\infty}$. Moreover, that article found an explicit asymptotic behaviour of the solutions with a decay as $O(\frac{1}{|x|^2})$ provided that $\mathbb{P}\Delta^{-1}f$ is bounded by $C/(1 + |x|^2)$, where $\mathbb{P}$ is the Leray projector. More precisely they showed the following expansion for the solution:

\begin{equation}
U(x) = \mathbb{P}\Delta^{-1}f(x) + m(x) : \int_{\mathbb{R}^3} U \otimes U + O\left(\frac{\ln|x|}{|x|^3}\right) \quad \text{as} \quad |x| \to \infty
\end{equation}

where $m(x)$ is an explicit function homogeneous of degree $-2$ and smooth outside 0. Observe that $\mathbb{P}\Delta^{-1}$ is a convolution operator whose kernel is an homogeneous function of degree $-1$. Therefore, if $f$ is sufficiently decaying at infinity, the condition that $|\mathbb{P}\Delta^{-1}f| \leq C/(1 + |x|^2)$ imposed in [1] holds true if and only if $\int_{\mathbb{R}^3} f = 0$. 

But since all terms in the expansion (1.2) are \(O(\frac{1}{|x|^2})\), it excludes all solutions which are homogeneous of degree \(-1\). In particular it excludes the very important case of Landau solutions. The Landau solutions were introduced by Landau in [10] and they are given by the explicit formula

\[
v_1^c(x) = 2\frac{c|x|^2 - 2x_1x + cx_1^2}{|x|(c|x| - x_1)^2}, \quad v_2^c(x) = 2\frac{x_2(cx_1 - |x|)}{|x|(c|x| - x_1)^2}, \quad v_3^c(x) = 2\frac{x_3(cx_1 - |x|)}{|x|(c|x| - x_1)^2}\]

with pressure

\[
p(x) = 4\frac{cx_1 - |x|}{|x|(c|x| - x_1)^2}.
\]

They verify (1.1) with forcing \(f = \beta \delta\) where

\[
\beta = \frac{8\pi c^3}{3(c^2 - 1)} \left( 2 + 6c^2 - 3c(c^2 - 1) \log \left( \frac{c + 1}{c - 1} \right) \right)
\]

and \(\delta\) is the Dirac mass in 0 (see [4]). It was even even shown by Šverák [12] that all homogeneous solutions of (1.1) on \(\mathbb{R}^3 \setminus \{0\}\) with vanishing forcing are the Landau solutions.

It appears then that the relevant asymptotic behaviour at infinity of the solutions of (1.1) should rather be of order \(O(1/|x|)\). And indeed, it was shown in [11] that small solutions of the stationary incompressible Navier-Stokes equations in an exterior domain of \(\mathbb{R}^3\) behave like \(v(x) + o(1/|x|)\) where \(v\) is some unknown vector field homogeneous of degree \(-1\). Moreover, Korolev and Šverák [9] observed that the asymptotic profile \(v\) must be a Landau solution. More precisely, they proved that if \(U\) is small and verifies (1.1) in the exterior of a ball with no boundary conditions required, then there exists \(a\) such that

\[
U = v^a + o(1/|x|) \quad \text{as} \quad |x| \to \infty.
\]

Let us also mention the work [5] where the authors study the stationary Navier-Stokes flow around a rotating body. They obtain again that the asymptotic behaviour of the solution is given by a Landau solution when the speed of rotation of the body is sufficiently small. In [8], the authors prove that the asymptotic behavior as \(|x| \to \infty\) of time-period solutions is also given by a Landau solution.

Since the relevant asymptotic behaviour at infinity is homogeneous of degree \(-1\) and since the forcing corresponding to a velocity homogeneous of degree \(-1\) is homogeneous of degree \(-3\), it makes sense to study the asymptotic behaviour of the solutions of (1.1) with a forcing whose asymptotic behaviour at infinity is homogeneous of degree \(-3\).

Let \(\alpha \in (0, 1)\) be fixed once and for all. We will assume in the rest of this paper that the forcing term is of the form

\[
(1.3) \quad f = \phi f_0 + f_1
\]

where

- \(f_0\) is homogeneous of degree \(-3\), locally bounded on \(\mathbb{R}^3 \setminus \{0\}\);
- we have that \(|f_1(x)| \leq C/(1 + |x|)^{3+\alpha}\) for some constant \(C\);
- \(\phi \in C^\infty(\mathbb{R}^3; [0, 1])\) is a radial cut-off function such that \(\phi(x) = 0\) for \(|x| \leq 1/2\) and \(\phi(x) = 1\) for \(|x| \geq 1\).
The questions that we ask ourselves are the following. Under what additional hypothesis on $f_0$ and $f_1$ there exists a solution $U$ of (1.1) such that $|U(x)| \leq C/|x|$ for some constant $C$? When such a solution exists, how does it behave at infinity? In short, we give the following answers. If such a $U$ exists then necessarily

$$\int_{S^2} f_0 = 0.$$  

And vice versa, if the above condition is satisfied and if $f_0, f_1$ are sufficiently small then there exists a unique $U$ which is small and bounded like $O(1/|x|)$. Moreover, we have the following asymptotic behaviour at infinity for the solution:

$$U(x) = U_0(x) + O(|x|^{-1-\alpha}),$$

where $U_0$ is the only small solution of the Navier-Stokes equation with forcing term $f_0 + (\int_{\mathbb{R}^3} f_1)\delta$ which is homogeneous of degree $-1$.

The precise results and the notation will be given in the next section. In Section 3 we prove a necessary condition for the existence of $O(1/|x|)$ solutions. We show next in Section 4 the existence and uniqueness of homogeneous solutions. We prove our main result in Section 5. Finally, in Section 6 we extend our results to exterior domains.

2 Main results and notation

We introduce the following function space for $a > 0$:

$$X_a = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^3 \setminus \{0\}) : |f(x)| \leq \frac{C}{|x|^a} \right\}$$

with norm

$$\|f\|_{X_a} = \sup_{x \in \mathbb{R}^3 \setminus \{0\}} |x|^a |f(x)|$$

We recall now some results on homogeneous distributions that can be found in the book of Hörmander [7, Section 3.2]. Let $v \in \mathcal{D}'(\mathbb{R}^3 \setminus \{0\})$ be a distribution homogeneous of degree $-3$. There exists a constant $S(v)$ such that the following relation holds true:

$$\langle v, \varphi \rangle = S(v) \int_0^{\infty} \frac{\varphi(r)}{r} \, dr$$

for all radial test functions $\varphi \in C^\infty_0(\mathbb{R}^3 \setminus \{0\})$. We define the integral of $v$ over the unit sphere to be the constant $S(v)$:

$$\int_{S^2} v \equiv S(v).$$

This definition is justified by the fact that the relation above holds true when $v$ is a continuous function as a consequence of the Fubini formula in polar coordinates.

The following result on extensions of homogeneous distributions is stated in [7, Theorem 3.2.4].
Proposition 2.1 ([7]). Let \( g \in \mathcal{D}'(\mathbb{R}^3 \setminus \{0\}) \) be homogeneous of degree \(-3\). There exists a distribution \( h \in \mathcal{D}'(\mathbb{R}^3) \) homogeneous of degree \(-3\) such that \( h|_{\mathbb{R}^3 \setminus \{0\}} = g \) if and only if \( \int_{S^2} g = 0 \).

Moreover, if in addition \( g \) is a locally bounded function and \( \int_{S^2} g = 0 \) then all such distributions \( h \) are given by

\[
h = \text{pv}(g) + C\delta
\]

where \( C \) is an arbitrary constant and the principal value of \( g \) is defined by

\[
\text{pv}(g) \in \mathcal{D}'(\mathbb{R}^3) : \quad \langle \text{pv}(g), \varphi \rangle = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} g \varphi \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^3).
\]

In what follows we will work with integrals of functions that decay like \( 1/|x|^3 \) at infinity or have local singularities at the origin like \( 1/|x|^3 \). We define such integrals in the principal value sense:

\[
\text{pv} \int_{\mathbb{R}^3} h = \lim_{R \to \infty} \lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < R} h
\]

provided of course that the limit exists.

We recall now the following well-known (and easy) lemma.

Lemma 2.2 ([3]). Let \( X \) be a Banach space and \( B : X \times X \to X \) a bilinear map. Assume that for all \( x_1, x_2 \in X \) one has

\[
\|B(x_1, x_2)\|_X \leq \eta \|x_1\|_X \|x_2\|_X.
\]

Then for all \( y \in X \) satisfying \( 4\eta \|y\|_X < 1 \), the equation

\[
x = y + B(x, x),
\]

has a solution \( x \in X \) satisfying and uniquely defined by the condition

\[
\|x\|_X \leq 2\|y\|_X.
\]

The proof of this lemma also shows that \( x = \lim_{k \to \infty} x_k \) where the approximate solutions \( x_k \) are defined by \( x_0 = y \) and \( x_k = y + B(x_{k-1}, x_{k-1}) \). Moreover \( \|x_k\|_X \leq 2\|y\|_X \) for all \( k \).

We denote by \( \mathbb{P} \) the Leray projector, i.e. the \( L^2 \) orthogonal projection on the subspace of divergence free vector fields. Applying \( \mathbb{P} \) to (1.1) and inverting the laplacian we obtain the following equivalent formulation of (1.1) where there is no pressure:

\[
(2.2) \quad U = \mathbb{P}\Delta^{-1} \text{div}(U \otimes U) - \mathbb{P}\Delta^{-1} f.
\]

In order to state our main theorem, we first need to establish a result of existence and uniqueness for homogeneous solutions.

Theorem 2.3. Let \( f_0 \in X_3 \) be homogeneous of degree \(-3\). If the equations (1.1) on \( \mathbb{R}^3 \setminus \{0\} \) with forcing \( f_0 \) admit a solution \( U_0 \) such that \( U_0 \in X_1 \) and \( U_0 \) is homogeneous of degree \(-1\), then necessarily

\[
(2.3) \quad \int_{S^2} f_0 = 0
\]

and there exists a constant vector \( \gamma \in \mathbb{R}^3 \) such that \( U_0 \) verifies (1.1) on \( \mathbb{R}^3 \) with forcing \( \text{pv}(f_0) + \gamma \delta \).
Conversely, assume that $\gamma \in \mathbb{R}^3$ and $f_0 \in X_3$ is homogeneous of degree $-3$ and that (2.3) is satisfied. Then $\mathbb{P}\Delta^{-1} \text{pv}(f_0)$ is well-defined in the principal value sense and there exist $\varepsilon_1$ such that if
\[
\|f_0\|_{L^\infty(S^2)} + |\gamma| < \varepsilon_1
\]
then there exists a unique solution $U_0 \in X_1$ of (2.2) on $\mathbb{R}^3$ with forcing $\text{pv}(f_0) + \gamma \delta$ such that $\|U_0\|_{X_1} \leq \varepsilon_1$. Moreover, $U_0$ is homogeneous of degree $-1$.

We are now able to state our main theorem:

**Theorem 2.4.** Let $f$ be as in (1.3). If there is a solution $U$ of (1.1) which belongs to $X_1$, then (2.3) must hold true.

Conversely, assume that (2.3) holds true. There exists $\varepsilon_2 > 0$ such that if $\|f_0\|_{L^\infty(S^2)} + \|f_1\|_{X_0 \cap X_3} \leq \varepsilon_2$

then the equation (2.2) admits a unique small solution $U \in X_1$. Moreover, this solution has the following asymptotic behaviour at infinity:
\[
U(x) = U_0(x) + O(|x|^{-1-\alpha}) \quad \text{as } |x| \to \infty
\]
where $U_0$ is the only $-1$ homogeneous solution of the Navier-Stokes equation with forcing term $\text{pv}(f_0) + (\text{pv} \int_{\mathbb{R}^3} f)\delta$ obtained in Theorem 2.3.

### 3 A necessary condition

The aim of this section is to prove that if the asymptotic behaviour at infinity of the forcing is homogeneous of degree $-1$, then a necessary condition for the existence of $O(1/|x|)$ solutions is that the asymptotic part of the forcing has vanishing integral on the unit sphere. More precisely, we prove the following result.

**Proposition 3.1.** Let $R_0 > 0$ and $U$ be a divergence free vector field defined on $\Omega_0 = \{ x \; ; \; |x| > R_0 \}$ such that $|U(x)| \leq C_0/|x|$ for some constant $C_0$. Let $f_0$ and $f_1$ be vector valued distributions such that $f_0 \in \mathcal{D}'(\mathbb{R}^3 \setminus \{0\})$ is homogeneous of degree $-3$ and $f_1$ is a (vector valued) bounded Radon measure on $\Omega_0$. If there exists some $p \in \mathcal{D}'(\Omega_0)$ such that
\[
(3.1) \quad -\Delta U + (U \cdot \nabla)U + \nabla p = f_0 + f_1 \quad \text{in } \mathcal{D}'(\Omega_0)
\]
then
\[
\int_{S^2} f_0 = 0.
\]

**Proof.** Let us choose two radial functions $\varphi_1$ and $\varphi_2$ such that:

- $\varphi_1 \in C^\infty(\mathbb{R}^3; [0,1])$ such that $\varphi_1(x) = 0$ for all $|x| \leq R_0 + 1$ and $\varphi_1(x) = 1$ for all $|x| \geq R_0 + 2$.
- $\varphi_2 \in C^\infty(\mathbb{R}^3; [0,1])$, $\varphi_2(x) = 1$ for all $|x| \leq 1$ and $\varphi_2(x) = 0$ for all $|x| \geq 2$. 


We define \( \varphi^R(x) = \varphi_2(x/R)\varphi_1(x) \) and we observe that \( \varphi^R \in C^\infty_0(\mathbb{B}^c_{R_0}) \). We apply the curl operator to (3.1) and we multiply by the test vector field \( V_R \equiv \varphi^R \left( \begin{array}{c} 0 \\ x_3 \\ -x_2 \end{array} \right) \) to obtain that
\[
-\langle \triangle \text{curl} \, U, V_R \rangle + \langle \text{curl} \, \text{div}(U \otimes U), V_R \rangle = \langle \text{curl}(f_0 + f_1), V_R \rangle
\]
which implies that
\[(3.2) \quad -\langle U, \text{curl} \, \triangle V_R \rangle - \langle U \otimes U, \nabla \text{curl} \, V_R \rangle = \langle f_0 + f_1, \text{curl} \, V_R \rangle.\]
Let us denote
\[
X = \left( \begin{array}{c} 0 \\ x_3 \\ -x_2 \end{array} \right) \quad \text{and} \quad e_1 = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right).
\]
We have that
\[
\begin{align*}
\text{curl} \, \triangle V_R &= \Delta(\nabla \varphi^R \times X + \varphi^R \text{curl} \, X) \\
&= \Delta(\nabla \varphi^R \times X - 2\varphi^R e_1) \\
&= \nabla \Delta \varphi^R \times X + 2 \sum_{i=1}^{3} \partial_i \nabla \varphi^R \times \partial_i X - 2\varphi^R e_1 \\
&= \nabla \Delta \varphi^R \times X + 2\partial_1 \nabla \varphi^R - 4\Delta \varphi^R e_1.
\end{align*}
\]
Observe next that
\[(3.3) \quad \nabla \varphi^R(x) = \frac{1}{R} \nabla \varphi_2(x/R)\varphi_1(x) + \varphi_2(x/R)\nabla \varphi_1(x) = \frac{1}{R} \nabla \varphi_2(x/R) + \nabla \varphi_1(x) \]
if \( R \geq R_0 + 2 \). Therefore, if \( R \geq R_0 + 2 \),
\[
\begin{align*}
\text{curl} \, \triangle V_R &= \frac{1}{R^3} \nabla \Delta \varphi_2(x/R) \times X + \frac{2}{R^2} \partial_1 \nabla \varphi_2(x/R) - \frac{4}{R^2} \Delta \varphi_2(x/R)e_1 \\
&\quad + \nabla \Delta \varphi_1 \times X + 2 \partial_1 \nabla \varphi_1 - 4\Delta \varphi_1 e_1.
\end{align*}
\]
Clearly
\[|\langle U, \frac{1}{R^3} \nabla \Delta \varphi_2(x/R) \times X \rangle| \leq \frac{C}{R^3} \int |\nabla \Delta \varphi_2(x/R)| = C\|\nabla \Delta \varphi_2\|_{L^1}
\]
and similarly
\[|\langle U, \frac{2}{R^2} \partial_1 \nabla \varphi_2(x/R) - \frac{4}{R^2} \Delta \varphi_2(x/R)e_1 \rangle| \leq C\|\nabla^2 \varphi_2/|x||_{L^1}
\]
We infer that
\[\langle U, \text{curl} \, \triangle V_R \rangle = O(1) \quad \text{as} \quad R \to \infty.
\]
It can be shown in a similar fashion that
\[\langle U \otimes U, \nabla \text{curl} \, V_R \rangle = O(1) \quad \text{as} \quad R \to \infty.
\]
We deduce from (3.2) that \( \langle f_0 + f_1, \text{curl} \, V_R \rangle \) must also be bounded as \( R \to \infty \). But
\[
\langle f_0 + f_1, \text{curl} \, V_R \rangle = \langle f_0 + f_1, -2\varphi^R e_1 + \nabla \varphi^R \times X \rangle \\
= -2\langle (f_0)_{1,}, \varphi^R \rangle - 2\langle (f_1)_{1,}, \varphi^R \rangle + \langle f_0, \nabla \varphi^R \times X \rangle + \langle f_1, \nabla \varphi^R \times X \rangle \\
\equiv I_1 + I_2 + I_3 + I_4.
\]
Recall that $\nabla \varphi^R$ can be expressed as in (3.3). By homogeneity, we observe that $I_3$ does not depend on $R$. Because $\varphi^R$ and $\nabla \varphi^R \times X$ are uniformly bounded and because $f_1$ is a bounded Radon measure, we have that $I_2$ and $I_4$ are bounded. We infer that $I_1$ must be bounded as $R \to \infty$. Since $f_0$ is homogeneous of degree $-3$ and $\varphi^R$ is radial, we have from (2.1) that

$$\int_{\mathbb{R}^3} (f_0)_1 \varphi^R = \int_0^\infty \frac{\varphi^R(r)}{r} \, dr \int_{S^2} (f_0)_1$$

But $\varphi^R$ is nonnegative and $\varphi^R(r) = 1$ for all $r \in [R_0 + 2, R]$ so

$$\int_0^\infty \frac{\varphi^R(r)}{r} \, dr \geq \int_{R_0+2}^R \frac{\varphi^R(r)}{r} \, dr \geq \int_{R_0+2}^R \frac{1}{r} \, dr = \ln(R) - \ln(R_0 + 2).$$

We conclude that for $\langle (f_0)_1, \varphi^R \rangle$ to be bounded it is necessary to have that $\int_{S^2} (f_0)_1 = 0$. The same argument can be applied to the other components of $f_0$, so we finally deduce that

$$\int_{S^2} f_0 = 0$$

must hold true. This completes the proof of the proposition.

\begin{proof}

\end{proof}

4 Homogeneous solutions

In this section, we prove Theorem 2.3.

Assume first that there exists $U_0 \in X_1$ homogeneous of degree $-1$ and some pressure $p_0$ such that

$$-\Delta U_0 + \text{div} (U_0 \otimes U_0) + \nabla p_0 = f_0 \quad \text{in} \ \mathbb{R}^3 \setminus \{0\}.$$ 

Proposition 3.1 immediately implies that (2.3) must hold true.

We prove now that there exists a constant vector $\gamma \in \mathbb{R}^3$ such that $U_0$ verifies (1.1) on $\mathbb{R}^3$ with forcing $\text{pv}(f_0) + \gamma \delta$.

Since $|U_0| \leq C/|x|$ we observe that $U_0 \in L^1_{\text{loc}}(\mathbb{R}^3)$. Therefore $U_0$ defines a distribution of $\mathcal{D}'(\mathbb{R}^3)$ which is homogeneous of degree $-1$. Taking the laplacian of this distribution implies that $\Delta U_0$ admits an extension to a distribution on the whole $\mathbb{R}^3$ which is homogeneous of degree $-3$. Similarly, $U_0 \otimes U_0$ is bounded by $C/|x|^2$ so it belongs to $L^1_{\text{loc}}(\mathbb{R}^3) \subset \mathcal{D}'(\mathbb{R}^3)$ so $\text{div}(U_0 \otimes U_0)$ also admits an extension to a distribution on the whole $\mathbb{R}^3$ which is homogeneous of degree $-3$.

Let us define $\overline{f}_0$ as follows

\begin{equation}
-\Delta U_0 + \text{div}(U_0 \otimes U_0) = \overline{f}_0.
\end{equation}

We have that $\overline{f}_0$ is a distribution on $\mathbb{R}^3$ which is homogeneous of degree $-3$. Clearly

$$\text{curl} \overline{f}_0 = \text{curl}(\Delta U_0 + \text{div}(U_0 \otimes U_0)) = \text{curl} f_0 \quad \text{in} \ \mathcal{D}'(\mathbb{R}^3 \setminus \{0\})$$

so $\text{curl} \overline{f}_0 - \text{curl} \text{pv}(f_0)$ is a distribution in $\mathcal{D}'(\mathbb{R}^3)$, homogeneous of degree $-4$ and supported in the origin. Therefore, there exist three constant vectors $A_1, A_2, A_3$ such that

\begin{equation}
\text{curl}(\overline{f}_0 - \text{pv}(f_0)) = A_1 \delta + A_2 \partial_2 \delta + A_3 \partial_3 \delta \quad \text{in} \ \mathcal{D}'(\mathbb{R}^3).
\end{equation}
Because of the identity \( \text{div} \, \text{curl} = 0 \) we infer that

\[
\text{div}(A_1 \partial_1 \delta + A_2 \partial_2 \delta + A_3 \partial_3 \delta) = 0
\]

so

\[
\sum_{i,j=1}^{3} a_{ij} \partial_{ij} \delta = 0
\]

where \( a_{ij} \) are the components of the matrix \( A \) whose columns are \( A_1, A_2, A_3 \). Because the derivatives of the Dirac mass are linearly independent we infer that the matrix \( A \) must be skew-symmetric. Therefore there exists some vector \( \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \) such that

\[
A = \begin{pmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix}.
\]

Then it can be easily checked that

\[
A_1 \partial_1 \delta + A_2 \partial_2 \delta + A_3 \partial_3 \delta = \text{curl}(\gamma \delta)
\]

so we deduce from (4.2) that

\[
\text{curl}(\mathcal{I}_0 - \text{pv}(f_0) - \gamma \delta) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3).
\]

We infer that there exists some distribution \( p \in \mathcal{D}'(\mathbb{R}^3) \) such that

\[
\mathcal{I}_0 - \text{pv}(f_0) - \gamma \delta = \nabla p
\]

so equation (4.1) can be rewritten under the form

\[
-\Delta U_0 + \text{div}(U_0 \otimes U_0) = \text{pv}(f_0) + \gamma \delta + \nabla p \quad \text{in } \mathcal{D}'(\mathbb{R}^3).
\]

This means that \( U_0 \) verifies the Navier-Stokes equation on \( \mathbb{R}^3 \) with forcing \( \text{pv}(f_0) + \gamma \delta \).

Assume now that \( \int_{\mathbb{S}^2} f_0 = 0 \). Then by Proposition 2.1 we have that \( \text{pv}(f_0) \) is well defined and homogeneous of degree \(-3\). It is not difficult to see that the Fourier transform of \( \text{pv}(f_0) \) is a bounded function, so \( \mathbb{P} \Delta^{-1} \text{pv}(f_0) \) can be easily defined in Fourier space. However, we need some estimates for \( \mathbb{P} \Delta^{-1} \text{pv}(f_0) \) in the space \( X_1 \) so we prefer to avoid using the Fourier transform and prove directly that \( \mathbb{P} \Delta^{-1} \text{pv}(f_0) \) is well defined and estimate it in \( X_1 \).

Let us recall that \( \mathbb{P} \Delta^{-1} \) is a convolution operator with kernel given by the following matrix (see [6]):

\[
G(x) = -\frac{1}{8\pi} \left( \frac{I_3}{|x|} + \frac{x \otimes x}{|x|^3} \right)
\]

where \( I_3 \) is the identity matrix.

We prove next that the convolution \( G \ast f_0 \) is well defined.
Lemma 4.1. The convolution $G \ast f_0$ is well defined in the principal value sense:

$$G \ast f_0(x) = \text{pv} \int_{\mathbb{R}^3} G(x - y)f_0(y) \, dy$$

and we have the bound

$$\|G \ast f_0\|_{X_1} \leq C\|f_0\|_{X_3} = C\|f_0\|_{L^\infty(S^2)}$$

for some universal constant $C$.

Proof. The existence of $G \ast f_0$ in the principal value sense follows from the estimates below. We decompose

$$\text{pv} \int_{\mathbb{R}^3} G(x - y)f_0(y) \, dy = \int_{|y|>|x|/2} G(x - y)f_0(y) \, dy + \int_{2|x|>|y|>|x|/2} G(x - y)f_0(y) \, dy$$

$$+ \text{pv} \int_{|x|/2>|y|} G(x - y)f_0(y) \, dy = \int_{|y|>|x|/2} G(x - y)f_0(y) \, dy + \int_{2|x|>|y|>|x|/2} G(x - y)f_0(y) \, dy$$

$$+ \int_{|x|/2>|y|} (G(x - y) - G(x))f_0(y) \, dy \equiv I_1 + I_2 + I_3$$

where we used that $f_0$ has vanishing mean on the unit sphere so

$$\text{pv} \int_{|x|/2>|y|} f_0(y) \, dy = 0.$$

We bound now each of these terms. Observe first that

$$|I_1| = \left| \int_{|y|>|x|/2} G(x - y)f_0(y) \, dy \right| \leq \int_{|y|>|x|/2} \frac{C\|f_0\|_{L^\infty(S^2)}}{|x - y||y|^3} \, dy$$

$$\leq C\|f_0\|_{L^\infty(S^2)} \int_{|y|>|x|/2} \frac{1}{|y|^4} \, dy \leq \frac{C\|f_0\|_{L^\infty(S^2)}}{|x|}.$$

Next

$$|I_2| = \left| \int_{2|x|>|y|>|x|/2} G(x - y)f_0(y) \, dy \right| \leq \int_{2|x|>|y|>|x|/2} \frac{C\|f_0\|_{L^\infty(S^2)}}{|x - y||y|^3} \, dy$$

$$\leq \frac{C\|f_0\|_{L^\infty(S^2)}}{|x|^3} \int_{|x-y|<|3x|} \frac{1}{|x-y|} \, dy \leq \frac{C\|f_0\|_{L^\infty(S^2)}}{|x|}.$$

We estimate now $I_3$. By the mean value theorem we have that $G(x - y) - G(x) = y \cdot \nabla G(\xi)$ for some $\xi \in [x, x - y]$. If $|y| < |x|/2$ then we can bound

$$|G(x - y) - G(x)| \leq |y| \sup_{\xi \in [x, x - y]} |\nabla G(\xi)| \leq C|y| \sup_{\xi \in [x, x - y]} \frac{1}{|\xi|^2} \leq C \frac{|y|}{|x|^2}.$$

Therefore

$$|I_3| = \left| \int_{|x|/2>|y|} (G(x - y) - G(x))f_0(y) \, dy \right| \leq \frac{C\|f_0\|_{L^\infty(S^2)}}{|x|^2} \int_{|x|/2>|y|} \frac{1}{|y|^2} \, dy \leq \frac{C\|f_0\|_{L^\infty(S^2)}}{|x|}.$$

Putting together the above estimates completes the proof of the lemma. □
Once this lemma is proved, it is not difficult to finish the proof of Theorem 2.3. Indeed, we have that $G * \delta = G \in X_1$ so $G * (\text{pv}(f_0) + \gamma \delta) \in X_1$. Therefore $\mathbb{P} \Delta^{-1}(\text{pv}(f_0) + \gamma \delta)$ is well defined, belongs to $X_1$ and we have the estimate

(4.4) \[ \| \mathbb{P} \Delta^{-1}(\text{pv}(f_0) + \gamma \delta) \|_{X_1} \leq C \| f_0 \|_{L^\infty(S^2)} + C |\gamma|. \]

To show the existence and the uniqueness of a small solution $U_0$ it suffices to appeal to the following result from [1, Theorems 2.2 and 3.1]:

**Theorem 4.2** ([1]). There exists an absolute constant $\varepsilon_0 > 0$ with the following property. If $f$ is such that $\mathbb{P} \Delta^{-1} f \in X_1$ and $\| \mathbb{P} \Delta^{-1} f \|_{X_1} < \varepsilon_0$ then there exists a unique solution $U \in X_1$ of (2.2) such that

\[ \| U \|_{X_1} \leq 2 \| \mathbb{P} \Delta^{-1} f \|_{X_1} \]

Because of (4.4), we can apply the above theorem and deduce that if $\| f_0 \|_{L^\infty(S^2)} + |\gamma|$ is sufficiently small, then there exists a unique small solution $U_0$ of (2.2) with forcing $\text{pv}(f_0) + \gamma \delta$. Moreover, we have that

(4.5) \[ \| U_0 \|_{X_1} \leq C \| f_0 \|_{L^\infty(S^2)} + C |\gamma|. \]

It remains to show that $U_0$ is homogeneous of degree $-1$. This follows from a standard scaling argument. By homogeneity, $\text{pv}(f_0) + \gamma \delta$ is invariant through the scaling $f \mapsto \lambda^3 f(\lambda x)$. Moreover, the $X_1$ norm of $\mathbb{P} \Delta^{-1} f$ is also invariant through this scaling. But if $U_0$ solves the Navier-Stokes equations with forcing $f$ then $U_{0,\lambda}(x) = \lambda U_0(\lambda x)$ also solves the Navier-Stokes equations with forcing $\lambda^3 f(\lambda x)$. By uniqueness of solutions, we must have that $U_{0,\lambda} = U_0$ for all $\lambda > 0$. Therefore the solution $U_0$ must be homogeneous of degree $-1$.

## 5 Asymptotic behaviour for general solutions

In this section we prove Theorem 2.4.

The necessity of the condition (2.3) for the existence of $U \in X_1$ is a consequence of Proposition 3.1.

We assume now that the condition (2.3) is verified. It can be proved exactly like in Lemma 4.1 that $\mathbb{P} \Delta^{-1}(\phi f_0) = G * (\phi f_0)$ is well defined and belongs to $X_1$ and that

\[ \| \mathbb{P} \Delta^{-1}(\phi f_0) \|_{X_1} \leq C \| f_0 \|_{L^\infty(S^2)} \]

for some universal constant $C$. Since $f_1 \in X_0 \cap X_{3+\alpha}$, we know from [1, relation (3.1)] that $\mathbb{P} \Delta^{-1} f_1 \in X_1$ and

\[ \| \mathbb{P} \Delta^{-1} f_1 \|_{X_1} \leq C \| f_1 \|_{X_0 \cap X_{3+\alpha}}. \]

We infer that $\mathbb{P} \Delta^{-1} f \in X_1$ and

\[ \| \mathbb{P} \Delta^{-1} f \|_{X_1} \leq C (\| f_0 \|_{L^\infty(S^2)} + \| f_1 \|_{X_0 \cap X_{3+\alpha}}) \leq C \varepsilon_2. \]

Let us recall now the following result, see [1, Lemma 3.6]:

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Lemma 5.1. Let $U_1 \in X_{1+\beta}$ and $U_2 \in X_{1+\gamma}$ with $\beta, \gamma \geq 0$ and $\beta + \gamma < 1$. We have that $\mathbb{P}\Delta^{-1}\text{div}(U_1 \otimes U_2) \in X_{1+\beta+\gamma}$ and

$$\|\mathbb{P}\Delta^{-1}\text{div}(U_1 \otimes U_2)\|_{X_{1+\beta+\gamma}} \leq C\|U_1\|_{X_{1+\beta}}\|U_2\|_{X_{1+\gamma}}.$$ 

Therefore the application $X_1 \times X_1 \ni (U_1, U_2) \mapsto \mathbb{P}\Delta^{-1}\text{div}(U_1 \otimes U_2) \in X_1$ is bilinear and continuous. Applying Lemma 2.2 we deduce that if $\varepsilon_2$ is sufficiently small, then there exists a unique solution $U$ of (2.2) which is small in $X_1$. We know from the proof of Lemma 2.2 that the solution $U$ is obtained as the limit of the approximate solutions defined by

$$U^0 = -\mathbb{P}\Delta^{-1}f, \quad U^{k+1} = \mathbb{P}\Delta^{-1}\text{div}(U^k \otimes U^k) - \mathbb{P}\Delta^{-1}f.$$ 

Moreover, we have that $\|U^k\|_{X_1} \leq C\varepsilon_2$ for all $k$.

Let $U_0$ be the only $-1$ homogeneous solution of the Navier-Stokes equation with forcing term $pv(f_0) + (pv \int_{\mathbb{R}^3} f)\delta$ as obtained in Theorem 2.3. Let us define $v = U - U_0$. To complete the proof of Theorem 2.4 we need to show that $v \in X_{3+\alpha}$. We have that $v$ solves the following equation:

$$v = \mathbb{P}\Delta^{-1}\text{div}(v \otimes v + v \otimes U_0 + U_0 \otimes v) - \mathbb{P}\Delta^{-1}f_2$$

where

$$f_2 = (\phi - 1)f_0 + f_1 - (\int_{\mathbb{R}^3} f_1)\delta.$$ 

We used above that $\int_{S^2} f_0 = 0$ and that $\phi$ is radial to deduce that $pv \int_{\mathbb{R}^3} \phi f_0 = 0$ so $\int_{\mathbb{R}^3} f_1 = pv \int_{\mathbb{R}^3} f$. Because $U$ is the limit of the approximate solutions $U^k$, we have that $v$ is the limit of the approximate solutions $v_k = U^k - U_0$ which solve

$$v_0 = -\mathbb{P}\Delta^{-1}f_2, \quad v_{k+1} = \mathbb{P}\Delta^{-1}\text{div}(v_k \otimes v_k + v_k \otimes U_0 + U_0 \otimes v_k) - \mathbb{P}\Delta^{-1}f_2.$$ 

We have that

$$\|v_k\|_{X_1} \leq C\varepsilon_2 \quad \forall k.$$ 

Indeed, we observed above that $\|U^k\|_{X_1} \leq C\varepsilon_2$ for all $k$ and we also know from the proof of Theorem 2.3 that $\|U_0\|_{X_1} \leq C\varepsilon_2$ (see relation (4.5)).

We will show by induction that there exists some small $\varepsilon_3$ such that if $\varepsilon_2$ is small enough then $v_k \in X_{1+\alpha}$ for all $k$ and

$$\|v_k\|_{X_{1+\alpha}} \leq \varepsilon_3.$$ 

The first step is to prove this bound for $k = 0$.

Lemma 5.2. We have that

$$\|\mathbb{P}\Delta^{-1}f_2\|_{X_{1+\alpha}} \leq C(\|f_0\|_{L^\infty(S^2)} + \|f_1\|_{X_{3+\alpha}}).$$
Proof. Recall that $\mathbb{P}\Delta^{-1}$ is a convolution operator with the kernel $G$ defined in (4.3). We write

$$\mathbb{P}\Delta^{-1}(\phi f_0 + f_1)(x) = \int_{\mathbb{R}^3} G(x - y) \cdot (\phi(y)f_0(y) + f_1(y))dy$$

where we assume that $I \leq |P|$. We can estimate

$$\int_{\mathbb{R}^3} G(x - y) \cdot (\phi(y)f_0(y) + f_1(y))dy$$

We deduce that

$$|\mathbb{P}\Delta^{-1}(f_1)(x)| \leq \left| \int_{|y| \leq 1} (G(x - y) - G(x)) \cdot (\phi - 1)f_0(y)dy \right| + \left| \int_{|y| > 1} (G(x - y) - G(x)) \cdot (\phi - 1)f_0(y)dy \right|$$

Therefore, we can bound $I_1$ by

$$I_1 \leq \int_{|y| \leq \min(|x|/2, 1)} |G(x - y) - G(x)||\phi - 1|f_0(y)dy$$

where $\zeta(x, y)$ is between $x$ and $x - y$, so $1/|\zeta| \leq C/|x|^2$

$$\leq C\|f_0\|_{L^\infty(S^2)} \int_{|y| \leq \min(|x|/2, 1)} \frac{|\nabla G(\zeta(x, y)) \cdot y|}{|y|^3}dy$$

Next, if $|x| \leq 2$ then we can bound $I_2$ exactly as in Lemma 4.1 to obtain that

$$|I_2| \leq \frac{C\|f_0\|_{L^\infty(S^2)}}{|x|} \leq \frac{C\|f_0\|_{L^\infty(S^2)}}{|x|^{1+\alpha}}.$$
where $\zeta(x, y)$ is between $x$ and $x - y$, so $1/|\zeta|^2 \leq C/|x|^2$

\[
\frac{C}{|x|^2} \int_{|y|\leq |x|/2} \frac{1}{|y|^{2+\alpha}} dy \\
\leq \frac{C}{|x|^{1+\alpha}}.
\]

Finally

\[
I_4 \leq \int_{|y|> |x|/2} |G(x - y) - G(x)||f_1(y)|dy \\
\leq \int_{|x|/2 \leq |y| \leq |x| | } |G(x - y)| |f_1(y)|dy + \int_{|y|> |x|/2} |G(x - y)| |f_1(y)|dy + \int_{|y|> |x|/2} |G(x)||f_1(y)|dy \\
\leq C||f_1||_{X^{3+\alpha}} \left( \frac{1}{|x|^{3+\alpha}} \int_{|x|/2 \leq |y| \leq |x| | } \frac{1}{|x - y|} dy + \int_{|y|> |x|/2} \frac{1}{|y|^{4+\alpha}} dy + \int_{|y|> |x|/2} \frac{1}{|x|} \frac{1}{|y|^{3+\alpha}} dy \right) \\
\leq \frac{C||f_1||_{X^{3+\alpha}}}{|x|^{1+\alpha}}.
\]

This completes the proof of the lemma.

We now go back to the proof of relation (5.2). From Lemma 5.2 we deduce that

\[
\|v_0\|_{X^{1+\alpha}} \leq C\varepsilon_2.
\]

We impose the condition $C\varepsilon_2 \leq \varepsilon_3$ so that (5.2) is verified for $k = 0$. Suppose that we have proved (5.2) for $v_k$, we want to prove it for $v_{k+1}$. We use relation (5.1) and Lemma 5.1 to estimate

\[
\|v_{k+1}\|_{X^{1+\alpha}} \leq C(\|v_k\|_{X^{1+\alpha}} \|v_k\|_{X^{1}} + \|v_k\|_{X^{1+\alpha}} \|U_0\|_{X^{1}}) + \|v_0\|_{X^{1+\alpha}} \\
\leq C\varepsilon_2\|v_k\|_{X^{1+\alpha}} + C\varepsilon_2 \\
\leq C\varepsilon_2(1 + \varepsilon_3).
\]

Clearly if $\varepsilon_2$ is sufficiently small then we can choose some $\varepsilon_3$ such that $C\varepsilon_2(1 + \varepsilon_3) \leq \varepsilon_3$. For such a choice of $\varepsilon_2$ and $\varepsilon_3$, relation (5.2) holds true for all $k$. Therefore it must hold true for the limit $v$. So $v \in X^{1+\alpha}$ and this completes the proof of Theorem 2.4.

6 The exterior domain case

In this section, we prove the following extension of Theorem 2.4 to exterior domains.

**Theorem 6.1.** Let $R > 0$ and consider $f, U$ and $p$ be defined for $|x| > R$. Assume that $U \in X_1$ and $f = f_0 + f_1$ with $f_0 \in X_3$ homogeneous of degree $-3$ and $f_1 \in X_{3+\alpha}$. We assume moreover that

\[
-\Delta U + (U \cdot \nabla) U + \nabla p = f, \quad \text{div } U = 0 \quad \text{in} \quad \{|x| > R\},
\]

that condition (2.3) holds true and that

\[
\int_{|x|=R_1} U \cdot x = 0
\]

(6.1)
for all $R_1 > R$. There exists $\varepsilon > 0$ such that, if

$$\|f_0\|_{X_{\delta}} + \|f_1\|_{X_{\alpha}} + \|U\|_{X_1} \leq \varepsilon$$

then there exists some small constant vector $m_0$ such that $U$ admits the following asymptotic behavior:

$$U(x) = U_0(x) + O\left(\frac{1}{|x|^{1+\alpha}}\right) \quad \text{as } |x| \to \infty$$

where $U_0$ is the only $-1$ homogeneous solution of the Navier-Stokes equation with forcing term $\text{pv}(f_0) + m_0\delta$ obtained in Theorem 2.3. Moreover, we can express $m_0$ under the following form

$$(6.2) \quad m_0 = \int_{|x| > R_1} f_1 + \int_{|x| = R_1} [-\partial_p U + (U \cdot \nu)U + pv] \quad \forall R_1 > R$$

where $\nu = x/|x|$.

**Remark 6.2.** The condition (6.1) is natural. It is automatically satisfied if $U$ verifies the Navier-Stokes equations together with homogeneous Dirichlet boundary conditions in an exterior domain $\Omega$. This can be easily checked by integrating the relation $\text{div} \ U = 0$ on $\Omega \cap \{|x| < R_1\}$. Moreover, a similar integration by parts shows that the integral in (6.1) is proportional to $R_1$ so it suffices to assume (6.1) for only one $R_1$.

To prove Theorem 6.1, we proceed as in [9]. We extend $U$ to the whole space, we study the additional forcing term that appears and we apply Theorem 2.4.

Let $\Omega = \{|x| > R\}$. First, let us observe that $U$ and $p$ are more regular than stated:

**Lemma 6.3.** We have that $(U, p) \in W^{2,q}_{\text{loc}}(\Omega) \times W^{1,q}_{\text{loc}}(\Omega)$ for any $1 < q < \infty$.

**Proof of the lemma.** The proof follows from the following interior regularity result for the stationary Stokes equation proved in [13]:

**Theorem 6.4 (\cite{13}).** Let $\Omega$ be a domain in $\mathbb{R}^3$, and $B_1 \subset B_2$ be concentric balls of radii $R$ and $2R$, strictly contained in $\Omega$. Let $1 < q < \infty$, and $f \in W^{-1,q}_{\text{loc}}(\Omega)$. If $(U, p) \in W^{1,q}_{\text{loc}}(\Omega) \times L^q_{\text{loc}}(\Omega)$ is a pair of solutions of the stationary Stokes system with forcing term $f$ in $\Omega$ (without any boundary condition), then

$$\|U\|_{W^{1,q}(B_1)} + \inf_{c \in \mathbb{R}} \|p - c\|_{L^q(B_1)} \leq C(\|U\|_{L^1(B_2 \setminus B_1)} + \|f\|_{W^{-1,q}(B_2)})$$

This theorem implies the following regularity result: if $U \in L^1_{\text{loc}}(\Omega)$ verifies the Stokes equation in $\Omega$ with forcing $f \in W^{-1,q}_{\text{loc}}(\Omega)$, then $U \in W^{1,q}_{\text{loc}}(\Omega)$. Indeed, we can approximate $U$ by using cut-off and convolution with an approximation of the identity. The regularized velocity verifies the Stokes equation on a sub-domain of $\Omega$ (where the cut-off function is 1), so by Theorem 6.4 it will be bounded in $W^{1,q}_{\text{loc}}$ of that sub-domain. So its limit $U$ will also belong to $W^{1,q}_{\text{loc}}$ of that sub-domain. Since the sub-domain can be an arbitrary bounded sub-domain of $\Omega$, we infer that $U \in W^{1,q}_{\text{loc}}(\Omega)$.

Going back to the proof of the lemma we observe that $f \in W^{-1,q}_{\text{loc}}(\Omega)$ and $\text{div}(U \otimes U) \in W^{-1,q}_{\text{loc}}(\Omega)$ for any $1 < q < \infty$. Since we have

$$-\Delta U + \nabla p = f - \text{div}(U \otimes U)$$
$$\text{div} U = 0$$

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in \( \Omega \), we can apply the above mentioned regularity result to obtain that \( U \in W^{1,q}_{loc}(\Omega) \) for any \( 1 < q < \infty \). Then we can derive these equations and we get:

\[
-\Delta(\partial_j U) + \nabla(\partial_j p) = \partial_j f + \text{div}(\partial_j(U \otimes U))
\]

\[\text{div}(\partial_j U) = 0\]

for any \( j \in \{1, 2, 3\} \). Again \( \partial_j f \in W^{-1,q}_{loc}(\Omega) \) for any \( 1 < q < \infty \). Moreover, since \( U \in L^\infty_{loc}(\Omega) \cap W^{1,q}_{loc}(\Omega) \), we have that \( \partial_j(U \otimes U) = \partial_j U \otimes U + U \otimes \partial_j U \in L^3_{loc}(\Omega) \) for any \( 1 < q < \infty \). So \( \text{div} \partial_j(U \otimes U) \in W^{-1,q}_{loc}(\Omega) \) for any \( 1 < q < \infty \). Reapplying the same regularity result implies that \( U \in W^{1,q}_{loc}(\Omega) \) for any \( 1 < q < \infty \). From the equation we also get that \( \nabla p \in L^3_{loc}(\Omega) \) so \( p \in W^{1,q}_{loc}(\Omega) \) for any \( 1 < q < \infty \). This completes the proof of the lemma.

Let \( R < R_0 < R_1 \) and consider a radial cut-off function \( \eta \in C^\infty(\mathbb{R}^3, [0, 1]) \) such that \( \eta = 0 \) on \( B(0, R_0) \) and \( \eta = 1 \) on \( B(0, R_1)^c \). We define the following extension of the solution \( (U, p) \):

\[
\tilde{U} = U,
\]

\[
\tilde{p} = p
\]

on \( B(0, R_1)^c \)

\[
\tilde{U} = \eta U + v,
\]

\[
\tilde{p} = \eta p
\]

on \( B(0, R_1) \)

where we extended \( \eta U \) and \( \eta p \) with zero values for \( |x| \leq R \). The vector field \( v \) is constructed in such a manner as to ensure that \( \tilde{U} \in L^\infty \cap X_1 \) and \( \text{div} \tilde{U} = 0 \) everywhere. Therefore \( v \) must verify the following problem:

\[
\text{div} v = -U \cdot \nabla \eta \text{ in } B(0, R_1)
\]

\[
v = 0 \text{ on } \partial B(0, R_1)
\]

This problem has many solutions, and a way to find one with good estimates is given by the Bogovskii operators, see [2]. In particular, we have the following result:

**Theorem 6.5** ([2]). Let \( g \in W^{k,q}_0(B) \) where \( B \) is a ball and \( k \in \mathbb{N}, 1 < q < \infty \). Assume that \( \int_B g = 0 \). Then there exists a solution \( V \in W^{k+1,q}_0(B) \) of the equation \( \text{div} V = g \), with the following estimate:

\[
\|V\|_{W^{k+1,q}_0(B)} \leq C(q, k, B)\|g\|_{W^{k,q}_0(B)}.
\]

We have that

\[
\int_{B(0, R_1)} U \cdot \nabla \eta = \int_{S(0, R_1)} U \cdot \nu \eta - \int_{B(0, R_1)} \eta \text{div} U = \int_{S(0, R_1)} U \cdot \nu = 0
\]

where we used (6.1). Moreover, because \( U \in W^{2,q}_{loc}(\Omega) \) for any \( 1 < q < \infty \) we have that \( U \cdot \nabla \eta \in W^{2,q} \) for any \( 1 < q < \infty \). Using Theorem 6.5 we infer that there exists \( v \in W^{3,q}_0(B(0, R_1)) \) for all \( 1 < q < \infty \) a solution of (6.3). We extend \( v \) to the whole space \( \mathbb{R}^3 \) by setting \( v = 0 \) for \( |x| > R_1 \) so that \( v \in W^{3,q}(\mathbb{R}^3) \).

We observe now that the extension \( (\tilde{U}, \tilde{p}) \) verifies the following stationary Navier-Stokes equation in the whole space:

\[
-\Delta \tilde{U} + (\tilde{U} \cdot \nabla) \tilde{U} + \nabla \tilde{p} = \eta f + F, \quad \text{div} \tilde{U} = 0
\]

where

\[
F = -\Delta v - \Delta \eta U - 2\Delta U \cdot \nabla \eta + \text{div}(\eta U \otimes v + \eta v \otimes U + v \otimes v + \eta^2 U \otimes U) - \eta \text{div}(U \otimes U) + p \nabla \eta
\]
is compactly supported in $B(0, R_1)$.

Given that $U \in W^{2,q}_{\text{loc}}$, $p \in W^{1,q}_{\text{loc}}$ and $v \in W^{3,q}$ for all $1 < q < \infty$, one can use Sobolev embeddings to deduce that $F$ is bounded. From the estimates of Theorems 6.4 and 6.5 we know that the $W^{2,q}_{\text{loc}}$ norms of $U$ and the $W^{3,q}$ norm of $v$ can be bounded in terms of the $L^\infty_{\text{loc}}$ norms of $U$ and $f$. Because these norms are assumed to be small, we conclude that $F$ is bounded, small and compactly supported. Moreover, $\bar{U}$ is also small in $X_1$. We can therefore apply Theorem 2.4 to (6.4) to deduce that $\bar{U}$ has the following asymptotic behavior:

$$\bar{U} = U_0 + O\left(\frac{1}{|x|^{1+\alpha}}\right)$$

where $U_0$ is the only $-1$ homogeneous solution of the Navier-Stokes equation with forcing term $pv(f_0) + m_0 \delta$ obtained in Theorem 2.3 and

$$m_0 = \int_{\mathbb{R}^3} \eta f_1 + \int_{\mathbb{R}^3} F.$$

It remains to prove relation (6.2). We integrate relation (6.4) on the ball $B(0, R_1)$ and use the Stokes formula to obtain that

$$\int_{B(0,R_1)} (\eta f_1 + F) = \int_{B(0,R_1)} \text{div}(-\nabla \bar{U} + \bar{U} \otimes \bar{U} + \bar{p} \text{Id}) = \int_{S(0,R_1)} [-\partial_\nu U + (U \cdot \nu) U + pv]$$

Since $F$ is compactly supported in $B(0, R_1)$ we have that

$$\int_{B(0,R_1)} F = \int_{\mathbb{R}^3} F$$

so relation (6.2) follows immediately.

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